Two hours

THE UNIVERSITY OF MANCHESTER

FRACTAL GEOMETRY

26 May 2017

14:00

Answer **ALL FIVE** questions in **Section A** (40 marks in total) and
TWO of the **THREE** questions in **Section B** (50 marks in total).
If more than two questions from Section B are attempted then credit will be given for the best two answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL FIVE** questions

A1.

(a) Let (X, d) be a metric space and $E \subset X$. Let $s \geq 0, \delta > 0$.
Define what is meant by a δ-*cover* of E. Define $\mathcal{H}^s_\delta(E)$.

[3 marks]

(b) Define $\mathcal{H}^s(E)$, and justify why any limits that you used in your definition exist.

[4 marks]

(c) Let $\delta > 0$ and let $0 < s < t$. Show that

$$\mathcal{H}^s_\delta(E) \geq \delta^{s-t}\mathcal{H}^t_\delta(E).$$

Prove a relationship between $\mathcal{H}^s(E)$ and $\mathcal{H}^t(E)$ and hence define $\dim_H(E)$.

[6 marks]

A2. Prove that if $A \subset B$ then $\dim_H(A) \leq \dim_H(B)$.

[4 marks]

A3.

(a) What does it mean for an outer measure μ on a space (X, d) to be a *mass distribution*? (You do not need to define outer measure.)

[3 marks]

(b) State without proof the mass distribution principle.

[4 marks]

(c) Use the mass distribution principle to prove that $\dim_H([0, 1]^2) \geq 2$ (using the Euclidean metric on $\mathbb{R}^2$). You do not need to prove that the mass distribution you use is a mass distribution.

[5 marks]

A4. What does it mean for a map $f : (X, d) \to (X', d')$ to be Lipschitz? Prove that if $f : (X, d) \to (X', d')$ is Lipschitz and $E \subset X$ then

$$\dim_H(f(E)) \leq \dim_H(E).$$

[6 marks]

A5. Let $g : [0, 1] \to \mathbb{R}$ be a continuous function. By constructing an appropriate Lipschitz map, prove that

$$\dim_H\{(x, g(x)) : x \in [0, 1]\} \geq 1.$$

You should prove that the map you construct is Lipschitz.

[5 marks]

SECTION B

Answer **TWO** of the **THREE** questions

B6.

(a) Let $\phi_1, \phi_3, \phi_5 : \mathbb{R} \to \mathbb{R}$ be given by

$$\phi_i(x) = \frac{x}{10} + \frac{i}{10}$$

for $i \in \{1,3,5\}$. Let E satisfy $E = \bigcup_{i\in\{1,3,5\}} \phi_i(E)$.
Show that

$$\overline{\dim}_B(E) \leq \frac{\log 3}{\log 10}$$

by finding for each $\delta > 0$ a δ-cover of E.

[5 marks]

(b) With E as above, use the mass distribution principle to prove that $\mathcal{H}^s(E) > 0$ for $s = \frac{\log 3}{\log 10}$. Hence prove that $\dim_B(E)$ exists, stating clearly any relationships between $\dim_H, \underline{\dim}_B$ and $\overline{\dim}_B$ from the course that you use.

[7 marks]

(c) Describe the set E in terms of decimal expansions.

[2 marks]

(d) Let (X, d) be a metric space and $\{A_k : k \in \mathbb{N}\}$ be a collection of subsets of X. Let

$$A = \bigcup_{k=1}^{\infty} A_k$$

where $\dim_H(A_k) \leq d$ for each $k \in \mathbb{N}$. Show from the definition of Hausdorff dimension that $\dim_H(A) \leq d$.

[5 marks]

(e) Let A be the set of points for which, after some point in their decimal expansion, only use digits $\{1,3,5\}$, i.e.

$$A := \{0.x_1x_2x_3\cdots : \text{ there exists } k \text{ such that } x_n \in \{1,3,5\} \text{ for all } n \geq k\}.$$

Prove that $\dim_H(A) \leq \frac{\log 3}{\log 10}$.

Hint: Given a word $a_1 \cdots a_k$ with each $a_i \in \{1, \cdots, 9\}$, construct a Lipschitz map from the set E in part (a) to the set

$$A_{a_1\cdots a_k} := \{0.a_1a_2\cdots a_kx_1x_2\cdots : x_n \in \{1,3,5\} \text{ for all } n \in \mathbb{N}\}.$$

[6 marks]

B7.

