Two hours

THE UNIVERSITY OF MANCHESTER

FRACTAL GEOMETRY

04 June 2018

09.45 - 11.45

Answer **ALL FIVE** questions in **Section A** (40 marks in total) and
TWO of the **THREE** questions in **Section B** (50 marks in total).
If more than two questions from Section B are attempted then credit will be given for the best two answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL FIVE** questions

A1.

(a) Let (X,d) be a metric space and $E \subset X$. Let $s \geq 0, \delta > 0$.
Define what is meant by a δ-*cover* of E. Define $\mathcal{H}^s_\delta(E)$.

[3 marks]

(b) Define $\mathcal{H}^s(E)$, and justify why any limits that you used in your definition exist.

[4 marks]

(c) Let $\delta > 0$ and let $0 < s < t$. Show that

$$\mathcal{H}^s_\delta(E) \geq \delta^{s-t}\mathcal{H}^t_\delta(E).$$

Prove a relationship between $\mathcal{H}^s(E)$ and $\mathcal{H}^t(E)$ and hence define $\dim_H(E)$.

[6 marks]

A2.

(a) Give the definition of an outer measure μ on a space (X,d).

[4 marks]

(b) Give the definition of a mass distribution μ on a space (X,d)

[2 marks]

(c) State without proof the mass distribution principle.

[3 marks]

(d) Use the mass distribution principle to prove that $\dim_H([0,1]) \geq 1$. You do not need to prove that any mass distributions that you define are mass distributions.

[3 marks]

A3. Let (X,d) denote $\mathbb{R}$ with the Euclidean metric. Let $A \subset [0,1]$. Use the definition of upper box dimension to show that

$$\overline{\dim}_B(A) = \overline{\dim}_B\left(A \cup \{0\}\right).$$

[4 marks]

A4. Let

$$G = \{x \in [0,1] : x \text{ has a decimal expansion containing only the digits } 0, 1, 5\}$$

Write down a collection $\phi_1 \cdots \phi_N$ of contractions of $[0,1]$ such that

$$G = \bigcup_{i=1}^{N} \phi_i(G).$$

[5 marks]

A5. Let (X,d) be $\mathbb{R}$ together with Euclidean metric, and E be a bounded subset of (X,d). Mark each of the following statements as true or false. If they are false, give a counter example. You do not need to prove that any counterexamples that you give are counterexamples.

(a) If $\overline{\dim}_B(E) > 0$ then $\dim_H(E) > 0$.

(b) $\dim_H(E+c) = \dim_H(E)$ where $c \in X$ and $E + c := \{x + c : x \in E\}$.

(c) If E is countable then $\dim_H(E) = 0$.

[6 marks]

SECTION B

Answer **TWO** of the **THREE** questions

B6.

(a) Define $\mathcal{N}_\delta(\mathcal{A})$ for a bounded set $A \subset (X, d)$. Hence define the upper box dimension and lower box dimension of A, and say what it means for the box dimension of A to exist.

[5 marks]

(b) Let A and B be bounded sets with $\overline{\dim}_B(A) = \alpha$, $\overline{\dim}_B(B) = \beta$. Prove that

$$\overline{\dim}_B(A \cup B) \leq \max\{\alpha, \beta\}.$$

[7 marks]

(c) Let (X, d) be a metric space and $A \subset X$ be a bounded set. Let

$$2A = \{2x : x \in A\}.$$

Prove that

$$\overline{\dim}_B(2A) = \overline{\dim}_B(A).$$

[7 marks]

(d) Prove that

$$\underline{\dim}_B \left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\} \geq \frac{1}{3}.$$

You may wish to begin by giving a lower bound for $\mathcal{N}_\delta(\{\frac{1}{n^2} : n \in \mathbb{N}\})$ when

$$\frac{1}{k^2} - \frac{1}{(k+1)^2} \leq \delta < \frac{1}{(k-1)^2} - \frac{1}{k^2}$$

for some $k \in \mathbb{N}$.

[6 marks]

B7.

(a) Let maps $\phi_1, \phi_2, \phi_3 : [0,1] \to [0,1]$ be given by $\phi_1(x) = \frac{x}{4}$, $\phi_2(x) = \frac{x+2}{4}$, $\phi_3(x) = \frac{x+3}{4}$. Let

$$E = \bigcup_{i=1}^{3} \phi_i(E)$$

be the attractor of the iterated function system $\{\phi_1, \phi_2, \phi_3\}$. Sketch the first two levels of the construction of E. What is the similarity dimension of E?

[5 marks]

(b) Prove that $\overline{\dim}_B(E) \leq \frac{\log 3}{\log 4}$ by using a covering argument.

[7 marks]

(c) Use the mass distribution principle to prove that $\dim_H(E) \geq \frac{\log 3}{\log 4}$.

[7 marks]

(d) By defining a suitable mass distribution, prove that $\dim_H(E \times [0,1]) \geq 1 + \frac{\log 3}{\log 4}$.

[6 marks]

B8.

Let (X, d) be a metric space and let the maps $\phi_1, \phi_2, \cdots \phi_N$ be contractive similarities on (X, d).

(a) Define the space $\mathcal{S}(X)$ and the map $\Phi : \mathcal{S}(X) \to \mathcal{S}(X)$ associated to $\{\phi_1, \cdots, \phi_N\}$. State Hutchinson's theorem.

[6 marks]

(b) Prove Hutchinson's theorem. You may use the fact that $\mathcal{S}(X)$ is a complete metric space.

[7 marks]

(c) Write down the open set condition and the definition of the similarity dimension of Φ. You may NOT assume that the maps ϕ_i all have the same contraction ratio.

[5 marks]

(d) Now let (X, d) be $\mathbb{R}$ together with Euclidean metric. Let $\phi_1(x) = \frac{x}{5}$, $\phi_2(x) = \frac{x+2}{3}$. Let

$$E = \bigcup_{i=1}^{2} \phi_i(E).$$

Prove that

$$\dim_H(E) \leq \dim_{sim}(E).$$

[7 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

TOPOLOGY

31 May 2018

14.00 - 16.00

Answer **ALL** SIX questions

University approved calculators may be used.

1.

(a) Define what is meant by a *topology* on a set X.

(b) Define what is meant by saying that a function $f\colon X \to Y$ between topological spaces is *continuous*. Define what is meant by saying that f is a *homeomorphism*.

(c) Consider the set $X = \{1, 2, 3\}$ and $\tau = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$. Show that τ defines a topology on X. Find all homeomorphisms $X \to X$.

[13 marks]

2.

(a) Define what is meant by saying that a topological space X is *path-connected*.

(b) What is meant by saying that path-connectedness is a *topological property*?

(c) Define what is meant by the set $\pi_0(X)$ of *path-components* of a topological space X.

(d) Prove that a continuous map of topological spaces $f\colon X \to Y$ induces a map $f_*\colon \pi_0(X) \to \pi_0(Y)$ between the sets of path-components, taking care to prove that your function is well-defined. Prove that if f is a homeomorphism then f_* is a bijection.

(e) Use (d) to show that path-connectedness is a topological property.

(f) Show that $\mathbb{R} \setminus \{0\}$ has exactly two path-components, i.e. $\#\pi_0(\mathbb{R} \setminus \{0\}) = 2$,

[15 marks]

3.

(a) Define what is meant by saying that a topological space is *Hausdorff*.

(b) Which of the following spaces are Hausdorff? Justify your answers.

(i) $\mathbb{Z}$ with the discrete topology $\tau_1 = \mathcal{P}(X)$,

(ii) $\mathbb{Z}$ with the indiscrete topology $\tau_2 = \{\emptyset, \mathbb{Z}\}$,

(iii) $\mathbb{Z}$ with the co-finite topology $\tau_3 = \{U \subset \mathbb{Z} \mid \mathbb{Z} \setminus U \text{ is finite}\} \cup \{\emptyset, \mathbb{Z}\}$.

(c) Suppose that X and Y are topological spaces. Define the *product topology* on the Cartesian product $X \times Y$. [It is not necessary to prove that this is a topology.]

(d) Let $\Delta = \{(x, x) \mid x \in X\} \subset X \times X$ be the diagonal. Prove that if Δ is closed in the product topology, then X is Hausdorff.

[15 marks]

4.

(a) Define what is meant by a *compact subset* of a topological space.

(b) Prove by using only the definition of compactness, that $[0, 1) \subset \mathbb{R}$ is not compact.

(c) Prove that every compact subset of a Hausdorff topological space is closed.

(d) Give an example to show that the Hausdorff condition in (c) cannot be omitted.

[15 marks]

5.

(a) Suppose that $q\colon X \to Y$ is a surjection from a topological space X to a set Y . Define the *quotient topology* on Y determined by q. State the *universal property* of the quotient topology.

(b) Suppose that $f\colon X \to Z$ is a continuous surjection from a compact topological space X to a Hausdorff topological space Z . Define an equivalence relation $\sim$ on X so that f induces a bijection $F\colon X/\!\sim\, \to Z$ from the identification space $X/\!\sim$ of this equivalence relation to Z. Prove that F is a homeomorphism. [State clearly any general results which you use.]

(c) Prove that

$$f\colon [-1, 1] \times \{-1, 1\} \to S^1;\ (t, a) \mapsto (t, a \cdot \sqrt{1 - t^2}).$$

induces a homeomorphism from the quotient space $([-1, 1] \times \{-1, 1\})/\!\sim$ to the circle $S^1 \subset \mathbb{R}^2$, where $\sim$ is generated by $(1, 1) \sim (1, -1)$ and $(-1, 1) \sim (-1, -1)$, i.e.

$$(t, a) \sim (t', a') \Leftrightarrow t = t' \text{ and either } a = a' \text{ or } |t| = 1.$$

[15 marks]

6.

(a) Prove that, if the product $\sigma_0 * \tau_0$ of two paths σ_0 and τ_0 in a topological space X is defined and the paths σ_1 and τ_1 are homotopic to σ_0 and τ_0 respectively, then the product $\sigma_1 * \tau_1$ is defined and is homotopic to $\sigma_0 * \tau_0$.

(b) Explain how a continuous function $f\colon X \to Y$ induces a homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(X, f(x_0))$. You should indicate why f_* is well-defined and why it is a homomorphism.

(c) Prove that, for topological spaces X and Y with points $x_0 \in X$, $y_0 \in Y$, there is an isomorphism of groups

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

[15 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY

06 June 2018

09.45 - 11.45

Answer **ALL FOUR** questions in Section A (50 marks in total).
Answer **TWO** of the THREE questions in Section B (40 marks in total).
If more than TWO questions in Section B are attempted, the credit will be given for the best TWO answers.

Electronic calculators may not be used.

Throughout the paper, where the index notation is used, the Einstein summation convention over repeated indices is applied if it is not explicitly stated otherwise.

SECTION A

Answer **ALL** FOUR questions

A1.

(a) Explain what is meant by saying that G is a Riemannian metric on a manifold M.

(b) Let $G = g_{ik}(x)dx^i dx^k$, $i, k = 1, \dots, n$ be a Riemannian metric on an n-dimensional manifold M.
Show that all diagonal components $g_{11}(x), g_{22}(x), \dots, g_{nn}(x)$ are positive functions.

(c) Consider the plane $\mathbf{R}^2$ with standard coordinates x, y equipped with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{1 + x^2 + y^2} \,.$$

Consider vectors $\mathbf{A} = 2\partial_x + \partial_y$ and $\mathbf{B} = \partial_x + 2\partial_y$ attached at the point $(2, 2)$.
Find the length of these vectors and the cosine of the angle between them (with respect to the metric G).

[10 marks]

A2.

(a) Explain what is meant by saying that a Riemannian surface is locally Euclidean.

(b) Consider a surface (the upper sheet of a cone) in $\mathbf{E}^3$

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = h \end{cases} , \quad h > 0, 0 \le \varphi < 2\pi \,.$$

Calculate the Riemannian metric on this surface induced by the canonical metric on Euclidean space $\mathbf{E}^3$.

(c) Show that this surface is locally Euclidean.

[10 marks]

A3.

(a) Explain what is meant by an affine connection on a manifold.

(b) Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u, v), all Christoffel symbols vanish except $\Gamma^u_{vv} = u$ and $\Gamma^v_{uu} = v$. Calculate the vector field $\nabla_{\mathbf{X}}\mathbf{X}$, where $\mathbf{X} = \frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$.

(c) Give an example of function $f = f(u, v)$ such that f does not vanish identically ($f \not\equiv 0$) and

$$\nabla_{\mathbf{X}}(f\mathbf{X}) = f\nabla_{\mathbf{X}}\mathbf{X}\,,$$

where $\mathbf{X}$ is a vector field considered above, $\mathbf{X} = \frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$.
Justify your answer.

[15 marks]

A4.

(a) Define a geodesic on a Riemannian manifold as a parameterised curve.
Write down the differential equation of geodesics in terms of the Christoffel symbols.
Explain what is meant by un-parameterised geodesic.

(b) What are the geodesics of the surface of a cylinder?
Justify your answer.

(c) Explain why the latitude, (curve $\theta = \theta_0$ in spherical coordinates θ, φ) is not a geodesic on the sphere when $\theta_0 \neq \frac{\pi}{2}$.
(You may wish use the fact that the acceleration vector of an arbitrary geodesic on a surface is orthogonal to the surface.)

[15 marks]

SECTION B

Answer **TWO** of the THREE questions

B5.

(a) Explain what is meant by saying that F is an isometry between two Riemannian manifolds.

(b) Consider the plane $\mathbf{R}^2$ with standard coordinates x, y and with the Riemannian metric

$$G_{(1)} = e^{-a(x^2+y^2)}(dx^2 + dy^2)\,,$$

and consider the same plane $\mathbf{R}^2$ with another Riemannian metric

$$G_{(2)} = be^{-u^2-v^2}(du^2 + dv^2)\,,$$

where a, b are parameters $a > 0$ and $b > 0$.
(We denote standard coordinates x, y in $\mathbf{R}^2$ in second formula by other letters u, v).
Show that the map $F\colon \begin{cases} u = \sqrt{a}x \\ v = \sqrt{a}y \end{cases}$ between these two Riemannian manifolds is an isometry in the case of $b = \frac{1}{a}$.

(c) Calculate the total area of the plane $\mathbf{R}^2$ with respect to the metric $G_{(1)}$, and the total area of the plane $\mathbf{R}^2$ with respect to the metric $G_{(2)}$.
(You may use the formula $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.)
Deduce why, in the case where $b \neq \frac{1}{a}$, there is no isometry between these Riemannian manifolds.

[20 marks]

B6.

(a) Give a detailed formulation of the Levi-Civita Theorem. In particular write down the expression for the Christoffel symbols Γ^i_{km} of the Levi-Civita connection in terms of the Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

(b) Consider the upper half-plane, $y > 0$ in $\mathbf{R}^2$ equipped with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2}\,,$$

(the Lobachevsky plane).
Calculate the Christoffel symbols of the Levi-Civita connection of this Riemannian manifold.

(c) Let ∇' be a symmetric connection on the Lobachevsky plane such that all the Christoffel symbols of this connection in coordinates (x, y) vanish identically.
Explain why this connection does not preserve the Riemannian metric of the Lobachevsky plane.

[20 marks]

B7.

(a) State the theorem about the result of parallel transport along a closed curve on a surface in the Euclidean space $\mathbf{E}^3$.

(b) Deduce that the Gaussian curvature of the surface depends only on the induced Riemannian metric on the surface, i.e. it is invariant under isometries (*Gauß Theorema Egregium*).

(c) Consider the sphere of radius R in $\mathbf{E}^3$. How does the Weingarten (shape) operator look for this sphere?
Justify your answer.
Calculate the Gaussian curvature of this sphere.
For an arbitrary point $\mathbf{p}$ of this sphere find local coordinates (u, v) such that the induced Riemannian metric on the sphere at the point $\mathbf{p}$ in these local coordinates is equal to

$$G\big|_{\mathbf{p}} = du^2 + dv^2 .$$

Explain why it is impossible to find coordinates such that this condition holds not only at the point $\mathbf{p}$ but in the vicinity of this point also.

[20 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

CODING THEORY

21 May 2018

14:00 – 16:00

Answer ALL THREE questions in Section A (40 marks in total). Answer TWO of the THREE questions in Section B (40 marks in total). If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

University approved calculators may be used

SECTION A

Answer **ALL** questions in this section (40 marks in total)

A1. (a) Define what is meant by:

— the *weight* $w(\underline{x})$ *of a vector* $\underline{x} \in \mathbb{F}_q^n$;

— a *linear code* C of length n over the field $\mathbb{F}_q$;

— the *weight* $w(C)$ *of a linear code* C;

— the *inner product* $\underline{x} \cdot \underline{y}$ of vectors $\underline{x}, \underline{y} \in \mathbb{F}_q^n$;

— the *dual code*.

(b) Consider the binary code $C = \{\underline{x} \in \mathbb{F}_2^n : \underline{x} \cdot \underline{x} = 0\}$ of length n where $n \geqslant 2$. Explain why C is a linear code. State without proof the cardinality, the dimension and the weight of C. Identify the code C by its well-known name.

[10 marks]

A2.

(a) Let an $(n-k) \times n$ check matrix H for a q-ary linear code C be given. What is meant by a *table of syndromes* for C? How many rows does such a table have? Explain how to decode a received vector using the table of syndromes.

From now on, let D be the binary linear code with parity check matrix $H = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

(b) Construct a table of syndromes for D and use it to decode the vector 11100.

(c) Use the table of syndromes constructed in (b) to show that if the code D is transmitted via $BSC(p)$, then the probability $P_{\mathsf{corr}}(D)$ that the received vector is decoded correctly is $(1-p)^3$.

(d) Write down the probability that an unencoded three-bit message, sent via $BSC(p)$, is received without errors. Compare this probability with $P_{\mathsf{corr}}(D)$ from part (c) and determine whether encoding three-bit messages using the code D improves *error correction* if transmitting via $BSC(p)$. Suggest one possible advantage and one disadvantage of using the code D versus transmitting unencoded messages of length 3.

[20 marks]

A3.

(a) Consider the Reed-Muller code $R(r, m)$ where $0 \leqslant r < m$. Write down the parameters $[n, k, d]_q$ of this code. State without proof which Reed-Muller code coincides with $R(r, m)^\perp$. Hence write down the condition on r and m equivalent to $R(r, m)$ being self-dual. Explain briefly why $R(r, m)$ is self-orthogonal, if and only if $r < m/2$.

(b) Use the result of (a) to prove that all Reed-Muller codes of dimension 2018 are self-orthogonal codes.

[10 marks]

SECTION B

Answer **TWO** of the three questions in this section (40 marks in total).
If more than TWO questions from this section are attempted, then credit will be given for the best TWO answers.

B4. (a) State without proof the *Hamming bound* for codes in $\mathbb{F}_q^n$ of minimum distance d. Define what is meant by a *perfect code*. Name a perfect code of minimum distance 9.

(b) Prove that if the minimum distance of a code is even, then the code is not perfect. You can use any other facts from the course without proof, but you should state the facts you use.

(c) Let $C \subseteq \mathbb{F}_q^n$ be a linear code. Define what is meant by a *generator matrix* of C. Assuming that C has a generator matrix G such that all rows of G have even weight: (i) Show that if $q = 2$, then C is not a perfect code. (ii) If $q = 3$, can such a code C be perfect? Justify your answer.

(d) A ternary linear code C has a generator matrix G with the following property: if the last row of G is removed, the remaining rows form a generator matrix of a ternary Golay code. Find all the possible values of the parameters $[n, k, d]_3$ of such codes C and justify your answer. Any results from the course can be used without particular comment. [20 marks]

B5. In this question, Z is the ternary linear code with generator matrix $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$.

(a) What is meant by saying that a generator matrix G of a linear code $C \subseteq \mathbb{F}_q^n$ is in *standard form*? Given G in standard form, explain how one can find a generator matrix for the dual code $C^\perp$. Define what is meant by the *weight enumerator* $W_C(x, y)$.

(b) List the codevectors of Z and find the weight enumerator $W_Z(x, y)$ of Z.

(c) State the *MacWilliams identity* for q-ary linear codes. Using the MacWilliams identity, or otherwise, for each $i = 0, 1, 2, 3, 4$ calculate the number of codevectors of weight i in the dual code $Z^\perp$.

(d) Let $D \subseteq \mathbb{F}_q^{2q}$ be a linear code which consists of the zero vector and $(q-1)^3$ vectors of weight q. Show that $w(D^\perp) = 1$. Find all q for which a code D with these properties exists. [20 marks]

B6. (a) What is a *cyclic code*? What are the properties required of a polynomial $g(x) \in \mathbb{F}_q[x]$ to be a generator polynomial of some cyclic code of length n over $\mathbb{F}_q$?

(b) Factorise $x^3 - 1$ into irreducible polynomials in $\mathbb{F}_3[x]$. Hence list all the cyclic codes in $\mathbb{F}_3^3$, stating the cardinality, the minimum distance, a generator polynomial and a generator matrix for each code.

(c) Let C be a linear code. Prove that C is cyclic, if and only if $C^\perp$ is cyclic.

(d) For a prime p, let $I_p = \{(a_1, \ldots, a_{p-1}) \in \mathbb{F}_p^{p-1} \mid \sum_{i=1}^{p-1} i a_i = 0 \text{ in } \mathbb{F}_p\}$.

i. What are the odd primes p for which I_p is an MDS code?
ii. What are the odd primes p for which $I_p^\perp \subseteq I_p$?
iii. What are the odd primes p for which I_p is a cyclic code?

Justify your answer in each case. Any facts from the course can be freely used. [20 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO ALGEBRAIC GEOMETRY

18 May 2018

14.00 - 16.00

Answer THREE of the FOUR questions.
If more than THREE questions are attempted, then credit will be given for the best THREE answers.

University approved calculators may be used.

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1. In this question K denotes an arbitrary field and unless specified otherwise, varieties are defined over K.

(a) (i) Define the *affine algebraic variety* $\mathcal{V}(J) \subseteq K^n$ determined by an ideal $J \lhd K[x_1, x_2, \ldots, x_n]$.

(ii) Define the *ideal* $\mathcal{I}(X)$ of a set $X \subseteq K^n$.

(iii) Show that if $X \subseteq K^n$ is an affine algebraic variety, then $\mathcal{V}(\mathcal{I}(X)) = X$.

(b) (i) Define what it means for an affine algebraic variety W to be *irreducible*.

(ii) Prove that every affine algebraic variety can be written as a union of finitely many irreducible affine algebraic varieties. (You may use any other results from the course that you need.)

(c) Find the irreducible components of the variety

$$W = \mathcal{V}(\langle xz(x-z), -2x^2 + y^2 + z^2 + 2x - 2z \rangle) \subset \mathbb{A}^3(\mathbb{C}).$$

(You should either identify geometrically what kind of variety each component is or prove directly that it is irreducible.)

(d) (i) Define the *co-ordinate ring* $K[V]$ of an affine algebraic variety $V \subseteq K^n$.

(ii) Let $V = \mathcal{V}(\langle x^2y^3 - 1 \rangle) \subset \mathbb{A}^2$. Show that $K[V]$ is not isomorphic to $K[\mathbb{A}^1]$ and hence deduce that V is not isomorphic to $\mathbb{A}^1$. (*Hint:* Consider the invertible elements. You may assume without proof that $\mathcal{I}(V) = \langle x^2y^3 - 1 \rangle$.)

[20 marks]

2. (a) Let K be a field. Let $V \subseteq \mathbb{A}^n(K)$ be an affine algebraic variety and let $P \in V$. Let f_1, f_2, $\ldots$, f_r be a set of generators for $\mathcal{I}(V) \lhd K[x_1, x_2, \ldots, x_n]$.

(i) State without proof how lines tangent to V at P can be characterised in terms of the Jacobian matrix of f_1, f_2, $\ldots$, f_r. Explain why this implies that the tangent space $T_P V$ is an affine subspace of K^n and state the formula for $\dim T_P V$.

(ii) Assume now that V is irreducible. Describe without proof how the dimension of V and singular points of V can be defined using the dimension of the tangent space at the points of P.

(b) This part of the question is over $\mathbb{C}$. Let x, y be co-ordinates on $\mathbb{A}^2(\mathbb{C})$ and let t be the co-ordinate on $\mathbb{A}^1(\mathbb{C})$. Let

$$f = x^4 + 4x^2y - y^3 + 2y^2 - y$$

and let $C = \mathcal{V}(\langle f \rangle) \subset \mathbb{A}^2(\mathbb{C})$. (You may assume without proof that f is an irreducible polynomial, therefore C is irreducible and $\mathcal{I}(C) = \langle f \rangle$.)

(i) Show that $\phi(t) = (t(1-t^2), t^4)$ is a morphism $\phi : \mathbb{A}^1(\mathbb{C}) \to C$.

(ii) Let $\psi(x, y) = \dfrac{x + xy}{1 - x^2 - y}$. Show that ψ defines a rational map $C \dashrightarrow \mathbb{A}^1(\mathbb{C})$ and that $\psi \circ \phi : \mathbb{A}^1(\mathbb{C}) \dashrightarrow \mathbb{A}^1(\mathbb{C})$ is the identity map.

(iii) Find the 3 singular points of C.

(iv) For each singular point P of C, find $t_1, t_2 \in \mathbb{C}$ such that $t_1 \neq t_2$ but $\phi(t_1) = \phi(t_2) = P$, where ϕ is the morphism defined in (b)(i).

[20 marks]

3. In this question K denotes an arbitrary field.

(a) (i) Let n be a positive integer and let A be an invertible $(n+1) \times (n+1)$ matrix with entries from K. Define the projective transformation Φ of $\mathbb{P}^n(K)$ corresponding to A and and show that Φ is a morphism $\mathbb{P}^n(K) \to \mathbb{P}^n(K)$.

(b) In this part of the question we identify $\mathbb{P}^1(K)$ with $K \cup \{\infty\}$ by identifying $(X_0 : X_1)$ with $X_0/X_1 \in K$ if $X_1 \neq 0$ and $(X_0 : 0)$ with ∞ as usual.
Recall that if z_1, z_2, z_3, $z_4 \in K \cup \{\infty\}$ with at least 3 of them pairwise distinct, their cross ratio is defined to be

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \in K \cup \{\infty\}$$

with appropriate rules for arithmetic involving ∞ and for division by 0.

(i) Let z_1, z_2, z_3, $z_4 \in K \cup \{\infty\}$ with at least 3 of them pairwise distinct. Show that $(z_1, z_2; z_3, z_4) = (\phi(z_1), \phi(z_2); \phi(z_3), \phi(z_4))$ for any projective transformation $\phi : \mathbb{P}^1 \to \mathbb{P}^1$.

(ii) Let z_1, z_2, z_3, w_1, w_2, w_3 be elements of $K \cup \{\infty\}$ such that z_1, z_2, z_3 are pairwise distinct and w_1, w_2, w_3 are also pairwise distinct. Prove that there exists a unique projective transformation $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\phi(z_i) = w_i$ for $i = 1, 2, 3$. (You may use standard properties of projective transformations without proof.)

(c) Find $a, b, c, d \in \mathbb{C}$ such that $\phi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, $\phi(z) = \dfrac{az+b}{cz+d}$ satisfies $\phi(1) = 1$, $\phi(2) = 6$ and $\phi(5) = 3$.

[20 marks]

4. (a) Let K be a field. Let $E \subset \mathbb{P}^2(K)$ be a non-singular cubic curve and let O be a point of E. (You should not assume that the equation of E is in a special form.)

(i) Describe without proof how one can define the addition operation on the points of E to obtain a commutative group in which O is the zero element.

(ii) Assume now that O is an inflection point of E. Show that $P \in E$ satisfies $3P = O$ if and only if P is an inflection point.

(b) Let F be the elliptic curve given by the affine equation

$$y^2 = x^3 + 4x^2 - 5x + 1$$

over $\mathbb{C}$ together with the point $O = (0 : 1 : 0)$ in homogeneous co-ordinates, which is taken to be the zero element of the group law as usual.

Let $A = (-1, -3)$, $B = (1, 1)$. Calculate $A + B$ and $2B$.

(c) Let $G \subset \mathbb{C}^2$ be the curve defined by the equation

$$y^2 = x^3 - 2x + \frac{1}{6}.$$

Calculate its Hessian and use it to find the two inflection points of G which have integer x co-ordinate.

[20 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

NUMBER THEORY

22 May 2018

14.00 - 16.00

Answer THREE of the FOUR questions.
(If more than three questions are attempted, then credit will be given for the best three answers.)

Electronic calculators are not permitted.

1.

(a) Give the definition of a finite simple continued fraction and explain what is meant by an infinite simple continued fraction. [4 marks]

(b) State a result describing exactly which real numbers have an infinite simple continued fraction representation and describe (without proof) an algorithm to find such a representation. [5 marks]

(c) Let $[x_0 : x_1, x_2, \ldots]$ be an infinite simple continued fraction. Define

$$\begin{array}{llll} p_{-2} = 0, & p_{-1} = 1, & \text{and} & p_n = x_n p_{n-1} + p_{n-2} \text{ for all } n \geq 0, \\ q_{-2} = 1, & q_{-1} = 0, & \text{and} & q_n = x_n q_{n-1} + q_{n-2} \text{ for all } n \geq 0. \end{array}$$

(i) For all $n \geq -2$, use induction to prove that

$$p_{n+1}q_n - p_n q_{n+1} = (-1)^n \qquad \text{and} \qquad p_{n+2}q_n - p_n q_{n+2} = (-1)^n x_{n+2}.$$

(ii) Deduce that $\gcd(p_n, q_n) = 1$ for all $n \geq -2$. [10 marks]

(d) Let $\omega = [\overline{a : a}]$, where $a \in \mathbb{N}$. Calculate the quadratic irrational which ω represents. [5 marks]

(e) Let $N > 2$. Prove that there are infinitely many primes p which satisfy $p \not\equiv 1 \bmod N$. [*Hint: Consider the polynomial $Nx - 1$.*] [6 marks]

[Total 30 marks]

2.

(a) Give the definition of a periodic simple continued fraction and state (without proof) a theorem describing precisely which numbers can be represented by a periodic simple continued fraction. [4 marks]

(b) Let d be a non-square positive integer. Describe (without proof) how to find *all* integer solutions of the equation $x^2 - dy^2 = 1$. [4 marks]

(c) Find the infinite simple continued fraction expansion of $\sqrt{5}$ and compute its first four convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$. [6 marks]

(d) For each of the following Diophantine equations find at least one *positive* integer solution or explain why there are none, stating clearly any results you use in your answer:

(i) $x^2 - 5y^2 = 1$, (ii) $x^2 + 5y^2 = 1$, (iii) $x^2 - 5y^2 = -1$, (iv) $25x^2 - 5y^2 = 1$, (v) $x^2 - 3y^2 = -1$. [12 marks]

(e) Let d be a non-square positive integer. Assume that the "negative Pell equation"

$$x^2 - dy^2 = -1 \qquad (\dagger)$$

has a solution (x_1, y_1) with $x_1, y_1 \in \mathbb{Z}$. For each integer n let $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$. For which integers n is (x_n, y_n) also a solution to the equation $(\dagger)$? Explain your answer. [4 marks]

[Total 30 marks]

3. Let n be a positive integer.

(a) Give the definition of a unit modulo n. For $a \in \mathbb{Z}$, show that $[a]_n \in \mathbb{Z}_n$ is a unit modulo n if and only if a and n are coprime. [5 marks]

(b) Give the definition of a primitive root modulo n. [2 marks]

(c) State (without proof) Hensel's lemma and Fermat's little theorem. [6 marks]

(d) For which positive integers m do the following congruences have a solution?
(i) $12x^{20} + x^2 + 4x + 1 \equiv 0 \bmod 6^m$, (ii) $x^{103} + 2 \equiv 0 \bmod 5^m$,
(iii) $x^4 - 2x^3 + 2x^2 - 2x + 1 \equiv 0 \bmod 3^m$. [9 marks]

(e) Let $n = p_1^{a_1} \cdots p_k^{a_k}$, where $p_1, \ldots, p_k$ are distinct primes and $a_1, \ldots, a_k$ are positive integers.
(i) Define Euler's φ function and give (without proof) a formula for $\varphi(n)$ in terms of the p_i and the a_i.
(ii) For which integers m do there exist infinitely many n with $m \mid \varphi(n)$? Explain your answer. [8 marks]

[Total 30 marks]

4. Let p be an odd prime.

(a) Give the definition of a quadratic residue modulo p and define the Legendre symbol $\left(\frac{a}{p}\right)$, where a is an integer. [5 marks]

(b) State (without proof) Euler's criterion and hence prove that

$$\left(\frac{-1}{p}\right) = \begin{cases} 1, & \text{if } p \equiv 1 \bmod 4, \\ -1, & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

[5 marks]

(c) (i) State (without proof) the law of quadratic reciprocity.

(iii) Compute the Legendre symbols $\left(\frac{106}{103}\right)$ and $\left(\frac{20}{37}\right)$. [8 marks]

(d) Determine the odd primes p for which -28 is a quadratic residue modulo p. [6 marks]

(e) Let a, b, c be integers with $p \nmid a$. Prove that

$$\left|\{x \in \mathbb{Z}_p : ax^2 + bx + c \equiv 0 \bmod p\}\right| = 1 + \left(\frac{b^2 - 4ac}{p}\right).$$

That is, show that $ax^2 + bx + c \equiv 0 \bmod p$ has exactly $1 + \left(\frac{b^2-4ac}{p}\right)$ solutions modulo p. [6 marks]

[Total 30 marks]

END OF EXAMINATION PAPER

UNIVERSITY OF MANCHESTER

GREEN'S FUNCTIONS, INTEGRAL EQUATIONS AND APPLICATIONS

31 May 2018

09.45 - 11.15

Answer ALL four questions (100 marks in total)

University approved calculators may be used.

Given equations:

- For functions f and g defined in either two or three dimensions, and a domain D with smooth boundary in either two or three dimensions
$$\int_D \left(f(\mathbf{x})\nabla^2 g(\mathbf{x}) - g(\mathbf{x})\nabla^2 f(\mathbf{x})\right)\, d\mathbf{x} = \int_{\partial D} \left(f(\mathbf{x})\nabla g(\mathbf{x}) - g(\mathbf{x})\nabla f(\mathbf{x})\right)\cdot \mathbf{n}(\mathbf{x})\, ds(\mathbf{x}) \qquad (1)$$
where the boundary ∂D is oriented by the outward pointing unit normal $\mathbf{n}(\mathbf{x})$ at each point $\mathbf{x}$ in ∂D.

1. You do not need to provide any proofs on the various parts of this problem. Just give the answers.

(a) Give the formula for the Neumann series solution of a Fredholm integral equation of the second kind:

$$u(x) = \lambda \int_a^b K(x,y)u(y)\ \mathrm{d}y + f(x).$$

Be sure to define the notation that you use.

[5 marks]

(b) Consider a second order linear differential operator

$$\mathcal{L}u = pu'' + ru' + qu$$

where p, r, and q are complex valued continuous functions. What are the requirements on p, r and q that imply $\mathcal{L}$ is formally self-adjoint?

[6 marks]

(c) Give the free-space Green's function for the Laplacian in two dimensions.

[5 marks]

(d) If $\{\lambda_n\}_{n=1}^{\infty}$ are the eigenvalues of a regular Sturm-Liouville eigenvalue problem, what is $\lim_{n\to\infty} \lambda_n$?

[4 marks]

[Total 20 marks]

2.

(a) Consider the following boundary value problem:

$$\begin{cases} u''(x) + 2u'(x) = f(x), \quad x \in (0, \ln(\sqrt{2})) \\ u(0) = 0, \quad u(\ln(\sqrt{2})) = 0. \end{cases}$$

Find the Green's function for this problem.

[12 marks]

(b) For a continuous function q, consider the following boundary value problem:

$$\begin{cases} u''(x) + 2u'(x) + q(x)u(x) = f(x), \quad x \in (0, \ln(\sqrt{2})) \\ u(0) = 0, \quad u(\ln(\sqrt{2})) = 0. \end{cases}$$

If u is the solution of this problem, use the Green's function from part (a) to find a Fredholm integral equation that u must also satisfy. What is the kernel of the integral operator in the Fredholm integral equation that you found?

[12 marks]

[Total 24 marks]

3. This question concerns the integral equation

$$u(x) = 2\lambda \int_{-\ln(\sqrt{2})}^{\ln(\sqrt{2})} e^{x+y} u(y)\ \mathrm{d}y + f(x). \tag{2}$$

(a) For what values of $\lambda \in \mathbb{C}$ is there a unique solution $u(x)$ to (2) for any continuous function f?

[7 marks]

(b) For the values of λ found in part (a), find a formula for the solution $u(x)$ of (2).

[7 marks]

(c) When λ is not one of the values found in part (a) find a condition on f so that there is a solution for (2). Also write a general solution for (2) in this case.

[6 marks]

[Total 20 marks]

4. The *free-space Green's function* for the Laplacian operator $\mathcal{L} = \nabla^2$ in three dimensions is

$$G_{3\infty}(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x} - \mathbf{x}_0|}$$

(a) Show that $\nabla^2_{\mathbf{x}} G_{3\infty}(\mathbf{x}, \mathbf{x}_0) = 0$ when $\mathbf{x} \neq \mathbf{x}_0$.

[8 marks]

(b) Let $\mathcal{D} \subset \mathbb{R}^3$ be the region in which the x coordinate is positive

$$\mathcal{D} = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 \ : \ x > 0\}.$$

Find a Green's function $G(\mathbf{x}, \mathbf{x}_0)$ that satisfies

$$\begin{cases} \nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0), & \mathbf{x}, \mathbf{x}_0 \in \mathcal{D}, \\ G(\mathbf{x}, \mathbf{x}_0)|_{\mathbf{x} \in \partial\mathcal{D}} = 0, & \mathbf{x}_0 \in \mathcal{D}. \end{cases}$$

[12 marks]

(c) Show how you can use the Green's function from part (b) to find a solution u of the boundary value problem

$$\begin{cases} \nabla^2 u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \mathcal{D}, \\ u(\mathbf{x})|_{\mathbf{x} \in \partial\mathcal{D}} = h(\mathbf{x}) \end{cases}$$

for given f and h.

[16 marks]

[Total 36 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

WAVE MOTION

25 May 2018

14:00 – 16:00

Answer **ALL** six questions.

Electronic calculators are permitted, provided they cannot store text.

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1. Consider a physical signal $y(x,t)$ in the form

$$y(x,t) = A\sin(kx - \omega(k)t) + A\sin((k+\delta)x - \omega(k+\delta)t)\,,$$

in the usual notation, where $\delta > 0$ is a perturbation to the wavenumber $k > 0$.

Show that when δ is small, the solution can be written as

$$y(x,t) = 2A\cos\left(\frac{\delta}{2}(x - \omega'(k)t)\right)\sin\left(kx - \omega(k)t\right) + O(\delta)\,.$$

Sketch the signal at a fixed instant in time, indicating the direction of propagation for $\omega > 0$. Indicate the envelope of the wave and the carrier wave in the sketch and give their speed of propagation.

<u>You may use:</u>

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)\,.$$

[8 marks]

2. An atmospheric wave has a velocity component

$$v(x,y,t) = \Re[V(y)\exp(i(kx - \omega t))]\,,$$

in the usual notation. You are given that V satisfies

$$V''(y) - \left(k^2 + \frac{\beta k}{\omega} + \frac{y^2}{\mu^2}\right)V(y) = 0\,,$$

where β and μ are positive constants and V is bounded for all y.

Look for a solution in the form

$$V(y) = H(Y)\exp\left(-\frac{Y^2}{2}\right)\,,$$

where $Y = y\mu^{-1/2}$ and give the corresponding equation for $H(Y)$. Hence give the dispersion relation for a solution of the form $H(Y) = C$ for constant C. Obtain the group velocity of this wave solution, and sketch the graph of ω vs k for positive k.

Find another solution for $H(Y)$ and the corresponding dispersion relation.

[12 marks]

3.

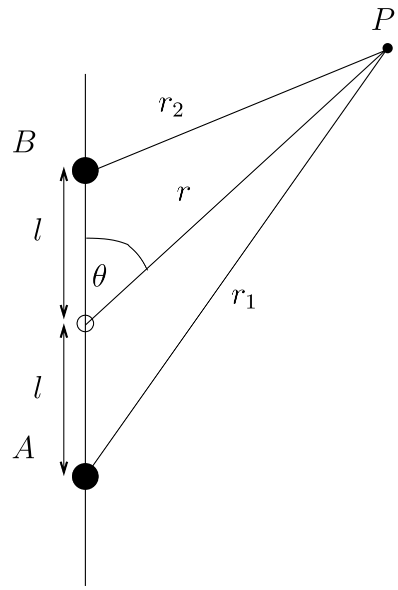

Figure 1: A schematic diagram for question 3.

Two monopoles (A, B), of equal and opposite sign, are separated by a distance $2l$ and placed in a compressible fluid of sound speed c, as shown in figure 1. A solution of the three-dimensional wave equation exists in the form of a velocity potential:

$$\phi(r, \theta, t) = \frac{1}{4\pi r_1} m(t - r_1/c) - \frac{1}{4\pi r_2} m(t - r_2/c)\,,$$

for a given function m, where θ, r, r_1 and r_2 are as shown in the figure.

By considering the limit of $l/r \to 0$, where $2l\, m(t) \to d(t)$, give a simplified expression for ϕ in terms of $d(t)$.

Hint: The cosine rule for a triangle of side lengths $\bar{a}$, $\bar{b}$, $\bar{c}$, gives

$$\bar{c}^2 = \bar{a}^2 + \bar{b}^2 - 2\bar{a}\bar{b}\cos\alpha\,,$$

where α is the angle between the sides of length $\bar{a}$ and $\bar{b}$.

[10 marks]

4. Consider planar sound waves in a slowly varying channel of cross-sectional area $A(x)$. You are given that the sound wave perturbations $\tilde{u}$, $\tilde{p}$, $\tilde{\rho}$ (to the velocity, pressure and density respectively) satisfy:

$$A(x)\frac{\partial \tilde{\rho}}{\partial t} = -\rho_o \frac{\partial}{\partial x}\left(A(x)\tilde{u}\right)\,,$$

and

$$\rho_o \frac{\partial \tilde{u}}{\partial t} = -\frac{\partial \tilde{p}}{\partial x}\,.$$

Here ρ_o is the undisturbed density of the compressible fluid in which the sound waves are travelling and $\tilde{\rho} = \gamma \tilde{p}$, for some constant $\gamma > 0$.