(a) Let (X, d) be a metric space. Define the space $\mathcal{S}(X)$ and the Hausdorff metric d_H on $\mathcal{S}(X)$. State a theorem from the course about the existence of attractors of iterated function systems.

[6 marks]

(b) Given an iterated function system $\Phi = \{\phi_1, \cdots, \phi_k\}$ where each ϕ_i is a similarity on $(\mathbb{R}^n, d)$ with contraction ratio $c_i < 1$, state the definition of the similarity dimension $\dim_{sim}(\Phi)$. (Note: not all of the c_i are the same.)

[2 marks]

(c) Let $E = \bigcup_{i=1}^{k} \phi_i(E)$, where the maps ϕ_i on (X, d) are similarities with contraction ratios $c_i < 1$. Let $\Phi = \{\phi_1, \cdots, \phi_k\}$. Prove that $\dim_H(E) \leq \dim_{sim}(\Phi)$. You may use without proof that there exists a set $Y \subset X$ such that $\phi_i(Y) \subset Y$ for all $i \in \{1, \cdots, n\}$.

[8 marks]

(d) State the open set condition for an iterated function system Φ on (X, d).

[3 marks]

(e) Let $\Phi_1 := \{\phi_1, \cdots, \phi_k\}$ be an iterated function system on (X, d). Let

$$\Phi_2 := \{\phi_i \circ \phi_j : 1 \leq i \leq k, 1 \leq j \leq k\}.$$

Express the similarity dimension of Φ_2 in terms of the similarity dimension of Φ_1. Show that Φ_1 and Φ_2 have the same attractor.

[6 marks]

 P.T.O.

B8.

(a) Define upper box counting dimension for a bounded subset E of a metric space (X, d).

[3 marks]

(b) Let $E \subset \mathbb{R}$. Show that

$$\overline{\dim}_B(E \times E) \leq 2\overline{\dim}_B(E).$$

[6 marks]

(c) Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. Prove that

$$\dim_B(A) = \frac{1}{2}.$$

[8 marks]

(d) Give an example of a set C for which $\dim_H(C) = 0$ but $\dim_B(C) = 1$.

[2 marks]

(e) Let $G \subset [0, 1]$ satisfy $\dim_B(G) = \frac{1}{2}$. Prove that

$$\overline{\dim}_B(G \times [0, 1]) \leq \frac{3}{2}.$$

[6 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

TOPOLOGY

2 June 2017

9:45-11:45

Answer **ALL** FOUR questions in Section A (40 marks in total). Answer **THREE** of the FOUR questions in Section B (45 marks in total). If more than THREE questions from Section B are attempted then credit will be given for the best THREE answers.

University approved calculators may be used.

SECTION A

Answer **ALL** FOUR questions.

A1.

(a) Define what is meant by a *topology* on a set X.

(b) Define what is meant by saying that a function $f\colon X \to Y$ between topological spaces is *continuous*. Define what is meant by saying that f is a *homeomorphism*.

(c) Prove that the closed disc $\mathbb{D}^2 = \{x \in \mathbb{R}^2 \mid |x| \leqslant 1\}$ with the usual topology is homeomorphic to the hemisphere $\{x = (x_1, x_2, x_3) \in S^2 \mid x_3 \geqslant 0\}$.

[Here S^2 denotes the unit sphere $\{x \in \mathbb{R}^3 \mid |x| = 1\}$ with the usual topology.]

[10 marks]

A2.

(a) Define what is meant by saying that a topological space X is *path-connected*.

(b) What is meant by saying the path-connectedness is a *topological property*?

(c) Prove that path-connectedness is a topological property.

(d) Prove that

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid |(x_1, x_2 - 1)| \leqslant 1 \text{ or } |(x_1, x_2 + 1)| \leqslant 1\} \subset \mathbb{R}^2$$

(with the usual topology) is path-connected.

[10 marks]

A3.

(a) Define what is meant by saying that a topological space is *Hausdorff*.

(b) Determine whether the set $S = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a, c\}, \{b\}, \{a, b, c\}\}$ is Hausdorff.

(c) Suppose that X and Y are topological spaces. Define the *product topology* on the Cartesian product $X \times Y$. [It is not necessary to prove that this is a topology.]

(d) Prove that if $\Delta \subset X \times X$ is closed in the product topology, then X is Hausdorff.

[10 marks]

A4.

(a) Suppose that X_1 is a subspace of a topological space X. Define what is meant by saying that X_1 is a *retract* of X.

(b) Use the functorial properties of the fundamental group to prove that, if X_1 is a retract of X, then, for any $x_0 \in X_1$, the homomorphism induced by the inclusion map

$$i_*\colon \pi_1(X_1, x_0) \to \pi_1(X, x_0)$$

is injective.

(c) Hence prove that S^1 is not a retract of the closed disc $\mathbb{D}^2$.

[You may quote any fundamental groups that you need, without proof.]

[10 marks]

SECTION B

Answer **THREE** of the FOUR questions.

B5.

(a) Suppose that $q\colon X \to Y$ is a surjection from a topological space X to a set Y . Define the *quotient topology* on Y determined by q. State the *universal property* of the quotient topology.

(b) Suppose that $f\colon X \to Z$ is a continuous surjection from a compact topological space X to a Hausdorff topological space Z . Define an equivalence relation $\sim$ on X so that f induces a bijection $F\colon X/{\sim} \to Z$ from the identification space $X/{\sim}$ of this equivalence relation to Z. Prove that F is a homeomorphism. [State clearly any general results which you use.]

(c) Prove that the quotient space $[0,1] \times [0,1]/\sim$ with $(0,s) \sim (1,s)$ is homeomorphic to the cylinder $[0,1] \times S^1 \subset \mathbb{R}^3$.

[15 marks]

B6.

(a) Define what is meant by a *compact subset* of a topological space and by a *compact topological space.*

(b) Prove that, if $f\colon X \to Y$ is a continuous function of topological spaces and $K \subset X$ is a compact subset, then $f(K)$ is a compact subset of Y.

(c) Given a non-compact Hausdorff space (X, τ) consider the set $X^* = X \sqcup \{\infty\}$ and the topology

$$\tau^* = \tau \cup \{(X \setminus C) \cup \{\infty\} \mid C \subset X \text{ compact}\}.$$

Show that (X^*, τ^*) is compact. [It is not necessary to prove that τ^* is a topology.]

[15 marks]

B7.

(a) Prove that, if the product $\sigma_0 * \tau_0$ of two paths σ_0 and τ_0 in a topological space X is defined and the paths σ_1 and τ_1 are homotopic to σ_0 and τ_0 respectively, then the product $\sigma_1 * \tau_1$ is defined and is homotopic to $\sigma_0 * \tau_0$.

(b) Explain how a continuous function $f\colon X \to Y$ induces a homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(X, f(x_0))$. You should indicate why f_* is well-defined and why it is a homomorphism.

(c) Prove that, for topological spaces X and Y with points $x_0 \in X$, $y_0 \in Y$, there is an isomorphism of groups

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

[15 marks]

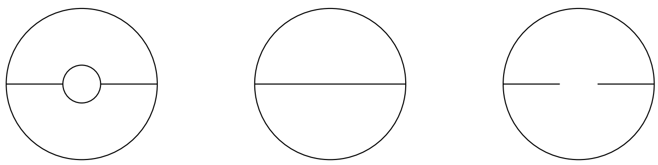
B8.

(a) Define what is meant by the *path-components* of a topological space. [You may assume the definition of a path and properties of paths.]

(b) Prove that a continuous map of topological spaces $f\colon X \to Y$ induces a map $f_*\colon \pi_0(X) \to \pi_0(Y)$ between the sets of path-components, taking care to prove that your function is well-defined. Prove that if f is a homeomorphism then f_* is a bijection.

(c) A pair of distinct points $\{p, q\}$ in a path-connected topological space X is called a cut-pair of type n when the subspace $X \setminus \{p, q\}$ has n path-components. Prove that a homeomorphism $f : X \to Y$ induces a bijection between the subsets of cut-pairs of type n for every $n \in \mathbb{N}$.

(d) Hence show, using cut-pairs of type 3 or otherwise, that no two of the following subspaces of $\mathbb{R}^2$ with the usual topology are homeomorphic. [Here, the radial line segments in (iii) are assumed to include the endpoints.]



(i) (ii) (iii)

[15 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY

1 June 2017

1400 - 1600

Answer **ALL FIVE** questions in Section A (50 marks in total).
Answer **TWO** of the THREE questions in Section B (30 marks in total).
If more than TWO questions in Section B are attempted, the credit will be given for the best TWO answers.

Electronic calculators may not be used.

Throughout the paper, where the index notation is used, the Einstein summation convention over repeated indices is applied if it is not explicitly stated otherwise.

SECTION A

Answer **ALL** FIVE questions

A1.

(a) Explain what is meant by saying that a Riemannian manifold is locally Euclidean.