By eliminating $\tilde{u}$ and $\tilde{\rho}$, obtain a single equation for $\tilde{p}$.

Hence, in the special case of $A(x) = \exp(\alpha x)$, show that for a wave of the form $\tilde{p}(x,t) = a(x)\exp(i\omega t)$ (for some fixed frequency ω), there is a critical value of α above which the waves cease to propagate through the channel.

[10 marks]

5. Consider the linearised problem for a free surface water wave in a domain with a sloping base, where $x \geq 0$, $z \in [0, x\tan\alpha]$, $0 < \alpha \leq \pi/2$ and $y \in (-\infty, \infty)$ (as shown in figure 2). The fluid is in a uniform gravitational field of acceleration g, with a mean free surface at $z = 0$ and z increases downwards through the fluid.

(i) The linear conditions at the free surface (in the usual notation) are

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}, \quad \text{and} \quad \frac{\partial \phi}{\partial t} = g\eta\,.$$

By eliminating η, derive the appropriate condition for the velocity potential ϕ.

(ii) State the equation to be satisfied within the fluid. Give the normal $\mathbf{n}$ to the sloping base and hence expand the boundary condition $\nabla\phi \cdot \mathbf{n} = 0$.

(iii) Consider a solution in the form

$$\phi(x, y, z, t) = \hat{\phi}(x, z)\cos(ky - \omega t)\,,$$

where k and ω are real constants. Solve for $\hat{\phi}(x, z)$ under the assumption that $\hat{\phi}$ decays with both x and z.

(iv) Draw a diagram that shows the dependence of ω/k on the wave length $\lambda = 2\pi/k$, showing how α affects the behaviour.

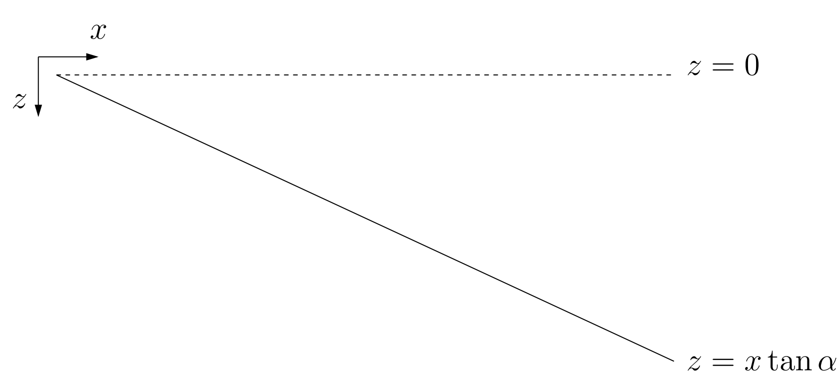

Figure 2: A cross section of the fluid domain at fixed y, for question 5.

[20 marks]

6. A rigid sphere of radius a is surrounded by an inviscid, compressible fluid of wave speed c. The sphere makes small rectilinear oscillations so that its centre is at a position $b(t)\mathbf{k}$ relative to an origin O, where $\mathbf{k}$ is a constant unit vector and $|b(t)| \ll a$.

(i) Write down the partial differential equation and boundary conditions to be satisfied by the velocity potential $\phi(\mathbf{r}, t)$ of any disturbance caused by the motion of the sphere in the fluid.

(ii) You are given that the problem can be solved in terms of a potential

$$\phi = \cos\theta \frac{\partial}{\partial r}\left\{\frac{1}{4\pi r} d(t - r/c)\right\},$$

where (r, θ, α) are spherical polar coordinates with $\theta = 0$ taken along the $\mathbf{k}$ direction. Assuming that the boundary condition can be applied at the mean location of the sphere surface, show that the function $d(t)$ satisfies

$$d(t) + t_0 d'(t) + \frac{t_0^2}{2} d''(t) = F(t)$$

where $t_0 = a/c$ and $F(t)$ is to be defined in your answer.

(iii) Hence obtain the transformed solution $\bar{d}(\omega)$, using the definition of the Fourier transform

$$\mathcal{F}\{d(t)\} = \bar{d}(\omega) = \int_{-\infty}^{\infty} d(t) e^{i\omega t}\, \mathrm{d}t\,.$$

(iv) Find the Fourier transform $\bar{f}(\omega)$ of

$$f(t) = e^{-t/t_0} \sin(t/t_0) H(t)\,,$$

where $H(t \geq 0) = 1$ and $H(t < 0) = 0$.

(v) Now use the general expression

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} f(t-\tau) g(\tau)\, \mathrm{d}\tau\right\} = \bar{f}(\omega)\bar{g}(\omega)\,,$$

with a suitable choice for g, to obtain a solution for $d(t)$ as a real integral.

[20 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL BIOLOGY

17 May 2018

09.45 - 11.45

Answer **ALL** THREE questions in Section A (30 marks in total).
Answer **TWO** of the THREE questions in Section B (50 marks in total).
If more than TWO questions from Section B are attempted,
then credit will be given for the best TWO answers.

University approved calculators may be used

SECTION A

Answer **ALL** 3 questions

A1. One way to control an insect pest is to release sterile insects. If, through steady release, we maintain a constant population n of sterile insects within the wild population, then a model for the population $N(t)$ of fertile insects is

$$\begin{aligned} \frac{dN}{dt} &= N\left(\frac{aN}{N+n} - b\right) - cN(N+n) \\ &= aN\left(\frac{N}{N+n}\right) - bN - cN(N+n) \end{aligned} \qquad \text{(A1.1)}$$

where $a > b > 0$, $c > 0$ and $n > 0$ are parameters.

(a) Consider the case where there are no sterile insects and show that (A1.1) reduces to the logistic growth law

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$

Your answer should include formulae for the growth rate r and the carrying capacity K in terms of the parameters a, b and c.

(b) Explain the role of the term $aN\left(\frac{N}{N+n}\right)$ in Eqn. (A1.1): what process does it model and why does it have this form?

(c) Determine the critical number n_c of sterile insects whose addition would guarantee the pest's extinction by ensuring that $dN/dt < 0$ for all $N \geq 0$.

[10 marks]

A2. Explain what is meant by the following terms:

- a *motif* in a genetic regulatory network
- a *coherent 3-node feed-forward loop*. Give two examples of such loops.

In a randomly-assembled regulatory network with $N = 7$ genes and $E = 10$ regulatory interactions, what is the probability of finding exactly 3 autoregulatory loops?

[10 marks]

A3. During the development of an embryo, certain cell-fate decisions are controlled by a morphogen M that diffuses through the tissue and is also degraded, so that its concentration $M(x,t)$ evolves according to the PDE

$$\frac{\partial M}{\partial t} = D\frac{\partial^2 M}{\partial x^2} - \alpha M^2, \tag{A3.1}$$

where D and α are positive real numbers. The morphogen is produced at one end of the embryo and is degraded so strongly that it is sufficient to study (A3.1) on the domain $x \geq 0$, $t \geq 0$ with the boundary conditions

$$M(0,t) = M_0 \qquad \text{and} \qquad \lim_{x\to\infty} M(x,t) = 0 \qquad \forall t \geq 0$$

(a) Verify that (A3.1) has a steady-state concentration profile of the form

$$M(x) = \frac{\gamma}{(x+\epsilon)^\nu}$$

and find expressions for γ, ϵ and ν in terms of M_0, D and α.

Cells at position x develop into one of three types—A, B or C—according to the steady-state concentration of the morphogen $M(x)$:

$$\text{cell type at } x \ = \begin{cases} A & \text{if } M(x) \geq \theta_1 \\ B & \text{if } \theta_1 > M(x) \geq \theta_2 \\ C & \text{if } \theta_2 > M(x) \end{cases}$$

where $\theta_1 > \theta_2 > 0$.

(b) Assume that $M_0 > \theta_1$, so that cells near $x = 0$ develop into type A. Find an expression for the position $x_1^\star$ of the boundary between cells of types A and B.

(c) Define $x_2^\star$ to be the position of the boundary between cells of types B and C. Show that $\Delta = x_2^\star - x_1^\star$ is independent of M_0 and discuss what this implies about the robustness of developmental patterning.

[10 marks]

SECTION B

Answer **2** of the 3 questions

B4. The following system of equations describes the progress of an epidemic of an infectious disease such as meningitis. Imagine that there is a total population of N_0 people that we can divide among two categories:

Susceptible people who do not currently have the disease

Infectious people who currently have the disease and so can infect susceptibles

If we say that the numbers of persons in these two groups are $S(t)$ and $I(t)$, respectively, then one commonly-used model is defined by the ODEs

$$\frac{dS}{dt} = -\beta IS + \gamma I, \qquad \frac{dI}{dt} = \beta IS - \gamma I \tag{B4.1}$$

where the parameters β and γ are positive real numbers.

(a) Show that if the initial population is $N_0 = S(0) + I(0)$ then $S(t) + I(t) = N_0$ for all times $t \geq 0$.

(b) Explain what the parameters β and γ mean and describe the course of the disease implied by (B4.1): in particular, what fates are possible for those who become infected?

(c) Show that by a suitable choice of dimensionless variables one can reduce the system (B4.1) to the form

$$\frac{du}{d\tau} = -(R_0 u - 1)v \qquad \text{and} \qquad \frac{dv}{d\tau} = (R_0 u - 1)v \tag{B4.2}$$

where $u(\tau)$ and $v(\tau)$ are nonnegative quantities satisfying $u(\tau) + v(\tau) = 1$. Your answer should include expressions for u, v, τ and R_0 in terms of the original variables and parameters.

(d) Using the results from parts (a)–(c), or by any other means, show that (B4.2) is equivalent to

$$\frac{dv}{d\tau} = (R_0 - 1)v - R_0 v^2 \tag{B4.3}$$

(e) A disease is said to be *endemic* if there is an equilibrium $(S_\star, I_\star)$ for (B4.1) in which $I_\star > 0$. Prove the following

Theorem (Threshold Theorem of Epidemiology)**.** *A disease obeying (B4.1) is endemic if and only if $R_0 > 1$, where R_0 is as in (B4.2).*

Your answer should include an analysis of the stability of the equilibria of (B4.3).

[25 marks]

B5. Consider the system of reactions

$$
\begin{aligned}
X + X &\underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} D \\
X + D &\underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} T
\end{aligned}
\qquad \text{(B5.1)}
$$

which describes the formation of dimers D and trimers T from monomers X. Here $k_{\pm 1}$ and $k_{\pm 2}$ are rate coefficients for the Law of Mass Action.

(a) Construct a stoichiometric matrix N for the system of reactions (B5.1), explaining your work thoroughly.

(b) Using the Law of Mass Action, write down a system of ODEs that describes the temporal evolution of the concentrations of the chemical species in (B5.1).

(c) By performing Gaussian elimination on N, or by any other means, find a complete set of conserved quantities of the form

$$a_1[X] + a_2[D] + a_3[T]$$

for (B5.1). Here square brackets indicate concentrations, so, for example, $[X]$ is the concentration of monomers.

(d) Find the rank r of the stoichiometric matrix N and decompose N as a product $N = LN_R$ where N_R, the reduced stoichiometric matrix, has full rank and L is of the form

$$L = \begin{bmatrix} I_r \\ L_0 \end{bmatrix},$$

where I_r is the $r \times r$ identity matrix. Your answer should include expressions for L, L_0 and N_R.

Now imagine that we are interested in a very small, isolated volume μ in which there are so few molecules that the ODEs found in part (b) become unrealistic and we are obliged to model the reactions using the Gillespie algorithm.

(e) Suppose that at $t = 0$ there are 4 molecules of the monomer X and none of species D or T. Using the notation (N_X, N_D, N_T), where N_X, N_D and $N_T \in \mathbb{N}$ are non-negative integers giving, respectively, the number of monomers, dimers and trimers, list the set of states the system can occupy.

(f) Using the notation $p_{i,j,k}(t)$ to indicate the probability that the system is, at time t, in a state with i copies of the monomer, j copies of the dimer and k copies of the trimer, write down a system of differential equations for $p_{i,j,k}(t)$.

[25 marks]

B6. Consider the following differential delay equation for a population whose birth rate at time t is controlled by the availability of food a certain time T in the past. Here $N(t)$ is the population at time t and it evolves according to the following differential delay equation

$$\frac{dN(t)}{dt} = b\,\Theta\left(1 - \frac{N(t-T)}{K}\right) - cN(t). \tag{B6.1}$$

where the delay T and the parameters b, K and c are positive real numbers, and $\Theta : \mathbb{R} \to \mathbb{R}$ is a function that satisfies

$$\Theta(x) \;=\; \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(a) Explain the role of the function Θ in the model (B6.1).

(b) Show that a suitable choice of dimensionless variables allows one to reduce the system (B6.1) to the form

$$\frac{du(\tau)}{d\tau} = \alpha\,\Theta(1 - u(\tau - \hat{T})) - u(\tau). \tag{B6.2}$$

Your answer should include formulae for u, $\tau, \hat{T}$ and α in terms of x, t, T and the parameters of the original problem.

(c) Show that the system (B6.2) has a constant solution $u(\tau) = u_\star$ and establish that $0 < u_\star < 1$.

(d) Linearise the system (B6.2) near $u(\tau) = u_\star$ and show that small perturbations of the form $u(\tau) = u_\star + \eta(\tau)$ evolve according to the linearized dynamics

$$\frac{d\eta(\tau)}{d\tau} = -\alpha\eta(\tau - \hat{T}) - \eta(\tau). \tag{B6.3}$$

(e) Consider perturbations of the form $\eta(t) = \eta_0 e^{(\mu + i\omega)t}$ evolving according to (B6.3) and prove

- if $\alpha < 1$, then $\mu < 0$ for all $\hat{T} > 0$ and so all such perturbations die away exponentially;
- if $\alpha > 1$, then there is a critical delay $\hat{T}_0$ such that if $\hat{T} > \hat{T}_0$, then $\mu > 0$, implying that the constant solution $u(\tau) = u_\star$ is unstable.

[25 marks]

END OF EXAMINATION PAPER

<u>Two Hours</u>

UNIVERSITY OF MANCHESTER

NUMERICAL ANALYSIS 2

30th May 2018

09:45–11:45

Answer **<u>ALL</u>** five questions in **<u>SECTION A</u>** (40 marks in total)
and
Answer **<u>TWO</u>** of the three questions in **<u>SECTION B</u>** (20 marks each).
If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

Electronic calculators are not permitted

SECTION A

Answer **ALL** five questions

A1. Let $T_n(x) = \cos(n\cos^{-1}(x))$ denote the Chebyshev polynomial of degree n.

(a) Show that the Chebyshev polynomials satisfy the recurrence relation

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x),$$

for $k \geq 1$.

(b) For a general family of polynomials $\phi_k(x)$ of degree k, define the term "orthogonality" with respect to a weight function $w(x) \geq 0$ on $[a, b]$, and prove that the Chebyshev polynomials are orthogonal with respect to the weight function $w(x) = 1/\sqrt{1-x^2}$ on $[-1, 1]$.

[9 marks]

A2. Let

$$r_{km}(x) = \frac{p_{km}(x)}{q_{km}(x)}.$$

be a rational function with p_{km} a polynomial of degree at most k, q_{km} a polynomial of degree at most m, and $q_{km}(0) = 1$.

(a) State the condition that r_{km} must satisfy if it is to be a $[k/m]$ Padé approximant of a given function $f(x)$.

(b) Compute the $[2/1]$ Padé approximant to $f(x) = \cos(x)$ and explain briefly why the result is not unexpected.

(c) Show that the Padé approximant $r_{km}(x)$ to the function $f(x) = \exp(x)$ satisfies $r_{km}(-x) = 1/r_{km}(x)$.

[9 marks]

A3. Use the method of undetermined coefficients to derive the 4-point closed Newton–Cotes rule

$$\int_0^1 f(x)\,dx \approx af(0) + bf(1/3) + cf(2/3) + df(1) := J(f).$$

That is, find the coefficients a, b, c and d. Given that the error is of the form $Af^{(4)}(\eta)$, for some $\eta \in [0, 1]$, determine the constant A.

[8 marks]

A4. Consider the integral

$$I(f) := \int_a^b w(x)f(x)\,dx,$$

where $w(x) \geq 0$ on $[a, b]$ is a weight function. Explain how to choose the nodes $\{x_i\}_{i=1}^n$ and weights $\{w_i\}_{i=1}^n$ so that the n-point Gauss rule

$$G_n(f) := \sum_{i=1}^{n} w_i f(x_i),$$

has degree of precision $2n - 1$. Show that, for this choice,

$$w_i = \int_a^b w(x) l_i(x)\, dx,$$

where l_i is a Lagrange interpolating polynomial.

[7 marks]

A5. The Backward Euler method for the initial value problem $y'(x) = f(x, y(x))$, $y(x_0) = y_0$ is written

$$y_{n+1} = y_n + h f_{n+1},$$

where h is the chosen stepsize, $f_{n+1} = f(x_{n+1}, y_{n+1})$ and y_n is the approximation to $y(x_n)$ at the grid point x_n for $n = 0, 1, \ldots$. Find the region of absolute stability and compare it with that of the standard Euler method. Which scheme has more favourable stability properties?

[7 marks]

SECTION B

Answer **TWO** of the three questions

B6.

(a) Define the L_∞ norm of $f \in C[a,b]$ and define what is meant by a *best L_∞ approximation* g from a subspace $G \subset C[a,b]$.

(b) Derive the best L_∞ approximation to $f \in C[a,b]$ from the set of polynomials of degree zero.

(c) State the Chebyshev equioscillation theorem, which gives a necessary and sufficient condition for a polynomial $p_n(x)$ of degree at most n to be a best L_∞ approximation to $f \in C[a,b]$ from the set of polynomials of degree at most n. In your answer, define fully the word "alternant".

(d) Using any of your previous answers, prove that the polynomial

$$q_n(x) = x^{n+1} - 2^{-n}T_{n+1}(x)$$

is a best L_∞ approximation to x^{n+1} on $[-1,1]$ from polynomials of degree at most n, where T_{n+1} is the Chebyshev polynomial of degree $n+1$.

(You may use any of the standard properties of Chebyshev polynomials without proof).

(e) Assume that $f \in C[-1,1]$ has $n+1$ continuous derivatives on $[-1,1]$ and let $p_n(x)$ be the polynomial of degree at most n that interpolates f at the $n+1$ distinct points $\{x_i\}_{i=0}^n$. Given that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}\prod_{i=0}^{n}(x - x_i),$$

for some (unknown) $\xi_x \in [-1,1]$, where $f^{(n+1)}$ is the $(n+1)$st derivative of f, explain why Chebyshev points are a good choice for $\{x_i\}_{i=0}^n$.

[20 marks]

B7. Consider the integral

$$I(f) = \int_a^b w(x)f(x)dx,$$

where $w(x) \geq 0$ on $[a, b]$ is a weight function. An n-point quadrature rule uses n weights w_i and n points x_i and has the form

$$J(f) = \sum_{i=1}^{n} w_i f(x_i).$$

(a) Let $w(x) = 1$. Define, and briefly compare, the n-point closed Newton-Cotes rule and the n-point Gauss rule for approximating $I(f)$. In particular, comment on: the choice of weights w_i, the choice of quadrature points x_i, and the degree of precision.

[6 marks]

(b) Derive the two-point Gauss-Chebyshev rule

$$I(f) = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \approx w_1 f(x_1) + w_2 f(x_2)$$

using the method of undetermined coefficients. That is, determine the choice of w_1, w_2 and x_1, x_2 to maximize the degree of precision.

HINT: You may find it useful to define a quadratic polynomial $\Phi(x) = x^2 + ax + b$ whose roots coincide with x_1 and x_2, and find a and b.

[9 marks]

(c) Newton-Cotes rules can be applied in repeated form by dividing $[a, b]$ into subintervals, and applying the rule on each one. Let $J_1(f)$ and $J_2(f)$ denote Newton-Cotes rules, such that $J_2(f)$ is more accurate than $J_1(f)$. Justify the following error estimate

$$|I(f) - J_2(f)| \approx |J_2(f) - J_1(f)|$$

and briefly explain how to construct an adaptive quadrature scheme.

[5 marks]

[20 marks]

B8. The ℓ-step Adams linear multistep methods for the initial value problem $y' = f(x, y)$, $x \geq x_0$, $y(x_0) = y_0$ have the form

$$y_{n+1} = \sum_{i=1}^{\ell} a_i y_{n+1-i} + h \sum_{i=0}^{\ell} b_i f_{n+1-i},$$

where a_i, b_i are given constants, h is the time step, and $f_n = f(x_n, y_n)$.

(a) Briefly explain the difference between the ℓ-step Adams–Bashforth method, and the ℓ-step Adams–Moulton method. Give an advantage and a disadvantage for each method.

(b) Using a suitable interpolating polynomial $p_1(x)$ of degree one, derive the two-step Adams–Bashforth method,

$$y_{n+1} = y_n + \frac{h}{2}\left[3f_n - f_{n-1}\right].$$

(c) Given that the two-step Adams–Moulton method has the form,

$$y_{n+1} = y_n + h\left[b_0 f_{n+1} + b_1 f_n + b_2 f_{n-1}\right],$$

derive conditions on the coefficients b_0, b_1, b_2 to achieve an order three method. Hence, derive the two-step Adams–Moulton method.