(b) Consider the surface of the cylinder $x^2 + y^2 = R^2$ in $\mathbf{E}^3$

$$\mathbf{r}(h,\varphi)\colon \quad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} , \quad -\infty < h < \infty\,, \quad 0 \le \varphi < 2\pi\,.$$

Calculate the Riemannian metric on this surface induced by the Euclidean metric on $\mathbf{E}^3$.

Show that this surface is locally Euclidean.

Is this surface isometric to $\mathbf{E}^2$?

[10 marks]

A2.

(a) Explain what is meant by an affine connection on a manifold.

Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u,v) we have that $\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} = \frac{\partial}{\partial v}$.

Calculate the Christoffel symbols Γ^u_{uu} and Γ^v_{uu}.

(b) Let ∇ be an affine connection on a 2-dimensional manifold M such that, in local coordinates (x,y), all Christoffel symbols vanish except $\Gamma^x_{xx} = xy$, $\Gamma^y_{xx} = -1$ and $\Gamma^y_{yy} = y$. Show that for the vector field $\mathbf{X} = \partial_x + x\partial_y$,

$$\nabla_{\mathbf{X}}\mathbf{X} = xy\mathbf{X}\,.$$

[10 marks]

A3.

(a) Formulate the Levi-Civita Theorem. In particular, write down the Christoffel symbols Γ^i_{km} in terms of a Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

(b) Calculate the Christoffel symbol $\Gamma^r_{\varphi\varphi}$ of the metric $G = dr^2 + r^2 d\varphi^2$. (It is the Euclidean metric of $\mathbf{E}^2$ written in polar coordinates.)

[10 marks]

A4.

(a) Consider the sphere of radius R in Euclidean space $\mathbf{E}^3$

$$\mathbf{r}(\theta,\varphi)\colon \quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} .$$

Let $\mathbf{e}, \mathbf{f}$ be unit vectors in the directions of the vectors $\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta}$ and $\mathbf{r}_\varphi = \frac{\partial \mathbf{r}}{\partial \varphi}$, and let $\mathbf{n}$ be a unit normal vector to the sphere.
Express these vectors explicitly.

(b) For the obtained orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ calculate the 1-forms a, b and c in the derivation formula

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} .$$

[10 marks]

A5.

(a) State the relation between the Riemannian curvature tensor of the Levi-Civita connection of a surface in $\mathbf{E}^3$ and its Gaussian curvature.

(b) For the sphere of radius ρ in $\mathbf{E}^3$ calculate all components of the Riemannian curvature tensor R_{ikmn} in spherical coordinates θ, φ at the points of the equator ($\theta = \frac{\pi}{2}$).
Explain why the sphere is not locally Euclidean.

[10 marks]

SECTION B

Answer **TWO** of the THREE questions

B6.

(a) Let $G = dx^2 + 2p dx dy + dy^2$ be a Riemannian metric on $\mathbf{R}^2$, where x, y are standard Cartesian coordinates, and p is a real constant.

Show that $|p| < 1$.

(b) Calculate the area of the disc $x^2 + y^2 \leq R^2$ with respect to this metric.

[15 marks]

B7.

(a) Consider the upper half-plane $y > 0$ with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2}$$

(the Lobachevsky plane).

Consider in the Lobachevsky plane the domain D defined by

$$D = \{x, y: \quad x^2 + y^2 \geq 1\,,\ 0 \leq x \leq a\}\,,$$

where a is a parameter ($0 < a < 1$).

It is known that the area of the domain D is equal to $\frac{\pi}{6}$.

Find the value of the parameter a.

(b) Consider the points $A_t = (0, t)$ and $B_t = (a, t)$ on the vertical rays delimiting the domain D. Show that the distance between these points tends to 0 if $t \to \infty$.

[15 marks]

B8.

(a) Let M be a surface in $\mathbf{E}^3$ with the Riemannian metric induced by the Euclidean metric on $\mathbf{E}^3$.

Prove that the induced connection on the surface M is equal to the Levi-Civita connection of the induced Riemannian metric.

(b) Explain why the acceleration vector of an arbitrary parameterised geodesic on a surface M is orthogonal to the surface.

Deduce that all geodesics of the sphere $S^2\colon\ x^2 + y^2 + z^2 = 1$ are great circles.

Let C be a geodesic on the sphere S^2, which passes through the North pole $N = (0, 0, 1)$. Show that the image of the geodesic C under the stereographic projection on the plane $z = 0$ with respect to N is a straight line.

[15 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

CODING THEORY

22 May 2017

09:45 – 11:45

Answer ALL THREE questions in Section A (40 marks in total). Answer TWO of the THREE questions in Section B (40 marks in total). If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

University approved calculators may be used

SECTION A

Answer **ALL** questions in this section (40 marks in total)

A1. (a) Define what is meant by:
— a word $\underline{x}$ of length n in a finite alphabet F;
— a code C of length n in a finite alphabet F;
— the Hamming distance $d(\underline{x}, \underline{y})$ between two words $\underline{x}$, $\underline{y}$ of length n;
— the minimum distance $d(C)$ of a code C;
— a binary code.

(b) Define the trivial binary code of length n and the binary repetition code of length n. For each code, write down the parameters $[n, k, d]_q$ and explain briefly why k, d, q are as you state.

[10 marks]

A2. In this question, C is the binary linear code with generator matrix $\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$.

(a) List all the codewords of C and check that $d(C) = 3$. How many bit errors can C detect? How many bit errors can C correct? Write down the generator matrix of C in standard form.

(b) Explain what is meant by a binary symmetric channel with bit error rate p, $BSC(p)$. If $0 \leq i \leq n$, state without proof the formula for the probability that i errors occur in a message of length n, transmitted down this channel.

(c) If the code C is transmitted via $BSC(p)$, calculate $P_{\text{undetect}}(C)$, the probability that the received vector contains an error that is not detected. Set $p = 0.01$ and show that the probability that an unencoded two-bit message sent via $BSC(0.01)$ is received with one or more errors is at least 5,000 times greater than $P_{\text{undetect}}(C)$.

(d) Explain why two vectors of weight 1 cannot lie in the same coset of C. Conclude that there are five cosets with coset leaders of weight 1. You are given that coset leaders have weight at most 2; use this information to find $P_{\text{corr}}(C)$, the probability that a codeword transmitted via $BSC(p)$ is decoded correctly (leave your answer as a polynomial in p).

[20 marks]

A3.

(a) Define the ISBN-10 code. State, with justification, the number of codewords and the weight of the code. Show that when two unequal symbols are swapped in a codeword, the resulting word does not belong to ISBN-10. (You may use the fact that ISBN-10 is a linear code and you do not have to prove this.)

(b) Alice writes down all the codewords of ISBN-10 which are made up of symbols 0 and 1. Bob then counts all the symbols that were written down and claims that there are more 0s than 1s. Do you agree with Bob? Justify your answer.

[10 marks]

SECTION B

Answer **TWO** of the three questions in this section (40 marks in total).

If more than TWO questions from this section are attempted, then credit will be given for the best TWO answers.

B4.

(a) Define the inner product on $\mathbb{F}_q^n$. Define the dual code of a linear $[n,k,d]_q$-code C. Define a check matrix of C and state how many rows and columns it has.

(b) State and prove the Distance Theorem for linear codes.

(c) Show that the 7-ary code C with generator matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}$ is self-dual, that is, $C = C^{\perp}$. Find $d(C)$ and justify your answer.

(d) Prove: if $D \subseteq \mathbb{F}_7^n$ is a self-dual code, then $2 < d(D) < (n+3)/2$. Any results from the course can be used without particular comment.

[20 marks]

B5.

(a) State without proof the Hamming bound and the Singleton bound for codes in $\mathbb{F}_q^n$ of minimum distance d. Define what is meant by a perfect code.

(b) Write down the length, dimension and weight of a Hamming code $\mathrm{Ham}(r,q)$. Prove that Hamming codes are perfect.

(c) Describe the decoding algorithm for a Hamming code with a check matrix H over $\mathbb{F}_q$. Explain briefly why the algorithm is valid.

(d) Prove that all 2017-ary perfect codes with fewer than 2017^{2017} codewords attain the Singleton bound. You may use any results from the course without giving a proof.

[20 marks]

B6.

(a) What is a cyclic code? What is meant by an ideal of a commutative ring R? Briefly explain how a cyclic code of length n over $\mathbb{F}_q$ can be viewed as an ideal of the ring $R_n = \mathbb{F}_q[x]/(x^n - 1)$.

You are given a factorisation of the polynomial $x^7 - 1$ into irreducible polynomials over the field $\mathbb{F}_2$:

$$x^7 - 1 = (x+1)(x^3 + x + 1)(x^3 + x^2 + 1).$$

(b) Show that there exist two binary cyclic codes of length 7 and dimension 3. Write down the generator polynomial and a generator matrix for each of these codes. By comparing the two generator matrices, or otherwise, show that the two codes are linearly equivalent.