[20 marks]

END OF EXAMINATION PAPER

MATH37012

Two hours

THE UNIVERSITY OF MANCHESTER

MARKOV PROCESSES

24 May 2018

09.45 - 11.45

Answer 3 of the 4 questions. If more than 3 questions are attempted, credit will be given for the best 3 answers. Each question is worth 20 marks.

Electronic calculators may be used, provided that they cannot store text.

Answer 3 questions

1. a)

(i) Define the terms "recurrent state" and "null-recurrent state" in the context of discrete time Markov Chains.

[2 marks]

(ii) Define the stationary distribution of a discrete time Markov Chain.

[2 marks]

(iii) Give an example of a Markov Chain for which a stationary distribution exists yet, for some i, j, $\lim_{n\to\infty} p_{ij}(n)$ does not exist.

[2 marks]

(iv) Give an example of a Markov Chain which has more than one stationary distribution.

[2 marks]

b) Consider the discrete time Markov Chain $\{X_n : n \geq 0\}$ with state space $\{1,2,3,4,5\}$ and transition probability matrix

$$\begin{bmatrix} 0.4 & 0 & 0 & 0.6 & 0 \\ 0 & 0.3 & 0 & 0 & 0.7 \\ 0.5 & 0.1 & 0.4 & 0 & 0 \\ 0.4 & 0 & 0 & 0.6 & 0 \\ 0 & 0.7 & 0 & 0 & 0.3 \end{bmatrix}$$

(i) Classify each state as either recurrent or transient.

[2 marks]

(ii) Determine $\lim_{n\to\infty} p_{11}(n)$ and $\lim_{n\to\infty} p_{25}(n)$.

[3 marks]

(iii) For $n \geq 1$ let $f_{ij}(n) = P(X_n = j,\, X_k \neq j \quad for \quad 1 \leq k < n \mid X_0 = i)$. Calculate $f_{31}(n)$. Deduce that $P(X_n = 1 \text{ for some } n \geq 1 \mid X_0 = 3) = \frac{5}{6}$.

[3 marks]

(iv) Determine $\lim_{n\to\infty} p_{34}(n)$.

[2 marks]

(v) Calculate the expected time of first visit to state 5 given $X_0 = 2$.

[2 marks]

2. a) Consider the following discrete time Markov chain. A particle moves among the state space $\{0, 1, ...\}$ according to the following transition probabilities;

$$p_{i,i+1} = p, \quad p_{ii} = 1 - p - q, \quad p_{i,i-1} = q \qquad i \geq 1;$$

$$p_{01} = p, \quad p_{00} = 1 - p.$$

where $0 < p, q < 1$ and $p + q < 1$.

(i) Draw the transition diagram.

[2 marks]

(ii) Write down the transition matrix for this Markov chain.

[2 marks]

(iii) Determine the pairs of values of p and q for which a stationary distribution exists. Calculate the stationary distribution.

[6 marks]

(iv) Stating carefully any result to which you appeal, determine, for each pair (p, q), the expected time of first return to state 0, given $X_n = 0$.

[3 marks]

(v) Calculate the expected time of the first visit to state 2, given initial state 0.

[4 marks]

b) Prove that the states of an irreducible, aperiodic, recurrent Markov chain, are either all positive recurrent, or all null-recurrent.

[3 marks]

3. a) Consider the Markov chain $\{X(t), t \geq 0\}$ with state space $\{0, 1\}$ where, the holding times for states 0 and 1 are independent exponential random variables with parameters λ and μ respectively. Upon leaving one state, the process enters the other.

(i) Write down the system of forward equations $P'(t) = P(t)Q$ for this process.

[4 marks]

(ii) With the initial condition $p_{00}(0) = 1$, use this system to show that, for $t \geq 0$,

$$p'_{00}(t) = -(\lambda + \mu)p_{00}(t) + \mu$$

and hence that

$$p_{00}(t) = \frac{1}{\lambda + \mu} \left\{\mu + \lambda \exp(-[\lambda + \mu]t)\right\}.$$

[6 marks]

b) Consider the continuous time Markov chain with state space $S = \{1, 2, 3\}$, where $q_j = \lambda$ for $j = 1, 2, 3$; i.e. all holding times are exponentially distributed with parameter λ. The jump-chain has transition matrix $R = (r_{ij})$ equal to

$$\begin{bmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

(i) Write down the matrix Q for this continuous time Markov chain.

[3 marks]

(ii) Show that the unique stationary distribution is given by
$(\pi_1, \pi_2, \pi_3) = (\frac{1}{3}, \frac{5}{18}, \frac{7}{18})$.

[3 marks]

(iii) Let a_{ij} denote the expected time of first visit to state j starting from state i. By conditioning on the destination of the first transition, show that

$$a_{31} = \frac{1}{\lambda}\frac{1}{2} + (a_{21} + \frac{1}{\lambda})\frac{1}{2}.$$

Hence, or otherwise, show that $a_{31} = \frac{2}{\lambda}$.

[4 marks]

4. Customers arrive at a two-server shop queue in a Poisson process rate λ per hour. If an arriving customer finds a server free, she presents herself for service. If she finds both servers occupied, but no-one else waiting, she either joins the queue with probability $\frac{1}{2}$, or leaves with probability $\frac{1}{2}$. If she finds both servers occupied and at least one other customer waiting, she does not join the queue. The service time distribution is exponential with mean μ^{-1} and the service times are independent and independent of the arrival process. Let $(\pi_0, \pi_1, \pi_2, \pi_3)$ denote the stationary distribution for the number of customers in the system.

(i) Explain why, when there are 2 customers in the system, the time to the next customer joining the queue is exponentially distributed with parameter $\frac{\lambda}{2}$.

[4 marks]

(ii) Using the result of part (i) and carefully stating any result to which you appeal, show that

$$\pi_0 = \tfrac{1}{1+\rho+\rho^2/2+\rho^3/8} \qquad \pi_1 = \tfrac{\rho}{1+\rho+\rho^2/2+\rho^3/8} \qquad \pi_2 = \tfrac{\rho^2}{2(1+\rho+\rho^2/2+\rho^3/8)}$$
$$\pi_3 = \tfrac{\rho^3}{8(1+\rho+\rho^2/2+\rho^3/8)}$$

where $\rho = \frac{\lambda}{\mu}$.

[6 marks]

(iii) Explain why, when the stationary distribution is the initial distribution, the mean number of customers lost per hour is equal to

$$\frac{\lambda(2\rho^2 + \rho^3)}{8(1 + \rho + \rho^2/2 + \rho^3/8)}.$$

[4 marks]

(iv) At present, $\lambda = 60$ and $\mu = 40$, and the average profit per customer served is *£1*. The shop keeper wishes to employ two replacement cashiers who are each, on average, able to serve twice as quickly ($\mu = 80$). To employ these two cashiers would increase wage costs by *£5* per hour. Calculate whether it would be profitable for the shop keeper to do so.

[6 marks]

END OF EXAMINATION PAPER

Statistical Tables to be provided

Two hours

THE UNIVERSITY OF MANCHESTER

Time Series Analysis

31 May 2018
09.45 – 11.45

Electronic calculators are permitted, provided they cannot store text.

Answer **ALL** Four questions in Section A (32 marks in all)

and

Two of the Three questions in Section B (24 marks each).

If more than Two questions from Section B are attempted, then credit will be given for the best Two answers.

SECTION A
Answer ALL four questions

A1. *Akaike's information criterion* is often used to choose between competing models fitted to an observed time series of length n. The AIC statistic can be defined by the formula

$$\text{AIC} = \ln \hat{\sigma}^2 + \frac{2k}{n},$$

where $\hat{\sigma}^2$ is an optimal (maximum likelihood or equivalent) estimate of σ^2 and k is the number of estimated parameters (apart from σ^2).

a) (3 marks) Explain concisely the general behaviour of the two terms of the AIC statistic as the number of parameters increases.

b) (2 marks)
The table below gives the value of the AIC statistic for several models fitted to an observed time series of length $n = 10$.

Model	AIC
$X_t = c + \varepsilon_t + \theta_1\varepsilon_{t-1}$	17
$X_t = c + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$	16.5
$X_t = c + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \theta_3\varepsilon_{t-3}$	18.5

Use the AIC criterion to select a model for this time series.

c) (3 marks) A modification, AIC corrected (AICC), of the AIC statistic is given by the following formula

$$\text{AICC} = \ln \hat{\sigma}^2 + \frac{2k}{n-k-1}.$$

Calculate AICC for the models in part (b) above and select a model according to AICC.

[8 marks]

A2. Let x_t, $t = 1, \ldots, n$ be an observed time series. The Box-Ljung test is based on the statistic

$$Q_{LB}(h) = n(n+2)\sum_{k=1}^{h} \frac{\hat{\rho}_k^2}{n-k},$$

where n is the length of the observed trajectory and $\hat{\rho}_k$ are sample autocorrelations of $\{x_t\}$. Under the null hypothesis of interest the distribution of $Q_{LB}(h)$ is approximately χ^2_h.

a) (3 marks) State the null and alternative hypotheses in the Ljung-Box test.

b) (3 marks)
What is the distribution of the test statistic, under the same null hypothesis, when x_t, $t = 1, \ldots, n$ represent the residuals from a fitted ARMA(p,q) model with intercept?

[6 marks]

A3. Let $\{X_t\}$ be an MA(q) time series with representation

$$X_t = \sum_{i=0}^{q} \theta_i \varepsilon_{t-i},$$

where $\theta_0 = 1$ and $\{\varepsilon_t\}$ is white noise with variance σ^2.

a) (3 marks) Show that $\operatorname{E} X_t \varepsilon_{t-k} = \theta_k \sigma^2$, for $k = 0, 1, \ldots, q$.

b) (7 marks) Let $\{X_t\}$ be a stationary time series with mean $\operatorname{E} X_t = 0$. Assume further that the autocovariances, γ_k, of $\{X_t\}$ are as follows:

$$\gamma_k = \begin{cases} 2 & \text{for } k = 0, \\ 0.8 & \text{for } k = 12, \\ 0 & \text{for } k \neq 0, \pm 12. \end{cases}$$

Identify a model for $\{X_t\}$ and compute its parameters. If more than one model is possible, choose an invertible one.

[10 marks]

A4. Consider the following model for a stationary process:

$$X_t = -\tfrac{3}{10} X_{t-1} + \tfrac{4}{10} X_{t-2} + \varepsilon_t - \tfrac{1}{2} \varepsilon_{t-1},$$

where $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$.

a) (3 marks) Write down the model in operator form.

b) (5 marks) Explain why this model is of higher order than necessary and give its reduced form.

[8 marks]

SECTION B
Answer 2 of the 3 questions

B5.

a) (12 marks) A stationary time series $\{X_t\}$ follows the AR(2) model

$$X_t = 0.01 + 0.2X_{t-2} + \varepsilon_t, \tag{1}$$

where $\{\varepsilon_t\} \sim \mathrm{WN}(0, 0.02)$.

i) What property of an AR model is referred to as *causality property*?

ii) Find the mean of $\{X_t\}$.

iii) Compute the first two autocorrelations, ρ_1 and ρ_2, and the variance, γ_0, of the process $\{X_t\}$.

iv) Assume that the values $x_{100} = -0.01$ and $x_{99} = 0.02$ of the series are observed at times $t = 100$ and $t = 99$. Compute $\hat{x}_{101|100,99,\ldots}$ and $\hat{x}_{102|100,99,\ldots}$, the 1- and 2-step ahead forecasts at the forecast origin $t = 100$.

v) Give the standard deviations of the forecast errors associated with the predictors in (iv).

b) (12 marks) Let $\{X_t\}$ be an AR(2) process with representation

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t, \qquad \varepsilon_t \sim \mathrm{WN}(0, \sigma^2). \tag{2}$$

Let $\{Y_t\}$ be the "reversed" process defined by $Y_t = X_{-t}$. Show that $\{Y_t\}$ is also an AR(2) process with the same parameters as that of the model for $\{X_t\}$.

[24 marks]

B6.

a) (8 marks) Let $\{V_t\}$ be an I(1) process and $\{Z_t\}$ be a stationary process.

i) Show that $V_t + Z_t$ is not stationary. (*Hint:* You may use the fact that a linear combination of stationary processes cannot be an I(1) process.)

ii) Assume further that the time series $\{V_t\}$ and $\{Z_t\}$ are independent of each other. Show that $V_t + Z_t$ is an I(1) process.

b) (10 marks) Consider a random walk $\{u_t\}$ and a stationary AR(1) time series $\{e_t\}$, which evolve according to the following equations:

$$\begin{aligned} u_0 &= 0, \\ u_t &= u_{t-1} + \varepsilon_{1,t}, \qquad \text{for } t \geq 1, \\ e_t &= \rho e_{t-1} + \varepsilon_{2,t}, \qquad \text{for } t \geq 1, \end{aligned}$$

where $|\rho| < 1$ and $\{\varepsilon_{1,t}\}$ and $\{\varepsilon_{2,t}\}$ are white noise sequences, independent of each other (so, $\mathrm{E}\,\varepsilon_{1,t}\varepsilon_{2,s} = 0$ for all t, s).

Consider also processes $\{X_t\}$ and $\{Y_t\}$ which obey the following equations:

$$\begin{aligned} X_t + \beta Y_t &= u_t \\ X_t + \alpha Y_t &= e_t, \end{aligned}$$

where α and β are some constants, $\alpha \neq \beta$, and $\{u_t\}$ and $\{e_t\}$ are the processes defined above.

i) Show that

$$X_t = \tfrac{\alpha}{\alpha-\beta}u_t - \tfrac{\beta}{\alpha-\beta}e_t$$
$$Y_t = \tfrac{-1}{\alpha-\beta}u_t + \tfrac{1}{\alpha-\beta}e_t.$$

ii) Identify the processes $\{u_t\}$, $\{e_t\}$, $\{X_t\}$ and $\{Y_t\}$ as I(d) processes (e.g. I(1), I(0)). Give your reasons concisely.

iii) For each of the following pairs of processes state, giving your reasons, if they are cointegrated of order one and, if so, give a cointegrating vector.

- $\{u_t\}$ and $\{e_t\}$,
- $\{X_t\}$ and $\{Y_t\}$.

c) (6 marks) An analysis of two time series, $\{X_t\}$ and $\{Y_t\}$, established that they are well described by ARIMA(0,1,0) models. The Johansen test for cointegration was performed with the following results:

```
######################
# Johansen-Procedure #
######################

Test type: maximal eigenvalue statistic (lambda max), with linear trend

Eigenvalues (lambda):
[1] 0.31366569 0.04020297

Values of test statistic and critical values of test:

          test 10pct  5pct  1pct
r <= 1 |  4.02  6.50  8.18 11.65
r = 0  | 36.89 12.91 14.90 19.19

Eigenvectors, normalised to first column:
(These are the cointegration relations)

             x        y
x    1.0000000 1.000000
y   -0.9989006 1.487965
##################################################################
```

Are the two time series cointegrated? If so, give a cointegrating vector and a cointegrating relation. Explain your decision.

[24 marks]

B7.
The number of overseas visitors to New Zealand is recorded for each month over the period 1977–1995. Let x_t be the number of overseas visitors in time period t (in months) and $z_t = \ln x_t$.

a) (3 marks) Comment on the main features of the plots of $\{x_t\}$ and $\{z_t\}$ (see Figure 7.1). Why one might wish to build a model for $\{z_t\}$ rather than directly for $\{x_t\}$?

b) (3 marks) Comment on the main features of the sample autocorrelation function of $\{z_t\}$ (see Figure 7.1).

c) (4 marks) An ARIMA(1,1,0) model was fitted to $\{z_t\}$. Comment on the quality of the fit, using the numerical and graphical information in Figure 7.2.

d) (4 marks) An ARIMA$(1,1,0)(0,1,0)_{12}$ model was fitted to $\{z_t\}$, see Figure 7.3. Why was this particular modification of the model in part (c) chosen? Did it achieve the desired effect?

e) (2 marks) Identify the model in Figure 7.6 as a seasonal ARIMA$(p,d,q)(p_s,d_s,q_s)_s$ model. Write it in operator form.

f) (4 marks) Choose the best fitting model among those given in Figures 7.2–7.10. Explain the reasons for your choice.

g) (4 marks) Comment on the quality of the fit of the model selected in part (f) above.

Note: Write verbal answers concisely. You normally need one sentence or phrase for each worthy thought. Credit will not be given for irrelevant material.

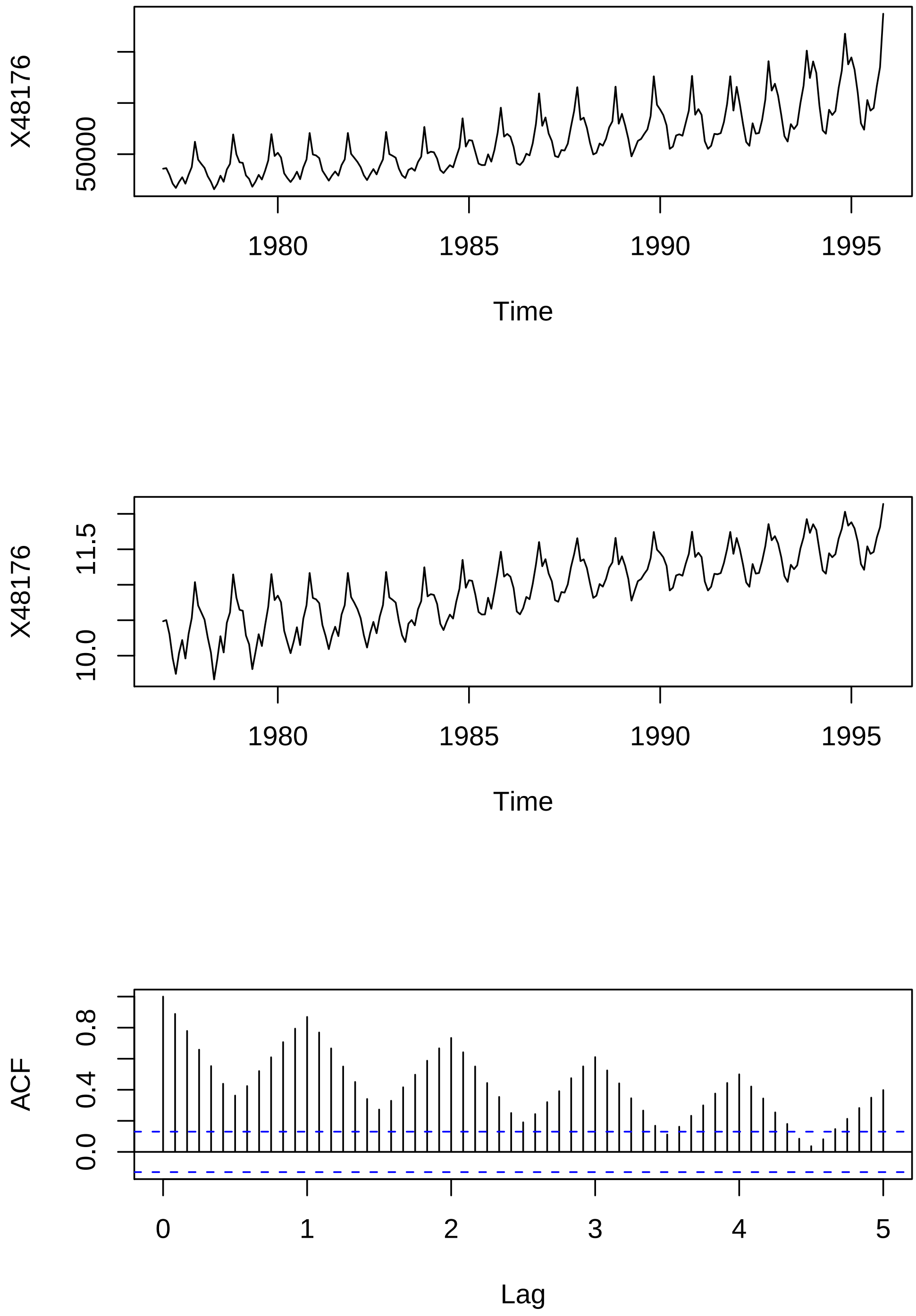

Figure 7.1: Visitors to New Zealand: the time series, $\{x_t\}$ (top), the logged time series, $\{z_t\}$ (middle), and the sample autocorrelation function of $\{z_t\}$ (bottom).

```
Call:
arima(x = visnzln, order = c(1, 1, 0))

Coefficients:
         ar1
      0.0255
s.e.  0.0667

sigma^2 estimated as 0.04985:  log likelihood = 18.17,  aic = -32.34
```