(c) Does either of the codes constructed in (b) contain vectors of weight 5? Justify your answer.

(d) Prove: if n is odd, a cyclic code $C \subseteq \mathbb{F}_2^n$ with $w(C) \geq 3$ has no codevectors of weight $n-2$.

[20 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO ALGEBRAIC GEOMETRY

6 June 2017
9.45 - 1145

Answer THREE of the FOUR questions.
If more than THREE questions are attempted, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text/transmit or receive information/display graphics.

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1. In this question K denotes an arbitrary field and unless specified otherwise, varieties are defined over K.

(a) (i) Define the *affine algebraic variety* $\mathcal{V}(J) \subseteq \mathbb{A}^n(K)$ determined by an ideal $J \lhd K[x_1, x_2, \ldots, x_n]$.

(ii) Define the *ideal* $\mathcal{I}(X)$ of a set $X \subseteq \mathbb{A}^n(K)$.

(iii) Let $J \lhd K[x_1, x_2, \ldots, x_n]$ be an ideal. Show that $\sqrt{J} \subseteq \mathcal{I}(\mathcal{V}(J))$ without any assumption on the field K.

(b) Let $J = \langle (x + y - z^2)^3, x^3 + xy^2 - xz^2 - 2xz + 2 \rangle \lhd \mathbb{C}[x, y, z]$.
Show that $\left.\dfrac{\partial h}{\partial x}\right|_{(1,0,1)} = 0$ for every $h \in J$. Deduce that $x + y - z^2 \in \mathcal{I}(\mathcal{V}(J))$, but $x + y - z^2 \notin J$.

(c) (i) Define what it means for an affine algebraic variety W to be *irreducible.*

(ii) Prove that an affine algebraic variety W is irreducible if and only if $\mathcal{I}(W)$ is a prime ideal.

(d) Find the irreducible components of the variety

$$W = \mathcal{V}(\langle xy^2 - xy, x^2 - y^2 - z^2 + 1 \rangle) \subset \mathbb{A}^3(\mathbb{C}).$$

(The irreducibility of certain varieties was proved in the course, you can use this if you identify correctly what kind of a variety each component is.)

[20 marks]

2. (a) Let K be a field. Let $V \subseteq \mathbb{A}^n(K)$ be an affine algebraic variety and let $P \in V$.

(i) Let $\mathbf{v} \in K^n$ be a non-zero vector. Let $\ell = \{P + t\mathbf{v} \mid t \in K\}$ be the line through P with direction vector $\mathbf{v}$. Explain what is meant by saying that ℓ is *tangent* to V at P and give the definition of the *tangent space* T_PV to V at P.

(ii) Let $f_1, f_2, \ldots, f_r$ be a generating set for $\mathcal{I}(V)$ and let

$$J_P = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}|_P & \frac{\partial f_1}{\partial x_2}|_P & \cdots & \frac{\partial f_1}{\partial x_n}|_P \\ \frac{\partial f_2}{\partial x_1}|_P & \frac{\partial f_2}{\partial x_2}|_P & \cdots & \frac{\partial f_2}{\partial x_n}|_P \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1}|_P & \frac{\partial f_r}{\partial x_2}|_P & \cdots & \frac{\partial f_r}{\partial x_n}|_P \end{pmatrix}$$

be the Jacobian matrix evaluated at $P \in V$. Show that the line $\ell = \{P + t\mathbf{v} \mid t \in K\}$ ($\mathbf{v} \in K^n \setminus \{\mathbf{0}\}$) is tangent to V at P if and only if $J_P\mathbf{v} = 0$.

(b) This part of the question is over $\mathbb{C}$. Let x, y be co-ordinates on $\mathbb{A}^2(\mathbb{C})$ and let t be the co-ordinate on $\mathbb{A}^1(\mathbb{C})$.
Let $C = \mathcal{V}(\langle x^5 - (y - x^2)^2 \rangle) \subset \mathbb{A}^2(\mathbb{C})$. (You may assume without proof that $x^5 - (y - x^2)^2$ is an irreducible polynomial, therefore C is an irreducible affine algebraic variety and $\mathcal{I}(C) = \langle x^5 - (y - x^2)^2 \rangle$.)

(i) Find the singular points of C or show that it has none.

(ii) Show that $\phi(t) = (t^2, t^5 + t^4)$ is a morphism $\mathbb{A}^1 \to C$.