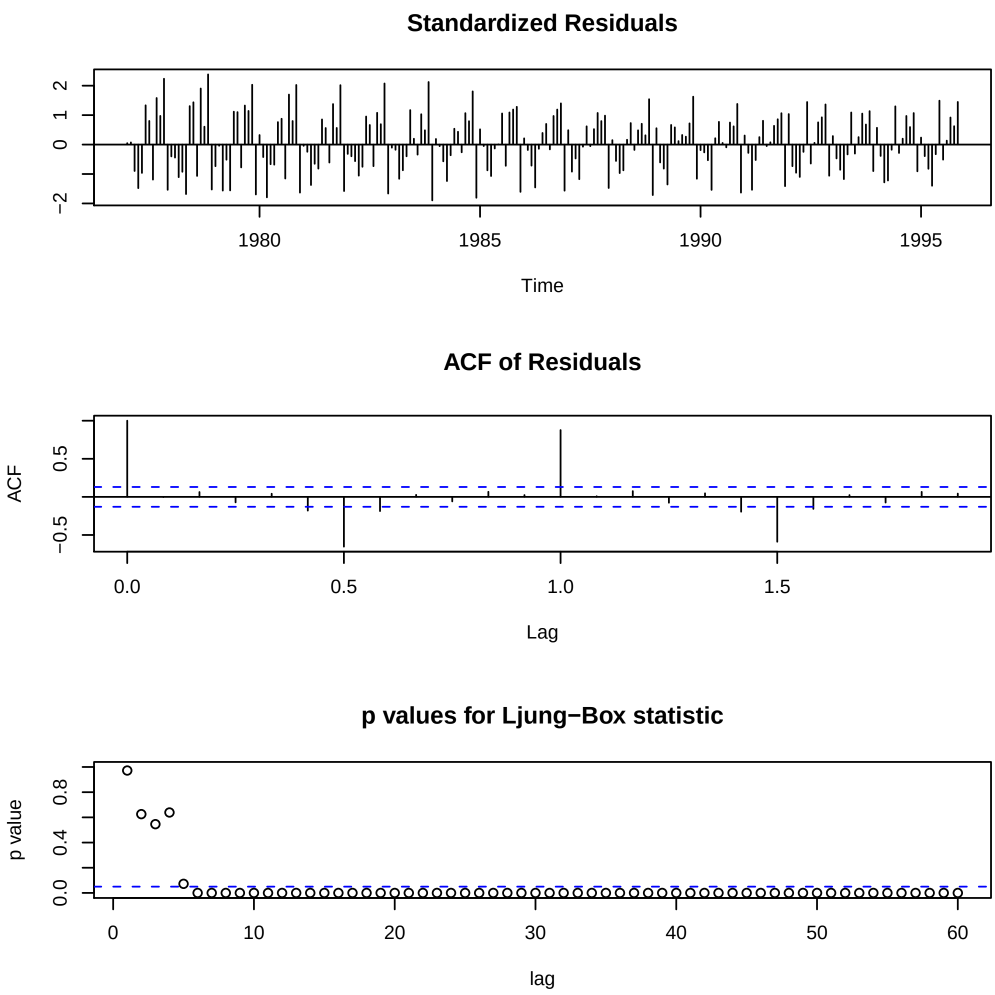

Figure 7.2: Visitors to New Zealand: Model 1

```
Call:
arima(x = visnzln, order = c(1, 1, 0), seasonal = list(order = c(0, 1, 0)))

Coefficients:
          ar1
      -0.4076
s.e.   0.0628

sigma^2 estimated as 0.005184:  log likelihood = 259.3,  aic = -514.6
```

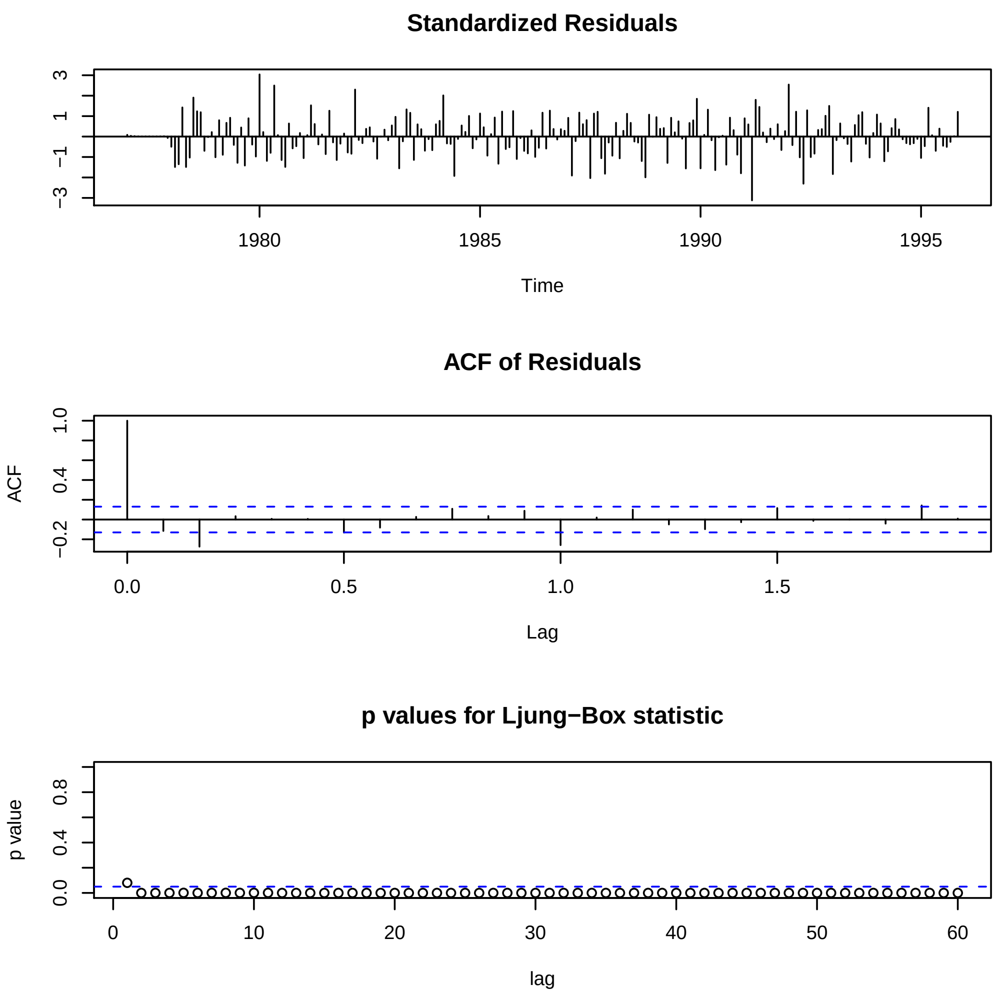

Figure 7.3: Visitors to New Zealand: Model 2

```
Call:
arima(x = visnzln, order = c(1, 1, 0), seasonal = list(order = c(1, 1, 0)))

Coefficients:
          ar1     sar1
      -0.3911  -0.2779
s.e.   0.0633   0.0672

sigma^2 estimated as 0.004784:  log likelihood = 267.44,  aic = -528.87
```

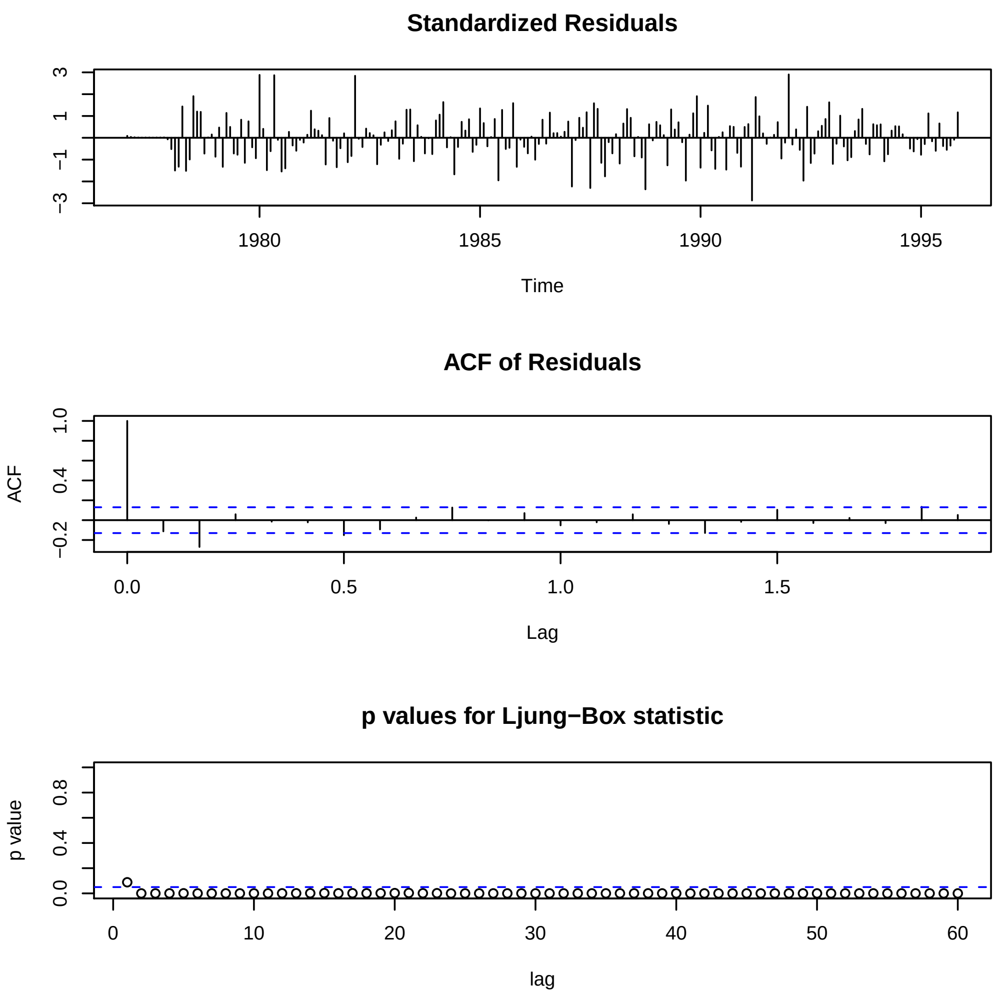

Figure 7.4: Visitors to New Zealand: Model 3


```
Call:
arima(x = visnzln, order = c(0, 1, 1), seasonal = list(order = c(0, 1, 1)))

Coefficients:
          ma1     sma1
      -0.6727  -0.3706
s.e.   0.0726   0.0725

sigma^2 estimated as 0.004050:  log likelihood = 284.64,  aic = -563.27
```

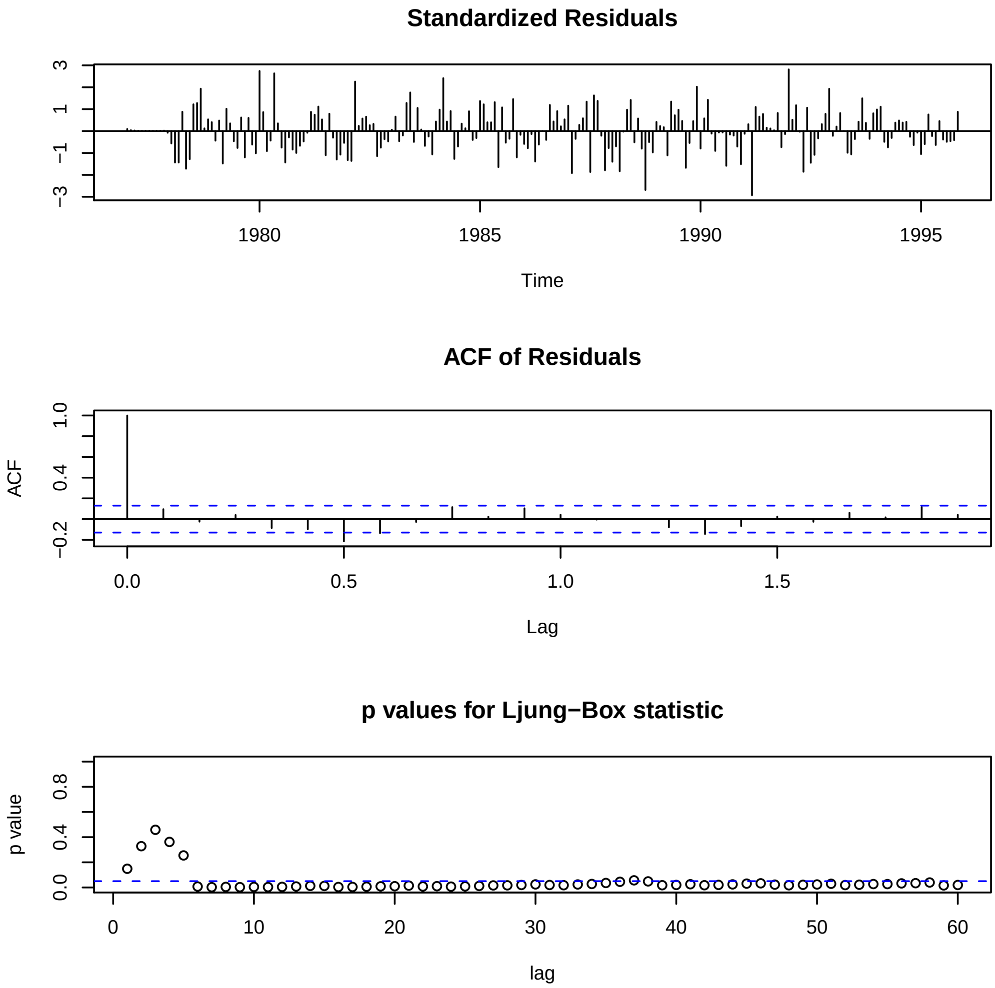

Figure 7.5: Visitors to New Zealand: Model 4

```
Call:
arima(x = visnzln, order = c(1, 1, 0), seasonal = list(order = c(0, 1, 1)))

Coefficients:
          ar1     sma1
      -0.3893  -0.4125
s.e.   0.0632   0.0706

sigma^2 estimated as 0.004556:  log likelihood = 272.01,  aic = -538.02
```

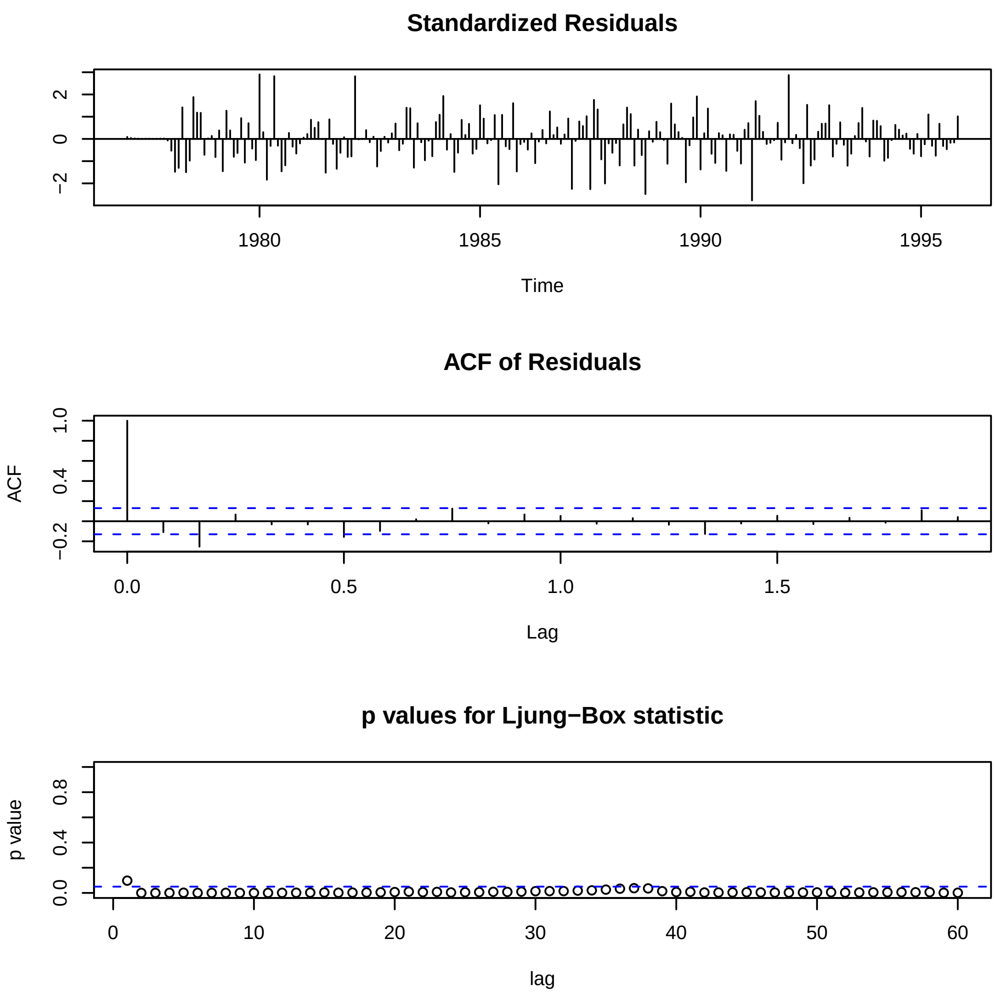

Figure 7.6: Visitors to New Zealand: Model 5

```
Call:
arima(x = visnzln, order = c(0, 1, 1), seasonal = list(order = c(1, 1, 0)))

Coefficients:
          ma1     sar1
      -0.6883  -0.2713
s.e.   0.0685   0.0674

sigma^2 estimated as 0.004182:  log likelihood = 281.6,  aic = -557.19
```

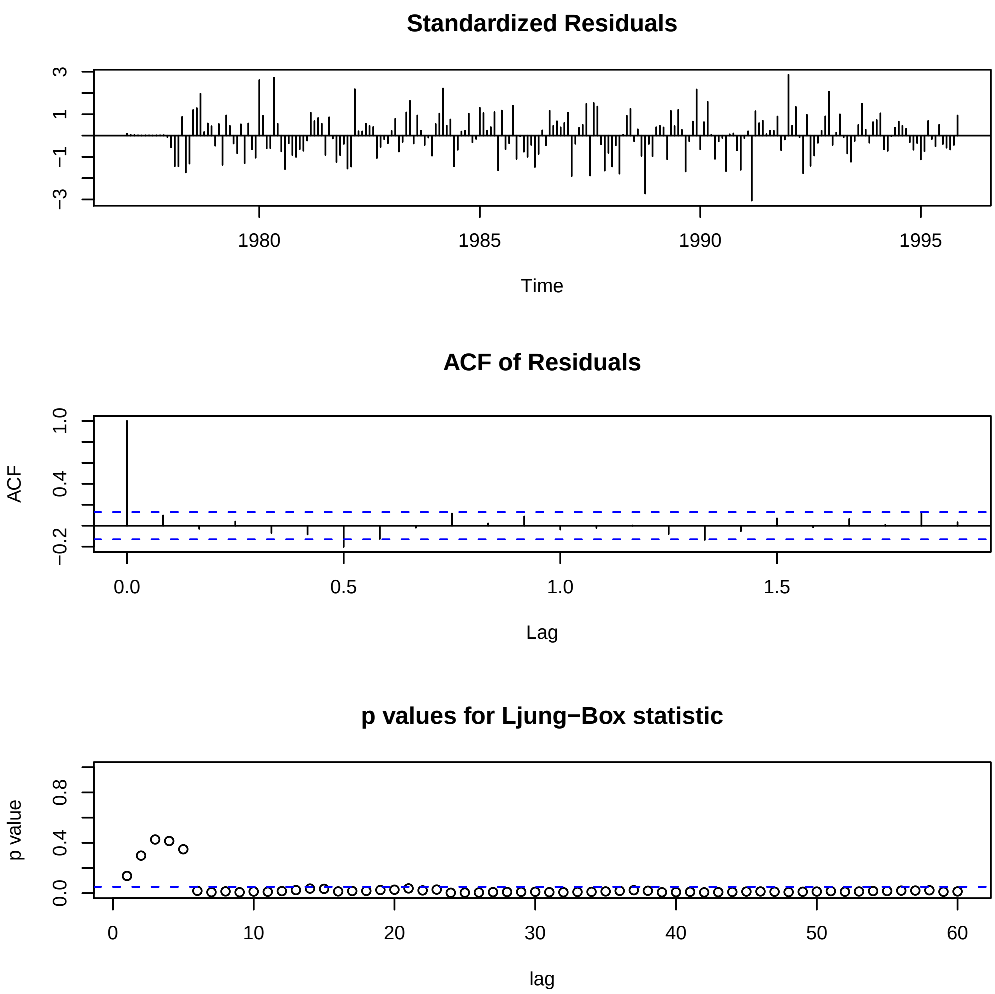

Figure 7.7: Visitors to New Zealand: Model 6

```
Call:
arima(x = visnzln, order = c(1, 1, 1), seasonal = list(order = c(1, 1, 1)))

Coefficients:
         ar1     ma1    sar1     sma1
      0.2546  -0.845  0.2297  -0.5866
s.e.  0.0962   0.059  0.1618   0.1349

sigma^2 estimated as 0.003897:  log likelihood = 288.35,  aic = -566.71
```

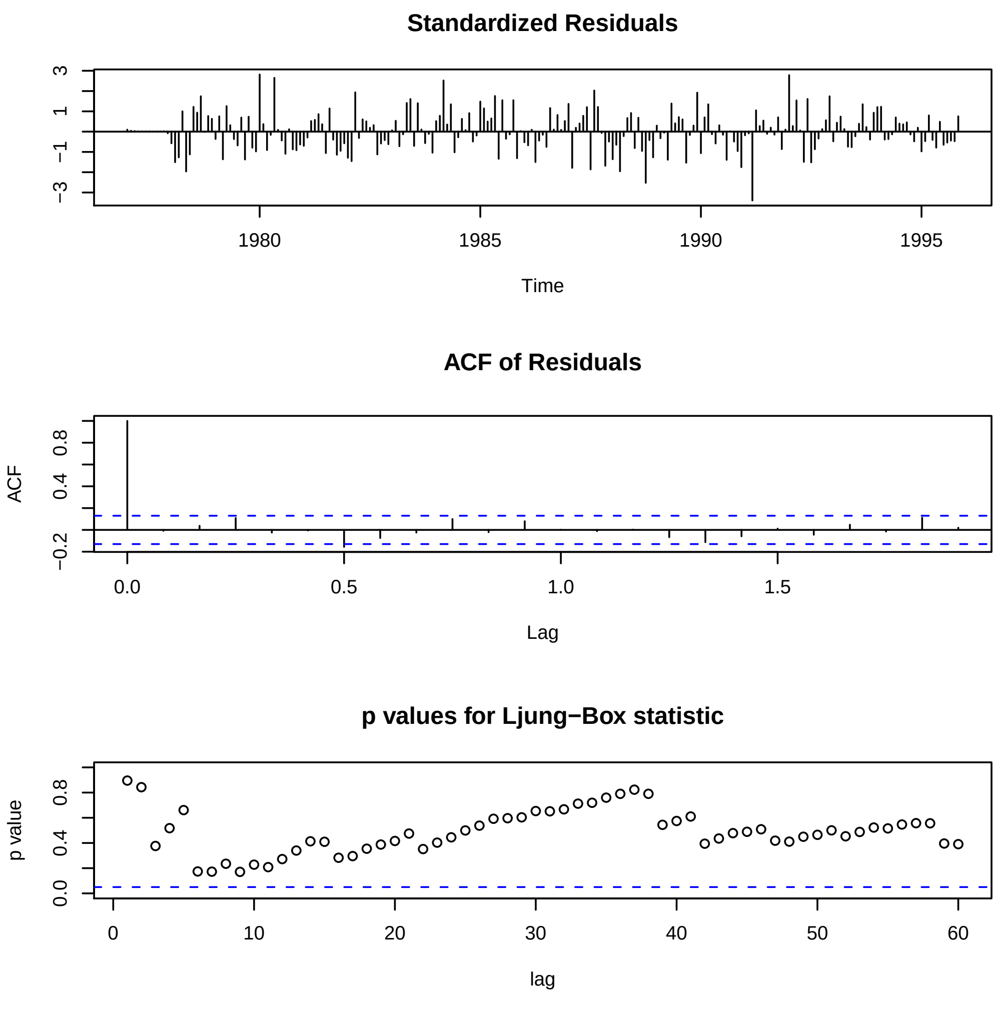

Figure 7.8: Visitors to New Zealand: Model 7

```
Call:
arima(x = visnzln, order = c(1, 1, 1), seasonal = list(order = c(1, 1, 0)))

Coefficients:
         ar1      ma1     sar1
      0.2396  -0.8409  -0.2868
s.e.  0.0972   0.0600   0.0669

sigma^2 estimated as 0.004077:  log likelihood = 284.12,  aic = -560.25
```

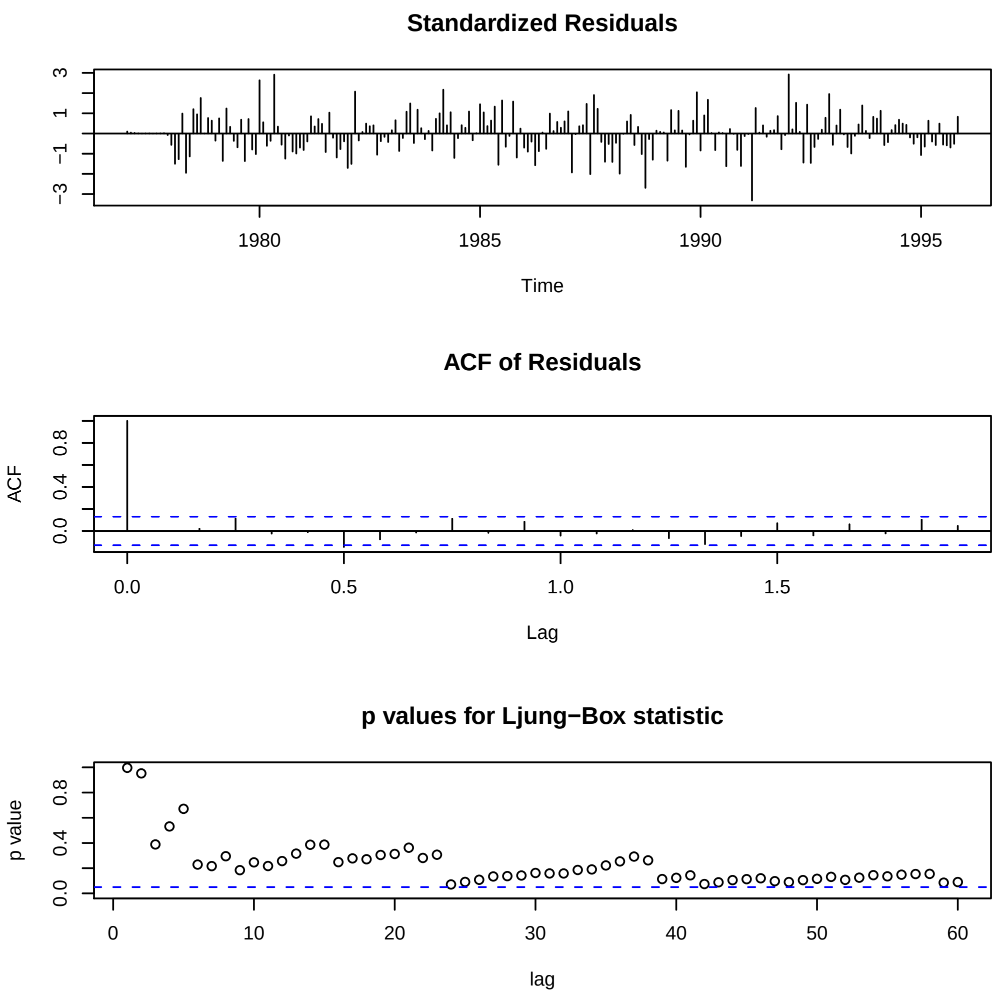

Figure 7.9: Visitors to New Zealand: Model 8

```
Call:
arima(x = visnzln, order = c(1, 1, 1), seasonal = list(order = c(0, 1, 1)))

Coefficients:
         ar1      ma1     sma1
      0.2527  -0.8393  -0.3909
s.e.  0.0948   0.0569   0.0717

sigma^2 estimated as 0.003935:  log likelihood = 287.44,  aic = -566.87
```

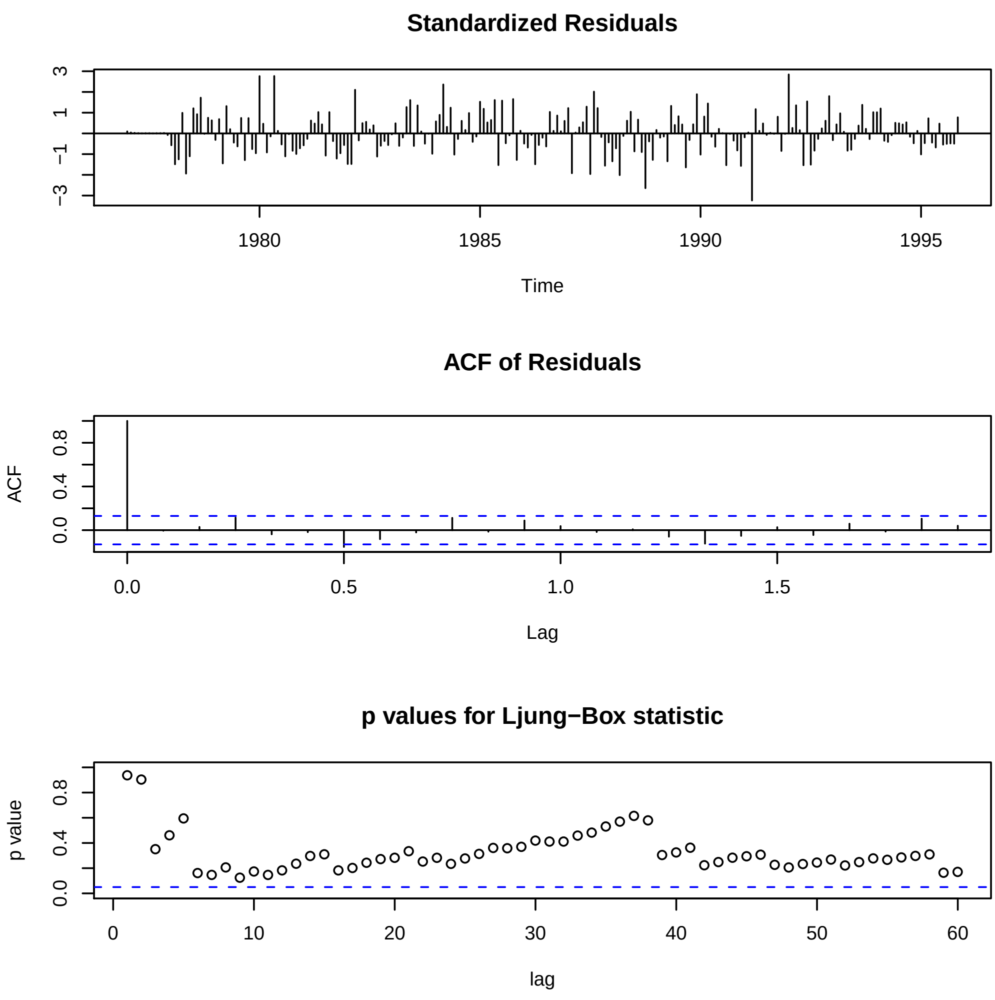

Figure 7.10: Visitors to New Zealand: Model 9

[24 marks]

END OF EXAMINATION PAPER

Two Hours

Statistical tables are provided at end of paper

THE UNIVERSITY OF MANCHESTER

GENERALISED LINEAR MODELS

25 May 2018

14:00 – 16:00

Answer ALL TWO questions in Section A.
Answer TWO of the THREE questions in Section B.

If more than TWO questions from section B are attempted
then credit will be given to the best TWO answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** of the two questions

A1.

(a) Show that the normal distribution $N(\mu, \sigma^2)$ with probability density function

$$f(y; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right\}, \quad -\infty < y < \infty$$

belongs to the exponential family. [4 marks]

(b) Use Property 1 of distributions from the exponential family to show that the mean of $N(\mu, \sigma^2)$ is μ and that the variance is σ^2. [4 marks]

(c) Define the variance function in general for any distribution from the exponential family and give an example of a discrete distribution that has variance function $V(\mu) = \mu$. Name the distribution and use the definition or otherwise to work out its variance function. [4 marks]

(d) Find the canonical link for a generalised linear model where the response variable follows the distribution in (c). [4 marks]

(e) Give the name of the algorithm for the fitting of generalised linear models and explain why it is known by such a name. You do not have to write down the updating equation. [4 marks]

[20 marks in total for this question.]

A2.

(a) Write down a generalised linear model with canonical link for binomial responses $y_i \sim B(n_i, \pi_i)$ on a single explanatory variable x with values x_i, $i = 1, \ldots, n$, and state any further assumption on $y_1, \ldots, y_n$. An intercept should be included in the linear predictor. [3 marks]

(b) Find the binomial probability π_i in terms of the linear predictor $\eta_i = \beta_0 + \beta_1 x_i$, $i = 1, \ldots, n$, for the model in (a). [3 marks]

(c) Work out the Fisher information matrix $I(\underset{\sim}{\beta})$ and explain how to use it to find approximate standard deviations of maximum likelihood estimates of $\underset{\sim}{\beta} = (\beta_0, \beta_1)'$ for the model in (a). You may use the formula

$$I(\underset{\sim}{\beta}) = X'WX$$

without proof, where X is the data matrix and W is a diagonal matrix with

$$w_i = n_i \pi_i (1 - \pi_i), \; i = 1, \ldots, n$$

on the main diagonal. The answer should be given in terms of β_0 and β_1. [6 marks]

(d) Derive the deviance of the model in (a) fitted to data and state without proof its approximate distribution when the sample size n is large. [6 marks]

(e) Can the deviance in (d) be used to assess model adequacy? Give reasons for your answer. [2 marks]

[20 marks in total for this question.]

SECTION B

Answer **TWO** of the three questions

B1. The inverse Gaussian distribution $\mathrm{IG}(\mu, \sigma^2)$ with density function

$$\begin{aligned} f(y;\mu,\sigma^2) &= \frac{1}{\sqrt{2\pi y^3}\sigma}\exp\left\{-\frac{1}{2\mu^2 y}\left(\frac{y-\mu}{\sigma}\right)^2\right\} \\ &= \exp\left\{\frac{y\theta - b(\theta)}{\sigma^2} - \frac{1}{2\sigma^2 y} - \frac{1}{2}\log(2\pi y^3\sigma^2)\right\}, \quad y > 0 \end{aligned}$$

belongs to the exponential family, where θ depends on μ and $b(\theta)$ is a function of θ.

(a) Find the canonical parameter θ in terms of μ and the function $b(\theta)$ in terms of θ. [4 marks]

(b) Show that the mean of $\mathrm{IG}(\mu, \sigma^2)$ is μ and find the variance function $V(\mu)$. [6 marks]

(c) A generalised linear model with IG response y, linear predictor $\eta = \alpha + \beta x + \gamma x^2$ and reciprocal link $g(\mu) = 1/\mu$ was fitted to $n = 48$ observations on (x, y). The deviance was 74.68 for the fitted model.

(i) Estimate σ^2 using the deviance. [4 marks]

(ii) The deviance increased to 84.95 for the reduced model with $\eta = \alpha + \beta x$. Test appropriate hypotheses at the 5% level to decide if the x^2 term is required. [6 marks]

[20 marks in total for this question.]

B2. In an experiment tobacco budworm moths are exposed to 6 different doses (mg) of the insecticide trans-cypermethrin. For each sex the number of moths killed and not killed at each dose is recorded in y. Logistic models are fitted to the data, where the base-2 logarithm of dose, Ldose=$\log_2$(dose), is taken as a covariate and the variable Sex as a factor. The models are referred to as fit1 and fit2.

```
> fit1<-glm(y ~ Sex * Ldose, family=binomial(link=logit))
> anova(fit1, test="Chisq")

  Analysis of Deviance Table
  Terms added sequentially (first to last)
               Df    Deviance Resid.   Df   Resid. Dev   P(>|Chi|)
  NULL                                 11    124.876
  Sex           1        6.077         10    118.799      0.014
  Ldose         1      112.042          9      6.757      3.499e-26
  Sex:Ldose     1        1.763          8      4.994      0.184

  > fit2 <- glm(y ~ Sex + Ldose, family=binomial(link=logit))
  > summary(fit2)
    Coefficients:
               Estimate  Std. Error  z value  Pr(>|z|)
  (Intercept)   -3.4732     0.4685    -7.413   1.23e-13
   SexMale       1.1007     0.3558     3.093   0.00198
   Ldose         1.0642     0.1311     8.119   4.70e-16
   Null deviance:       124.876 on 11 degrees of freedom
   Residual deviance:     6.757 on  9 degrees of freedom
```

(a) Write down the model fit1 as a generalised linear model and explain what the parameters represent. [4 marks]

(b) What is the deviance of the model fit1? Use it to assess the goodness of fit. [4 marks]

(c) What is the deviance of the model fit2? Is there a significant difference in deviance at 5% between the two models? [4 marks]

(d) Is there a significant difference between males and females? Test appropriate hypotheses at the 5% level. [4 marks]

(e) Explain what is meant by the median lethal dose (LD50). Based on the model fit2, for each sex estimate the LD50 value. [4 marks]

[20 marks in total for this question.]

B3.

(a) Let y_{ij} be the number of observations in row i column j of a contingency table, where $i = 1, \ldots, I$, $j = 1, \ldots, J$. Assuming a multinomial distribution with $n = \sum_{i,j} y_{ij}$ fixed, write down the log-likelihood of $\{y_{ij}\}$ in terms of the cell probabilities π_{ij}, $i = 1, \ldots, I$, $j = 1, \ldots, J$. [4 marks]

(b) What is the distribution of y_{ij} in (a) for each combination of i and j and are the random variables $\{y_{ij}\}$ in (a) independent? Give a reason for your answer to the latter question. [4 marks]

(c) Write down a generalised linear model for $\{y_{ij}\}$ with independent Poisson distributions $\text{Pois}(\lambda_{ij})$ and log link even though the distribution of y_{ij} in (a) is not Poisson. How do you make a connection between the parameters $\{\lambda_{ij}\}$ and the parameters $\{\pi_{ij}\}$? [4 marks]

(d) Show that the hypothesis of no association between row and column classifications leads to an additive Poisson model. [4 marks]

(e) A random sample of 100 motor vehicles stopped at roadside were classified according to type and mechanical condition, and a Poisson additive model was fitted to the data.

```
> summary(fit)
Call: glm(formula = y ~ type + condition, family = poisson)
Coefficients:
                      Estimate Std. Error z value Pr(>|z|)
(Intercept)             3.2261     0.1610  20.035  < 2e-16 ***
typelorry              -0.5152     0.1863  -2.765 0.005701 **
typevan                -0.7603     0.2019  -3.766 0.000166 ***
conditionpoor          -0.2136     0.2075  -1.029 0.303259
conditionsatisfactory   0.2231     0.1861   1.199 0.230387

Null deviance: 37.889  on 8  degrees of freedom
Residual deviance: 16.236  on 4  degrees of freedom
```

What conclusion can be made about the relationship, if any, between vehicle type (car, lorry, van) and condition (good, satisfactory, poor)? Use a 5% significance level. [4 marks]

[20 marks in total for this question.]

df	$q - value$										
ν	0.005	0.01	0.025	0.05	0.10	0.50	0.90	0.95	0.975	0.99	0.995
1	$.0^{4}39$	$.0^{3}16$	$.0^{3}98$	.0039	.0158	.455	2.706	3.841	5.024	6.635	7.879
2	.0100	.0201	.0506	.103	.211	1.386	4.605	5.991	7.378	9.210	10.597
3	.0717	.115	.216	.352	.584	2.366	6.251	7.815	9.348	11.345	12.838
4	.207	.297	.484	.711	1.064	3.357	7.779	9.488	11.143	13.277	14.860
5	.412	.554	.831	1.145	1.610	4.351	9.236	11.070	12.832	15.086	16.750
6	.676	.872	1.237	1.635	2.204	5.348	10.645	12.592	14.449	16.812	18.548
7	.989	1.239	1.690	2.167	2.833	6.346	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	7.344	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	8.343	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	9.342	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	10.341	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	11.340	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	12.340	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	13.339	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	14.339	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	15.338	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	16.338	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	17.338	25.989	28.869	31.526	34.805	37.156
19	6.844	7.638	8.907	10.117	11.651	18.338	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	19.337	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	20.337	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	21.337	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.688	13.091	14.848	22.337	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	23.337	33.196	36.415	39.364	42.980	45.558
25	10.520	11.524	13.120	14.611	16.473	24.337	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	25.336	35.563	38.885	41.943	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	26.336	36.741	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	27.336	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	28.336	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	29.336	40.256	43.733	46.979	50.892	53.672
31	14.458	15.656	17.539	19.281	21.434	30.436	41.422	44.985	48.232	52.191	55.003
32	15.134	16.362	18.291	20.072	22.271	31.336	42.585	46.194	49.480	53.486	56.328
33	15.815	17.074	19.047	20.867	23.110	32.336	43.745	47.400	50.725	54.776	57.649
34	16.501	17.879	19.806	21.664	23.952	33.336	44.903	48.602	51.966	58.061	58.964
35	17.192	18.509	20.569	22.465	24.797	34.336	46.059	49.802	53.203	57.342	60.275
36	17.887	19.233	21.336	23.269	25.643	35.336	47.212	50.999	54.437	58.619	61.581
37	18.586	19.960	22.106	24.075	26.492	36.336	48.363	52.192	55.668	59.893	62.883
38	19.289	20.691	22.879	24.884	27.343	37.335	49.513	53.384	56.896	61.162	64.181
39	19.996	21.462	23.654	25.695	28.196	38.335	50.660	54.572	58.120	62.428	65.476
40	20.707	22.164	24.433	26.509	29.051	39.335	51.805	55.759	59.342	63.691	66.766
42	22.139	23.650	25.999	28.144	30.765	41.335	54.090	58.124	61.777	66.206	69.336
44	23.584	25.148	27.575	29.788	32.487	43.335	56.369	60.641	64.201	68.710	71.893
46	25.041	26.657	29.160	31.439	34.215	45.335	58.641	62.830	66.617	71.201	74.437
48	26.511	28.177	30.755	33.098	35.949	47.335	60.907	65.171	69.023	73.683	76.969
50	27.991	29.707	32.357	34.764	37.689	49.335	63.167	67.505	71.420	76.154	79.490
55	31.735	33.571	36.398	38.958	42.060	54.335	68.796	72.312	77.381	82.292	85.749
60	35.535	37.485	40.482	43.188	46.459	59.335	74.397	79.082	82.298	88.379	91.952
65	39.383	41.444	44.603	47.450	50.883	64.334	79.973	84.821	89.177	94.422	98.105
70	43.275	45.442	48.758	51.739	55.329	69.334	85.527	90.531	95.023	100.43	104.22