(iii) Prove that the rational map $\psi : C \dashrightarrow \mathbb{A}^1$, $\psi(x, y) = \dfrac{y - x^2}{x^2}$ is an inverse to ϕ considered as a rational map.

(iv) Show that ϕ is a bijection, but C is not isomorphic to $\mathbb{A}^1(\mathbb{C})$.

[20 marks]

3. In this question K denotes an arbitrary field.

(a) Let n be a positive integer, let $a_{ij} \in K$ for $0 \le i \le n$, $0 \le j \le n$ and assume that the matrix $A = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{pmatrix}$ is invertible.

State the definition of the *projective transformation* $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ determined by A and prove that Φ is in fact an isomorphism $\mathbb{P}^n \to \mathbb{P}^n$.

(b) In this part of the question we identify $\mathbb{P}^1(K)$ with $K \cup \{\infty\}$ by identifying $(X_0 : X_1)$ with $X_0/X_1 \in K$ if $X_1 \neq 0$ and $(X_0 : 0)$ with ∞ as usual.
Let z_1, z_2, z_3, w_1, w_2, w_3 be elements of $K \cup \{\infty\}$ such that z_1, z_2, z_3 are pairwise distinct and w_1, w_2, w_3 are also pairwise distinct. Prove that there exists a projective transformation $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\phi(z_i) = w_i$ for $i = 1, 2, 3$.
(Recall that if z_1, z_2, z_3, $z_4 \in K \cup \{\infty\}$ with at least 3 of them pairwise distinct, their cross ratio is defined by the formula

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \in K \cup \{\infty\}$$

with appropriate rules for arithmetic involving ∞ and for division by 0.)

(c) Find $a, b, c, d \in \mathbb{C}$ such that the projective tranformation $\phi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, $\phi(z) = \dfrac{az + b}{cz + d}$ satisfies $\phi(-1) = 1$, $\phi(2) = 3$ and $\phi(4) = 11$. (You may use standard properties of the cross ratio if necessary.)

(d) Show that any isomorphism $\Phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a projective transformation.

[20 marks]

4. (a) Let K be a field. Let $E \subset \mathbb{P}^2(K)$ be a non-singular cubic curve and let O be a point of E. (You should not assume that the equation of E is in a special form.)

(i) Describe without proof how one can define the addition operation on the points of E to obtain a commutative group in which O is the zero element.

(ii) Let $A \in E$. Describe without proof how $-A$ can be calculated.

(iii) Assume now that O is an inflection point of E. Show that a point $P \in E$, $P \neq O$ satisfies $2P = O$ if and only if the tangent line to E at P passes through O.

(b) Let F be the elliptic curve given by the affine equation

$$y^2 = x^3 - 2x^2 - 8x + 16$$

over $\mathbb{C}$ together with the point $O = (0 : 1 : 0)$ in homogeneous co-ordinates, which is taken to be the zero element of the group law as usual.

(i) Let $A = (2, 0)$, $B = (-2, 4)$. Calculate $A + B$ and $2B$.

(ii) Find the points of order 2 of F.

[20 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

NUMBER THEORY

24 May 2017

09.45 – 11.45

Answer THREE of the FOUR questions.
(If more than three questions are attempted, then credit will be given for the best three answers.)

Electronic calculators are not permitted.

1.

(a) Give the definition of a finite simple continued fraction and explain what is meant by an infinite simple continued fraction.

(b) State a result describing exactly which real numbers have an infinite simple continued fraction representation and describe (without proof) an algorithm to find such a representation.

(c) (i) Find a particular solution $(x, y) \in \mathbb{Z}^2$ to the Diophantine equation

$$24x + 19y = 1.$$

(ii) Find the finite simple continued fraction expansion of $\frac{24}{19}$.

(d) Let $[x_0 : x_1, x_2, \ldots]$ be an infinite simple continued fraction. Define

$$\begin{aligned} p_{-2} &= 0, \quad p_{-1} = 1, \quad \text{and} \quad p_n = x_n p_{n-1} + p_{n-2} \text{ for all } n \geq 0, \\ q_{-2} &= 1, \quad q_{-1} = 0, \quad \text{and} \quad q_n = x_n q_{n-1} + q_{n-2} \text{ for all } n \geq 0. \end{aligned}$$

(i) For all $n \geq -2$, use induction to prove that

$$p_{n+1}q_n - p_n q_{n+1} = (-1)^n \qquad \text{and} \qquad p_{n+2}q_n - p_n q_{n+2} = (-1)^n x_{n+2}.$$

(ii) Deduce that

$$\frac{p_{2n+2}}{q_{2n+2}} > \frac{p_{2n}}{q_{2n}} \quad \text{and} \quad \frac{p_{2n+3}}{q_{2n+3}} < \frac{p_{2n+1}}{q_{2n+1}}$$

for all $n \geq 0$.