Table 1: **Percentage points $\chi^2_{\nu,q}$ of the chi-squared distribution. The percentage point $\chi^2_{\nu,q}$ of the chi-squared distributed random variable χ^2_ν is defined as** $\Pr(\chi^2_\nu \le \chi^2_{\nu,q}) = q$

	Percentage points of the F-distribution $q = 0.95$																		
	Numerator degrees of freedom ν_1																		
ν_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.54	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.50	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
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22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
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60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.86	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Table 2: **The percentage point $F_{\nu_1,\nu_2;q}$ of the F distributed random variable F_{ν_1,ν_2} with ν_1 and ν_2 degrees of freedom is defined as $\Pr(F_{\nu_1,\nu_2} \leq F_{\nu_1,\nu_2;q}) = q$.**

END OF EXAMINATION PAPER

Statistical tables are attached

Two Hours

UNIVERSITY OF MANCHESTER

01 June 2018

14.00 – 16.00

Medical Statistics

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i) What was the outcome in the trial? [1 mark]

The clinician states that they are only going to report the analysis corresponding to the outcome at 1 month post-randomisation, because this was the only result that was statistically significant.

ii) Explain why the clinician's approach may introduce bias. [2 marks]

iii) State *two* things that the clinician could have done (other than seeking advice from a statistician) to prevent this bias. [2 marks]

[Total 5 marks]

A2. A randomised controlled trial compared a new method of surgery to improve eyesight with the usual procedure. The outcome was the score on a sight test, measured by the number of letters that the patient could read. Thirty participants were allocated to each group. In the intervention group, the mean score on the sight test was 21 letters, with a standard deviation of 6, compared to a mean of 17 letters, with a standard deviation of 5, in the control group. An improvement of 5 letters is deemed to be a clinically important effect.

i) Calculate the pooled standard deviation for the trial. [1 mark]

ii) Stating your null and alternative hypotheses clearly, conduct a suitable hypothesis test at a 5% significance level. What are the assumptions of the test (you do not have to verify them)? [5 marks]

iii) Calculate a 95% confidence interval for your estimate of the treatment effect. [3 marks]

iv) Using your results from part ii) and iii) what do you conclude about the effect of the new surgery compared to the usual procedure? [3 marks]

[Total 12 marks]

A3. A randomised placebo-controlled trial is being planned to assess the effectiveness of a new drug for couples struggling to get pregnant. You have been asked to calculate the sample size for the trial, which will have an equal allocation ratio. The pregnancy rate in the control group is believed to be 20%, and the investigators would like to achieve 90% power to detect an improvement to 35% in the treatment group, using a test of proportions at a 5% significance level.

i) Using the fact that

$$n = \frac{(z_{\frac{\alpha}{2}}\sqrt{2\pi(1-\pi)} + z_{\beta}\sqrt{\pi_T(1-\pi_T) + \pi_C(1-\pi_C)}}{\tau^2}$$

and assuming there will be no drop out, calculate the total number of participants that the investigators should randomise to the trial. You do not need to comment on the assumptions underlying the calculation. [2 marks]

The investigators anticipate that, of all patients screened to be in the study, 70% will be eligible to be included in the trial. Moreover, of all those eligible, only 50% will actually consent to be included. Finally, it is anticipated that 20% of the randomised participants will drop out before completing the trial.

ii) In light of this information, how many patients should be randomised? [1 mark]

iii) How many patients should be screened? [2 marks]

Two years later, the trial is completed. The numbers of pregnancies in each group are shown below.

	Drug	Placebo	Total
Pregnant	46	37	83
Not Pregnant	114	126	240
Total	160	163	323

iv) Calculate the rate (risk) difference for the effect of the drug on pregnancy [1 mark]

v) Calculate an odds ratio for the effect of the drug on pregnancy [1 mark]

vi) Using your answer to part v) estimate the probability of pregnancy when taking the drug in a population with a 10% chance of getting pregnant when taking a placebo

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A4. You are approached by a researcher who needs help with the treatment allocation for their randomised controlled trial. They state that duration of disease (measured in years) and sex are important prognostic factors.

i) Explain how you would carry out stratified randomisation in this case. [5 marks]

ii) An alternative method of treatment allocation is minimisation. Give one advantage and one disadvantage of stratified randomisation compared to minimisation. [2 marks]

[Total 7 marks]

A5. The table shows baseline data from a randomised controlled trial. The investigator has performed statistical tests comparing the baseline characteristics of the two study groups.

Characteristic	Treatment (n = 40)	Control (n = 40)	P-value
Age (years): Mean (SD)	46 (8)	43 (8)	0.049
Sex , female: n (%)	21 (53%)	17 (43%)	0.184
Body Mass Index: Mean (SD)	27.4 (8)	27.2 (6)	0.456

On the basis of these analyses, the investigator decides to control for age in the analysis of the primary outcome in the trial.

Explain what is wrong with the investigator's use of baseline data from the trial. [4 marks]

[Total 4 marks]

B6. In a randomised control trial comparing a treatment (T) to a control (C) suppose *Y* is a Normally distributed outcome variable and *X* is the same variable measured at baseline. Let τ be the treatment effect such that:

$Y = \mu_y + \epsilon_y$ and $X = \mu_x + \epsilon_x$ for treatment C

$Y = \mu_y + \tau + \epsilon_y$ and $X = \mu_x + \epsilon_x$ for treatment T

With $E[\epsilon_x] = E[\epsilon_y] = 0,\ Var[\epsilon_y] = \sigma_y^2,\ Var[\epsilon_x] = \sigma_x^2,\ and\ Cov[\epsilon_x, \epsilon_y] = \sigma_{xy}$

Suppose that $\bar{x}_T,\ \bar{x}_C, \bar{y}_T,$ and $\bar{y}_C$ are the sample means of *X* and *Y* for each treatment.

Define $\hat{\tau}(\theta) = (\bar{y}_T - \theta\, \bar{x}_T) - (\bar{y}_C - \theta\, \bar{x}_C)$.

i) Show that $E[\hat{\tau}(\theta)] = \tau$. [4 marks]

ii) Show that $Var[\hat{\tau}(\theta)] = \lambda^2(\sigma_y^2 + \theta^2\sigma_x^2 - 2\theta\sigma_{xy})$ where $= \sqrt{\frac{1}{n_T} + \frac{1}{n_C}}$, n_T is the numbers of patients allocated to the new treatment and n_C is the number allocated to the control treatment. [7 marks]

iii) Show that $Var[\hat{\tau}(\theta)]$ has a minimum when $\theta = \frac{\sigma_{xy}}{\sigma_x^2}$. [4 marks]

iv) Three statistical analyses that could be used to estimate and test the treatment effect in this setting are:

a) A t-test using the outcome variable *Y*
b) A t-test based on the change score *Y-X*
c) A linear model (ANCOVA) of the outcome variable *Y* with treatment group and *X* as covariates.

In light of the results given in i) and iii), discuss the choice between the three options. [3 marks]

v) A common approach in randomised trials is to conduct tests of within group change. Why is this not appropriate? [2 marks]

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B7 In a parallel group *equivalence* trial a new treatment T is being compared with a control treatment C on a continuous outcome measure Y. Let $\bar{y}_T$, $\bar{y}_C$, μ_T and μ_C be the sample and population means of Y for each treatment, n_T and n_C be the sample sizes, and s be the common within-group sample standard deviation of Y. Define the treatment effect $\tau = \mu_T - \mu_C$.

(i) Explain why a significance test of the hypothesis $H_0 : \tau = 0$ vs $H_1 : \tau \neq 0$ would be inappropriate in an equivalence trial. [3marks]

(ii) Suppose that the null hypothesis $H_0 : |\mu_T - \mu_C| \geq \delta_E$ is rejected if the (1-2α)*100% confidence interval, given by $\bar{y}_T - \bar{y}_C \pm z_\alpha \lambda s$ with $\lambda = \sqrt{1/n_T + 1/n_C}$, is within the interval ($-\delta_E$, $+\delta_E$). Show that

$$\Pr[\text{Reject } H_0|\tau] = \Phi\left(\frac{(\delta_E - \tau)}{s\lambda} - z_\alpha\right) - \Phi\left(-\frac{(\delta_E + \tau)}{s\lambda} + z_\alpha\right)$$

where Φ is the cumulative distribution function of the standard normal distribution, and z_α is the value defined as $Pr(Z \leq z_\alpha) = \alpha$, for a Standard Normal random variable Z.

[6 marks]

(iii) Show that Pr[Reject $H_0|\tau$] has a maximum under H_0 when $\tau = -\delta_E$ or $\tau = \delta_E$. Hence show that this procedure has a type I error $\leq \alpha$.

[7 marks]

(i) Twenty-five patients are allocated to each group in the trial. The table below summarizes the outcome Y for each group. A difference of 5 was considered to be clinically important.

Treatment Group			*Control Group*		
Mean	*S.D.*	*N*	*Mean*	*S.D.*	*N*
135.5	14.7	25	136.1	15.1	25

The 90% confidence interval for the mean difference between the *treatment* group and the *control* group is -7.9 to 9.1. What do you conclude about the equivalence of the treatments?

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B8 A meta-analysis of randomised controlled trials is conducted to investigate the effect of a cholesterol-reducing drug compared to a placebo for the treatment of high cholesterol. The treatment effect for each study ($\hat{\theta}_i$, i = 1,2,3) is the difference in mean cholesterol between treatment and control. $Var\left[\hat{\theta}_i\right]$ is the sample variance estimate for the i^{th} study.

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$$Var\left[\hat{\theta}_{MV}\right] = \frac{1}{\sum_{i=1}^k \frac{1}{Var\left[\hat{\theta}_i\right]}}$$

[4 marks]

The table below summarizes the outcome of three trials in the meta-analysis

Study	*Difference in blood cholesterol,* $\hat{\theta}_i$	$\hat{Var}\left[\hat{\theta}_i\right]$
Dyson1996	-0.6	0.1
Thomson 2002	-0.7	0.5
Smith 1989	-0.6	0.2

(iii) Compute the minimum variance estimate of the overall treatment effect, $\hat{\theta}_{MV}$ and, assuming a normal distribution, determine its 95% confidence interval.

[7 marks]

(iv) What do you conclude from the meta-analysis regarding the effectiveness of the drug treatment? [1 mark]

(v) In the context of meta-analysis, explain what is meant by the term *publication bias*. [3 marks]

(vi) What type of plot might one use to investigate possible publication bias in a meta-analysis? [1 mark]

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[4 marks]

The table below summarizes the outcome of three trials in the meta-analysis

Study	*Difference in blood cholesterol,* $\hat{\theta}_i$	$\hat{Var}\left[\hat{\theta}_i\right]$
Dyson1996	-0.6	0.1
Thomson 2002	-0.7	0.5
Smith 1989	-0.6	0.2

(iii) Compute the minimum variance estimate of the overall treatment effect, $\hat{\theta}_{MV}$ and, assuming a normal distribution, determine its 95% confidence interval.

[7 marks]

(iv) What do you conclude from the meta-analysis regarding the effectiveness of the drug treatment? [1 mark]

(v) In the context of meta-analysis, explain what is meant by the term *publication bias*. [3 marks]

(vi) What type of plot might one use to investigate possible publication bias in a meta-analysis? [1 mark]

[Total 20 marks]

END OF EXAMINATION PAPER

2 hours

Tables of the cumulative distribution function are provided.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL MODELLING IN FINANCE

05 June 2018

14.00 - 16.00

Answer **all** 4 questions in **Section A** (60 marks in all) and
2 of the 3 questions in **Section B** (20 marks each). If all three questions from Section B are attempted then credit will be given for the two best answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** 4 questions

A1. An asset pays a dividend of £1 at the end of each of the next 3 years. The current asset price is £200 and the constant risk-free interest rate is 5% per annum with continuous compounding. Suppose that a zero value forward contract on this asset with delivery date in 4 years and delivery price of £245 is available. Does this situation create any arbitrage opportunities, and if so, how could an investor exploit this?

[15 marks]

A2.

(i) Consider a contract with value function $V(S,t)$ which satisfies the Black-Scholes equation, namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

in the usual notation, with constant r and σ, with a payoff at expiry $t = T$ given by

$$V(S,T) = S^n,$$

where $n \geq 0$. By seeking a solution of the form $V(S,t) = S^n f(t)$, determine $V(S,t)$.

(ii) By using (i) above, find (in the form of an infinite series), the value for a contract with payoff

$$V(S,T) = \exp(S)$$

[15 marks]

A3. Draw the expiry payoff diagrams for each of the following portfolios (be sure to label axes and indicate gradients on the diagrams):

(i) Short two shares, short two calls, both with an exercise price X.

(ii) Short one share, long three puts all with exercise price X_1, and short one call with exercise price X_2. Consider each of the cases $X_1 > X_2$, $X_1 = X_2$, $X_1 < X_2$.

[15 marks]

A4. Consider a general stochastic process with the dynamics

$$dS = A(S,t)dt + B(S,t)dW,$$

where W is a Brownian motion.

(i) Let the function $f(S,t)$ have a continuous second derivative with respect to S, and a continuous first derivative with respect to t. Show that

$$df = \left[A(S,t)\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}B^2(S,t)\frac{\partial^2 f}{\partial S^2} \right] dt + B(S,t)\frac{\partial f}{\partial S}dW.$$

(ii) Increments in S, a share price, are described by the following variant on geometric Brownian motion:

$$dS = \mu S dt + \sigma S^\beta dW,$$

where μ is the drift (constant) and σ is the volatility (constant), and β is also constant. Using the result from part (i), find the dynamics dV followed by the option price $V(S,t)$.

(iii) Consider a portfolio comprising one option $V(S,t)$ and $-\Delta$ of the underlying asset, S, which follows the dynamics given in (ii) above. Show that by choosing Δ such that

$$\Delta = \frac{\partial V}{\partial S},$$

and using arbitrage arguments, leads to the following variant of the Black-Scholes partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\beta}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where r is the risk-free interest rate, assumed constant.

[15 marks]

SECTION B

Answer **2** of the 3 questions

B5.

(i) Starting with the Black-Scholes equation for an option on a non-dividend paying asset, in the usual notation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where the volatility σ and interest rate r are constants, if $S = \exp F$, show that $V(F,t)$ satisfies

$$\frac{\partial V}{\partial t} + c\frac{\partial^2 V}{\partial F^2} + d\frac{\partial V}{\partial F} - rV = 0,$$

where you are to determine c and d.

(ii) By writing $V = g_1(t) + g_2(t)F$, determine the value of an option whose payoff at time $t = T$ is $V(S,T) = A\log S + B$, where A and B are constants.

[20 marks]

B6. You may assume that the Black-Scholes equation (in the usual notation) for a put option on a non-dividend paying asset is

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0,$$

where the interest rate r and the volatility σ are both constant.

(i) A binary put option $P_b(S,t)$ has the payoff

$$P_b(S,T) = \begin{cases} 0 & \text{if} \quad S > X \\ K & \text{if} \quad \text{S} < \text{X} \end{cases}.$$

Show that its value is given by

$$P_b(S,t) = K\frac{\partial P}{\partial X},$$

where $P(S,t;X)$ is the value of a European put option with strike X.

(ii) Given that the value of a European put option is

$$P(S,t;X) = Xe^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where $N(x)$ is the cumulative distribution function of the standard normal distribution, namely

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy,$$

$$d_1 = \frac{\log\frac{S}{X} + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

show that the price of a European binary put option is

$$P_b(S,t;X) = Ke^{-r(T-t)}N(-d_2).$$

(iii) Consider a stock whose price (6 months from the expiration of a binary put option) today is £10.50. The option pays £0.5 if the stock value at expiry is less than £11, and nothing if the stock value at expiry is greater than £11. The risk-free interest rate is 4% per annum (fixed) and the volatility (constant) is 15% per $(\text{annum})^{\frac{1}{2}}$. Using the formula above for $P_b(S,t;X)$, determine its value today. The value of $N(x)$ may be determined by interpolation using the tables of the cumulative distribution function provided.

[20 marks]

B7.

Consider the Black-Scholes equation for a call option, C, where the underlying asset pays a continuous known constant dividend yield, D, namely

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S\frac{\partial C}{\partial S} - rC = 0,$$

in the usual notation, where r and σ are both constants.

(i) By neglecting the time derivative in the above equation, seek solutions of the form

$$C(S) = AS^\alpha,$$

where A is a constant. Show that the two possible values of α are

$$\alpha = \frac{1}{2}\left\{ -\frac{2}{\sigma^2}(r - D - \frac{1}{2}\sigma^2) \pm \left[\frac{4}{\sigma^4}(r - D - \frac{1}{2}\sigma^2)^2 + \frac{8r}{\sigma^2}\right]^{\frac{1}{2}} \right\}.$$

(ii) Consider a *perpetual* American call option (on an asset paying a continuous known constant dividend yield, D), i.e. an option with no expiry date but where it is possible at any point in time to exercise and receive $S - X$, where X is the exercise price. This can be valued using the time-independent solutions in (i) above. If the exercise boundary is denoted by S_f, write down the two conditions to be imposed on S_f; justify your answer.

(iii) State why only one of the solutions in (i) is retained, and which one it is.

(iv) Use the conditions at $S = S_f$ to determine both the value of the constant A and S_f itself. Hence determine the value of the perpetual call option $C(S)$.

(v) What happens to S_f as $D \to 0$; explain your answer.

[20 marks]

END OF EXAMINATION PAPER

Two hours

Statistical Tables provided

THE UNIVERSITY OF MANCHESTER

ACTUARIAL MODELS 2

Answer ALL FIVE questions.
The total number of marks in the paper is 90.

University approved calculators may be used.

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1. Consider two homogeneous groups of individuals, group a and group b who were observed for a period of 50 weeks. The hazard function of the survival time of an individual from group a, respectively b, is denoted by $\mu_a(t)$, respectively $\mu_b(t)$. You can assume here that the genuine survival times of the individuals are independent and that there is independent (right-)censoring and no truncation. The observed survival times (in weeks) of the two groups are as follows (here + denotes a censored survival time):

Group a	16	22	31+	33	50+	50+
Group b	2+	5	9	23	32+	35

(a) Assume that $\mu_a(t) = \mu_b(t)$ for all $t \geq 0$ and consider all the individuals from both groups together as one aggregate group of individuals. Use the data of all individuals combined to calculate the Kaplan-Meier estimate of the survival function of the survival time of an arbitrary individual from this aggregate group and plot your estimate (you should show the details of your calculations).

[9 marks]

(b) Explain why it is not sensible to ignore/discard the censored survival times and then only work with the genuine observed survival times for estimating the survival time distribution.

[3 marks]

(c) Assuming the same setting as in part (a), estimate, by using the Kaplan-Meier estimate, the probability that an arbitrary individual from the aggregate group fails before week 40 given that he/she has not failed in the first 20 weeks.

[4 marks]

(d) We now want to check whether the assumption that $\mu_a(t) = \mu_b(t)$ for all t in a suitable time region, is appropriate. To this end, we want to test the null hypothesis,

$$H_0 : \mu_a(t) = \mu_b(t) \quad \text{for all } t \in [0, \tau_R],$$

versus $H_A : \mu_a(t) \neq \mu_b(t)$ for some $t \in [0, \tau_R]$, where τ_R is the first time when there are no more individuals at risk in one of the two groups. The log-rank test says to reject H_0 at significance level α if $|Z/\sqrt{V}| > z_{\alpha/2}$, where

- $Z = \int_0^{\tau_R} \frac{R_s^a R_s^b}{R_s^a + R_s^b} \mathrm{d}\left(\widehat{A}_a(s) - \widehat{A}_b(s)\right)$ with R_t^j denoting the number of individuals at risk just before time t in group j and $\widehat{A}_j(t)$ denoting the Nelson-Aalen estimator at time t of the cumulative hazard function of an individual from group j;
- $V = \int_0^{\tau_R} \frac{R_s^a R_s^b}{(R_s^a + R_s^b)^2} \mathrm{d}\left(N_s^a + N_s^b\right)$ with N_t^j denoting the number of individuals in group j that are observed to have failed before or at time t;
- $z_\alpha > 0$ is such that $\Phi(z_\alpha) = 1 - \alpha$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Given the survival data at the beginning of the question, carry out the log-rank test at significance level 0.05 and report your conclusions.

[9 marks]

[Total 25 marks]

2. A study took place in a rehabilitation centre for treating a certain drug addiction. For this study the survival time of interest was the time from admission to the centre until recovery from the addiction. During the study some patients decided to leave the centre as they felt the treatment was not helping them to get rid of the addiction. The survival time recorded for those patients was the duration of stay in the rehabilitation centre and was treated as a censored survival time for the purpose of the study as these patients were not recovered from the addiction at the time they left the centre.

Explain briefly what independent censoring is and explain whether or not the particular type of censoring described above is likely to be independent or not.

[Total 5 marks]

3. Consider the following observed values of the possibly left-truncated and right-censored survival times of five independent homogeneous individuals, where + denotes a censored value:

$$(0,\, 5] \quad (0,\, 9+] \quad (1,\, 6] \quad (1,\, 8+] \quad (3,\, 8]$$

Assume the following parametric form of the common hazard function of the individuals:

$$\mu(t) = \mu(t;\beta) = \beta t^2,$$

with $\beta \geq 0$. Assume further that the censoring times of the individuals are (i) independent, (ii) are independent of the genuine failure times of the individuals and (iii) do not depend on the parameter β.

(a) Show that the log-likelihood function of the parameter β given the observations is given by

$$\ell(\beta) = C\beta^3 \mathrm{e}^{-688.33\beta},$$

where $C > 0$ does not depend on β. Here you should explain your steps.

[9 marks]

(b) Determine the maximum likelihood estimate (mle) of β.

[3 marks]

(c) Determine a 95% asymptotic confidence interval for β.

[4 marks]

(d) Use the mle of β to estimate the probability that an individual does not fail within the first 10 units of time.

[3 marks]

[Total 19 marks]

4. A 20-year long study has been conducted to see how the amount of bacon one eats affects mortality. The survival time of interest was the time from the start of the study until death. The study has recorded the survival times in years of 10 persons together with the average amount of bacon each person has eaten during the study. The data can be found below. Here + denotes a censored survival time.

survival time (in years)	4	4+	5+	7	10	11	14	19	20+	20+
average amount of bacon eaten (in kilos per year)	10	5	6	15	0	8	8	0	9	0

(a) Consider a Cox proportional hazards model for modelling the force of mortality over time with the average amount of bacon eaten acting as a covariate and with β denoting the regression coefficient corresponding to this covariate.

Derive an explicit expression in terms of β for the partial likelihood in this model given the above data.

[9 marks]

(b) The maximum likelihood estimate of β was found to be $\hat{\beta} = 0.107$. Estimate the relative risk of a bacon lover who eats on average 10 kilos of bacon per year in comparison to a vegetarian.

[3 marks]

(c) Suppose you want to test $H_0 : \beta = 0$ versus $H_A : \beta \neq 0$, where recall β is the regression coefficient.

(i) Describe how one performs the likelihood ratio test for this particular hypothesis test. Note that you do not have to carry out this test, i.e. you do not have to compute a numerical value for the test statistic.

(ii) For both possible outcomes of this test explain what you can conclude from it within the context of this particular study.

[5 marks]

(d) Critics of the study have complained that the study does not take into account the age of the persons involved and whether or not they are smokers. Construct a new Cox proportional hazards model which takes into account these two characteristics as well.

[4 marks]

[Total 21 marks]

5. Assume a proportional frailty model for the mortality of a certain population where the frailty random variable Z has a probability density function given by

$$f_Z(z) = z\mathrm{e}^{-z}, \quad z > 0,$$

and the individual hazard function is given by

$$\mu_1(t|Z) = \begin{cases} Z & \text{if } t \in [0,5), \\ 3Z & \text{if } t \geq 5. \end{cases}$$

(a) Show that the population hazard function is given by

$$\mu_1(t) = \begin{cases} \frac{2}{t+1} & \text{if } t \in [0,5), \\ \frac{2}{t-3} & \text{if } t \geq 5. \end{cases}$$

[7 marks]

(b) Within this particular proportional hazards model, show that the expected value of the frailty level corresponding to the people who are still alive at time t is a decreasing function of t.

[5 marks]

(c) Explain in words why the phenomenon described in part (b) should appear not only in this particular example but in any proportional frailty model.

[3 marks]

(d) Consider a second population whose population hazard function is given by

$$\mu_2(t) = \frac{4t}{2t^2+1}, \quad t \geq 0.$$

Explain briefly what a crossover phenomenon is for the population hazard ratio of two populations and investigate whether or not there is such a phenomenon in this particular case.

[5 marks]

[Total 20 marks]

END OF EXAMINATION PAPER

2 Hours

UNIVERSITY OF MANCHESTER

CONTINGENCIES 2

18 May 2018

14.00 - 16.00

Answer ALL EIGHT questions
The total number of marks in the paper is 100.

University approved calculators may be used.

Please note the following before you get started.

- You can find a formula sheet on the last page of this exam paper.
- The marking is based on your answer booklet only, so make sure to write down your working there.

1.

male age x	q_x	female age y	q_y
30	0.0100	30	0.0080
31	0.0105	31	0.0085
32	0.0110	32	0.0090
33	0.0115	33	0.0095
34	0.0120	34	0.0100

a) Using the above extract from a simplified mortality table, express the probabilities below in terms of single life probabilities and then calculate their value (in all cases assume the first life is male and the second is female):

i) $p_{31:30}$

[2 marks]

ii) $p_{\overline{31:30}}$

[2 marks]

iii) ${}_3q_{31:30}$

[3 marks]

iv) ${}_{2|}q_{31:30}$

[2 marks]

b) Using the same mortality table extract (again assume the first life is male and the second is female) and assuming an interest rate of 3% p.a., calculate:

$\ddot{a}_{31:30:\overline{3}|}$

[4 marks]

[Total 13 marks]

2. An insurance company issues a policy to two female lives both aged 60 exact which pays a sum assured of £50,000 at the end of the year in which the first death occurs. Premiums are payable annually in advance until the first death occurs but with a maximum period of 20 years. Assuming interest at 4% p.a. and mortality of PFA92C20 for both lives, calculate:

a) The net premium for the policy.

[8 marks]

b) The variance of the death benefits payable.

[4 marks]

[Total 12 marks]

3. Consider a 3-state sickness model with states H (healthy), S (sick) and D (dead). You are given the following dependent probabilities:

Age exact x	P_x^{HS}	${}_2P_x^{HS}$	${}_3P_x^{HS}$
40	0.01	0.02	0.035

a) Write down an integral expression for the expected present value (EPV) of a sickness benefit of £5,000 p.a. payable while continuously sick for a healthy life aged 40 exact. Assume the sickness policy has a term of 3 years.

[3 marks]

b) Using the probability data you are given and assuming an interest rate of 5 % p.a., calculate an estimated value for the EPV using an approximate integration method.

[5 marks]

[Total 8 marks]

4. Consider a 3-state multiple decrement model with states a (active), r (retired) and d (dead), where the dependent probabilities are defined by the following multiple decrement table.

Age exact x	$(al)_x$	$(ad)_x^r$	$(ad)_x^d$
60	10,000	500	100
61	9,400	300	100
62	9,000	200	110

a) Calculate the independent probabilities of dying and of retiring when aged 61.

[10 marks]

b) State two assumptions necessary to make your calculations.

[4 marks]

[Total 14 marks]

5. A pension scheme provides a lump sum benefit on ill health retirement, calculated as follows:

$$\text{Lump sum payable on retirement due to ill health at age } x = 0.20 \times TS$$

where TS = total salary earned over their working life through to age x.

So, for example, if an individual works for three years, earns £20,000 in year 1, £21,000 in year 2 and £24,000 in year 3 and then retires in ill health, then the lump sum payable is £13,000.

Now, consider a person aged 63 exact who joined the pension scheme 20 years ago. You are told that the person will earn a salary of £30,000 in the following year and has earned a total of £400,000 over the 20 years that he/she has been in the pension scheme. The person's next salary increase will be at age 64 exact.

a) Write down a vector showing an estimate of the amount of lump sum payable on ill health retirement between age 63 exact and 65 exact, the Scheme's normal retirement date. Use the salary scale provided in the R data and functions provided.

[10 marks]

b) Using the data and functions provided in R, write down the vector for the expected present value (EPV) of a lump sum payment of £1 on ill health retirement between the age of 63 exact and age 65 exact. Use an interest rate of 4% p.a.

[5 marks]

c) Then calculate the EPV of the benefits calculated in part a).

[2 marks]

[Total 17 marks]

6. An insurer writes a 10 year unit linked endowment policy for a life aged 50. Level premiums are paid yearly in advance. Benefits are paid at the end of the year.

a) Construct a table setting out the development of the Unit Fund during the **second** year of the policy and then a table showing the cash flows for the second year for the Non-Unit fund. Briefly show your workings so that the derivation of each item in the tables can be seen and state what the profit for the year is.Details are as follows:

- Premium = £ 3,000 p.a..
- Allocation is 90% in each year.
- Bid offer spread = 4%.
- Management charges 1.0% deducted from the unit fund at the end of the year, before any claim payments are made.
- Renewal expenses are £ 80 on payment of the annual premium.
- Dependent probability of death = 0.01 in each year.
- The death benefit is equal to the Unit Fund value at the end of the year but subject to a minimum sum assured of £10,000.
- The value of the Unit Fund at the end of year 1 is £1,500 and the Non-Unit Fund shows a loss of £150.
- Assume a rate of return of 4% p.a. on the Unit Fund and 2% p.a. on the Non-Unit Fund.

[9 marks]

b) Calculate the number of years after the end of year 2 before the Unit Fund exceeds the minimum payment of £10,000.

[4 marks]

[Total 13 marks]

7. You are given the following profit vector for a policy taken out for a life aged 30 exact

$$(100, -70, 60, -90, 100)$$

You are also given the following mortality table:

Age x	q_x
30	0.0100
31	0.0105
32	0.0110
33	0.0115
34	0.0120

a) Assuming an interest rate on reserves of 3% p.a. and using the above mortality table extract, zeroise the profit vector for a life aged 30 exact.

[8 marks]

b) Calculate the Profit Signature.

[4 marks]

c) Using an assumption of 5% p.a. for the risk discount rate, calculate the Net Present Value of the policy.

[4 marks]

[Total 16 marks]

8. You have the following information for a subgroup of people compared to that for the whole population.

Age	Number of lives in the subgroup	Number of lives in the whole population	Number of deaths in the whole population
50	16,000	1,000,000	2,500
51	18,000	900,000	2,700
52	20,000	800,000	2,900

You are also given that the total number of deaths for the subgroup as a whole is 200 and for the whole population is 8,100 (both these totals are for the three ages shown).

Calculate:

a) The crude mortality rate for both the sub group and the whole population

[2 marks]

b) The indirectly standardised mortality rate for the sub group.

[5 marks]

[Total 7 marks]

Formula sheet Contingencies 2

1. Compound interest

i is the interest rate per annum, $v = (1+i)^{-1}$ and $\delta = \log(1+i)$

$a_{\overline{n}|} = v + v^2 + \ldots + v^n$ is the present value of an annuity certain in arrears with term n years

2. Mortality tables and laws

l_x is the number of lives at age x

$d_x = l_x - l_{x+1}$ is the number of lives who die between ages x and $x+1$

${}_tp_x$ is the probability that a life aged x will survive at least t years and ${}_tq_x$ is the probability that a life aged x will die within t years

${}_{m|n}q_x$ is the probability that a life aged x will die between ages $x+m$ and $x+m+n$

K_x is the curtate future lifetime for a life aged x; $e_x = \mathbb{E}[K_x]$

T_x is the complete (exact) future lifetime for a life aged x; $\mathring{e}_x = \mathbb{E}[T_x]$

μ_x is the force of mortality at age x:

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s} \cdot ds\right)$$

3. Assurances

A_x is the EPV of a whole life assurance that pays at the end of the year of death of a life aged x

$\bar{A}_x$ is the EPV of a whole life assurance that pays at the moment of death of a life aged x

$$\mathbb{E}\left[v^{K_x+1}\right] = A_x;\ \mathrm{Var}\left(v^{K_x+1}\right) = A_x^{i^*} - (A_x)^2$$

$$\mathbb{E}\left[v^{T_x}\right] = \bar{A}_x;\ \mathrm{Var}\left(v^{T_x}\right) = \bar{A}_x^{i^*} - (\bar{A}_x)^2$$

(where $i^* = (1+i)^2 - 1$. Similar relationships hold for other assurances.

A_{xy} is the EPV of a whole life assurance that pays at the end of the year in which the first death of two lives aged x and y occurs.

$A_{\overline{xy}}$ is the EPV of a whole life assurance that pays at the end of the year in which the second death of two lives aged x and y occurs.

A^1_{xy} is the EPV of a whole life assurance that pays at the end of the year in which a life aged x dies assuming that a life aged y remains alive at that time.

A^2_{xy} is the EPV of a whole life assurance that pays at the end of the year in which a life aged x dies assuming that a life aged y is already dead at that time.

4. Annuities

a_x is the EPV of an annuity for life in arrears for a life aged x

$\ddot{a}_x = 1 + a_x$ is the EPV of an annuity for life in advance for a life aged x

$\bar{a}_x$ is the EPV of a continuously paying annuity for life for a life aged x

$$\mathbb{E}[\ddot{a}_{\overline{K_x+1}|}] = \ddot{a}_x;\ \mathrm{Var}(\ddot{a}_{\overline{K_x+1}|}) = \frac{A_x^{i^*} - (A_x)^2}{(1-v)^2}$$

$$\mathbb{E}[\bar{a}_{\overline{T_x}|}] = \bar{a}_x;\ \mathrm{Var}(\bar{a}_{\overline{T_x}|}) = \frac{\bar{A}_x^{i^*} - (\bar{A}_x)^2}{\log(v)^2}$$

(where $i^* = (1+i)^2 - 1$. Similar relationships hold for temporary annuities with status ${}_{x:\overline{n}|}$)

a_{xy} is the EPV of an annuity for the joint lifetime of two lives aged x and y, paid in arrears.

$a_{\overline{xy}}$ is the EPV of an annuity payable in arrears so long as at least one of two lives aged x and y is alive.

$a_{x|y}$ is the EPV of a reversionary annuity payable to a life aged y after the death of a life aged x, paid in arrears.

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m};$$

$$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - v^n\, {}_np_x\right).$$

$$A_x = 1 - (1-v)\ddot{a}_x;\quad \bar{A}_x = 1 - \delta\bar{a}_x$$

END OF EXAMINATION PAPER

Two hours

UNIVERSITY OF MANCHESTER

RISK THEORY

1 June 2018

14:00 – 16:00

Answer ALL FIVE questions
The total number of marks in the paper is 90.

University approved calculators may be used

Note: attached to this exam paper you find the list of distributions that you have previously used during the course.

1. For a portfolio of products, an insurance company models the evolution of capital levels by means of a Cramér-Lundberg capital process denoted $(U_t)_{t\geq 0}$. That is, the amount of capital at time t is given by

$$U_t = u + ct - \sum_{k=1}^{N_t} X_k \quad \text{for } t \geq 0,$$

where time t is measured in years, $u \geq 0$ is the initial capital, $c > 0$ the premium rate, $(N_t)_{t\geq 0}$ a Poisson process with intensity $\lambda > 0$, and finally $\{X_k\}_{k=1,2,\ldots}$ is a sequence of iid positive random variables representing the claim sizes.

For this portfolio of products we know that $c = 3$, $\lambda = 1$ and that the claim sizes follow a Gamma$(2, 1)$ distribution.

(a) What is the expected number of claims that arrive during the first two years?

[3 marks]

(b) Compute the expected profit (i.e. change in amount of capital) for the first two years.

[4 marks]

(c) Determine Lundberg's coefficient R for this capital process.

[7 marks]

(d) Using part (c), give an upper bound for the ruin probability $\psi(u)$ for all $u \geq 0$. Your upper bound should be strictly less than 1 for all $u > 0$. (Note: if you could not do part (c), then use $R = 1$ for the sake of this question).

[3 marks]

(e) Suppose now that the initial capital is $u = 0$. The insurance company is considering to buy reinsurance for this portfolio, with the aim of minimising the ruin probability. The reinsurance contract on offer is of the following type. For a retention level $a \in [0, 1]$, a portion $1 - a^2$ of each claim is paid by the reinsurer, at a cost for the insurer of $5(1 - a)$ per year. Note that $a = 0$ corresponds to full reinsurance, while $a = 1$ corresponds to no reinsurance at all. Determine the optimal retention level for the insurance company.

[8 marks]

[Total 25 marks]

2. In this question we consider a second insurance company, that also uses a Cramér-Lundberg capital process as defined in question 1 above to model the evolution of their capital levels. Their initial capital is $u \geq 0$, their premium rate $c > 0$, the intensity of their Poisson process $\lambda > 0$, and their claim sizes are $\text{Exp}(\alpha)$ distributed for some $\alpha > 0$.

(a) Consider the following two probabilities:

(i) the probability that ruin happens during the first year,

(ii) the probability that $U_1 < 0$.

Without doing any computations, which of these two is larger? Briefly motivate your answer.

[4 marks]

(b) Show that the probability that ruin happens due to the first incoming claim equals

$$e^{-\alpha u} \frac{\lambda}{\lambda + c\alpha}.$$

[8 marks]

(c) Determine the probability that neither the first nor the second incoming claim leads to ruin.

Hint: make clever use of part (b), and further you may without proof make use of the following identity:

$$\mathbb{E}\left[\mathbf{1}_{\{U_{J_1} \geq 0\}} e^{-\alpha U_{J_1}}\right] = \frac{\lambda\alpha}{\lambda + c\alpha}\left(u + \frac{c}{\lambda + c\alpha}\right) e^{-\alpha u},$$

where $\mathbf{1}_A$ as usual denotes the indicator of an event A.

[9 marks]

[Total 21 marks]

3. This question concerns Premium principles.

(a) State the Monotonicity property for a Premium principle π.

[3 marks]

(b) An insurance company holds a portfolio of 1,000 identical products. It is assumed that the present value of the outgo per product can be modelled by a Unif$(0, 100)$ distribution, and that the outgoes are independent. Furthermore the company uses the Expected Value principle to compute the premium per product. Using the Central Limit Theorem, how large should the loading factor be so that the probability that the company suffers a loss on this portfolio of products is at most 1%?

Hint: you may use that $\Phi^{-1}(0.99) \approx 2.33$, where Φ denotes the cdf of a $\mathcal{N}(0, 1)$ distribution.

[6 marks]

(c) Prove that the Exponential principle satisfies the No Rip-off property.

[5 marks]

[Total 14 marks]

4. Consider a Bayesian experiment in which $\Theta \sim \mathcal{N}(0, 1)$ and given $\Theta = \theta$, $X \sim \mathcal{N}(\theta, 1)$. A sample x from X is observed.

(a) Show that the posterior distribution is $\mathcal{N}(x/2, 1/2)$.

[6 marks]

(b) Determine Bayes estimate for θ under squared error loss.

[4 marks]

(c) Now determine Bayes estimate for θ under the loss function $l : \mathbb{R}^2 \to \mathbb{R}$ given by

$$l(\theta, a) = \begin{cases} 0 & \text{if } |\theta - a| \leq c \\ 1 & \text{if } |\theta - a| > c \end{cases}$$

for some constant $c > 0$.

Hint: you may find it helpful to sketch the graph of the pdf of the posterior distribution, and to use its well known properties as part of your answer.

[8 marks]

[Total 18 marks]

5. It is assumed that the number of times a particular breed of pet dog asks its owner to throw the ball for him to fetch during their daily walk is given by a Binomial$(10, \theta)$ distribution. Here $\theta \in (0, 1)$ is a parameter that measures overall well being and is unknown for a given dog. It is further assumed that the parameter values are distributed over all pet dogs of this breed according to the following pdf:

$$f(\theta) = \begin{cases} 12\theta(1-\theta)^2 & \text{if } \theta \in (0,1) \\ 0 & \text{if } \theta \notin (0,1). \end{cases}$$

You have been recording the number of times your pet dog Kevin asked you to throw the ball during the past 5 walks, which resulted in an average of 7.2. Find the Bühlmann credibility estimate for the number of times Kevin will ask you to throw the ball during the walk tomorrow.

[12 marks]

List of distributions

Discrete distributions

Name	Notation	Parameters	Pmf	Mean	Variance
Bernoulli	Bernoulli(p)	$p \in (0,1)$	$f(0) = 1-p,\ f(1) = p$	p	$p(1-p)$
Binomial	Binomial(n,p)	$n \in \{1,2,\ldots\}$ $p \in (0,1)$	$f(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0,1,\ldots,n$	np	$np(1-p)$
Geometric	Geometric(p)	$p \in (0,1)$	$f(k) = (1-p)^{k-1} p$ for $k = 1,2,\ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Poisson(λ)	$\lambda > 0$	$f(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ for $k = 0,1,\ldots$	λ	λ

Continuous distributions

Name	Notation	Parameters	Pdf	Mean	Variance
Beta	Beta(α,β)	$\alpha,\beta > 0$	$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$ for $x \in (0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Exponential	Exp(λ)	$\lambda > 0$	$f(x) = \lambda e^{-\lambda x}$ for $x > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma	Gamma(α,β)	$\alpha,\beta > 0$	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ for $x > 0$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
Normal	$\mathcal{N}(\mu,\sigma^2)$	$\mu \in \mathbb{R}$ $\sigma^2 > 0$	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-(x-\mu)^2/(2\sigma^2)\right)$ for $x \in \mathbb{R}$	μ	σ^2
Uniform	Unif(a,b)	$a,b \in \mathbb{R}$ $a < b$	$f(x) = \frac{1}{b-a}$ for $x \in (a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$

Notes

- For the pdf's above we understand $f(x) = 0$ for those $x \in \mathbb{R}$ for which no value is specified. For example, in the case of a Beta(α,β) distribution we have that $f(x) = 0$ for $x \le 0$ and for $x \ge 1$.
- Γ denotes the Gamma function which can be defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t \quad \text{for } x > 0.$$

  Note that for integers $x \in \{1,2,\ldots\}$ we have that $\Gamma(x) = (x-1)!$.

- B denotes the Beta function which can be defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, \mathrm{d}t \quad \text{for } x,y > 0.$$

  Note that for integers $x,y \in \{1,2,\ldots\}$ we have that

$$B(x,y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}.$$

END OF EXAMINATION PAPER