(iii) Deduce also that

$$\frac{p_{2n}}{q_{2n}} < \frac{p_{2n+1}}{q_{2n+1}}$$

for all $n \geq 0$.

[30 marks]

2.

(a) Give the definition of a periodic simple continued fraction and state (without proof) a theorem describing precisely which numbers can be represented by a periodic simple continued fraction.

(b) Let d be a non-square positive integer. Describe (without proof) how to find *all* integer solutions of the equation $x^2 - dy^2 = 1$.

(c) Find the infinite simple continued fraction expansion of $\sqrt{26}$ and compute the first four convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$ of $\sqrt{26}$.

(d) For each of the following Diophantine equations find at least one *positive* integer solution or explain why there are none, stating clearly any results you use in your answer:

(i) $x^2 - 26y^2 = 1$, (ii) $20x^2 - 26y^2 = 1$, (iii) $x^2 - 26y^2 = -1$, (iv) $4x^2 - 9y^2 = 1$.

(e) Prove that $[\overline{a : b}] = \dfrac{ab + \sqrt{ab(ab+4)}}{2b}$, for all integers $a, b \geq 1$.

[30 marks]

3.

Let n be a positive integer.

(a) Give the definition of a unit modulo n. For $a \in \mathbb{Z}$, show that $[a] \in \mathbb{Z}_n$ is a unit modulo n if and only if a and n are coprime.

(b) Give the definition of a primitive root modulo n.

(c) For which of the following integers n does there exist a primitive root modulo n? Find a primitive root if it exists; otherwise prove that there are none.

(i) 7, (ii) 8.

(d) State (without proof) Lagrange's polynomial congruence theorem and Fermat's little theorem.

(e) Find all solutions to each of the following polynomial congruences, stating clearly any results you use in your answer:

(i) $x^{14} - 1 \equiv 0 \bmod 13$, (ii) $169x^{102232} + 25x^4 \equiv 0 \bmod 13$, (iii) $x^{122} + 39x^{44} + 12 \equiv 0 \bmod 26$.

(f) Let $n = p_1^{a_1} \cdots p_k^{a_k}$, where $p_1, \ldots, p_k$ are distinct primes and $a_1, \ldots, a_k$ are positive integers.
(i) Define Euler's φ function and give (without proof) a formula for $\varphi(n)$ in terms of the p_i and the a_i.
(ii) Prove that $\varphi(n)$ is a prime number if and only if $n = 3, 4$, or 6.

[30 marks]

4. Let p be an odd prime.

(a) Give the definition of a quadratic residue modulo p and define the Legendre symbol $\left(\frac{a}{p}\right)$, where a is an integer.

(b) State (without proof) Euler's criterion and hence prove that

$$\left(\frac{-1}{p}\right) = \begin{cases} 1, & \text{if } p \equiv 1 \bmod 4, \\ -1, & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

(c) (i) State (without proof) the law of quadratic reciprocity.
(ii) Find all quadratic residues modulo 7.
(iii) Compute the Legendre symbols $\left(\frac{20}{7}\right)$ and $\left(\frac{15}{31}\right)$.

(d) Let $[a] \in U_p$ be a primitive root of the group U_p of units modulo p.

(i) For all integers i show that
$$\left(\frac{a^i}{p}\right) = (-1)^i.$$

(ii) Deduce that the number of quadratic residues in U_p equals $\frac{p-1}{2}$.

(e) Show that 3 is a quadratic residue modulo p if and only if $p \equiv \pm 1 \bmod 12$.

[30 marks]

END OF EXAMINATION PAPER

TWO HOURS

UNIVERSITY OF MANCHESTER

GREEN'S FUNCTIONS, INTEGRAL EQUATIONS AND APPLICATIONS

7 June 2017
9.45 - 11.45

Answer ALL six questions (100 marks in total)

University approved calculators may be used.

Given equations:

- For functions f and g defined in either two or three dimensions, and a domain D with smooth boundary in either two or three dimensions

$$\int_D (f(\mathbf{x})\nabla^2 g(\mathbf{x}) - g(\mathbf{x})\nabla^2 f(\mathbf{x}))\ d\mathbf{x} = \int_{\partial D} (f(\mathbf{x})\nabla g(\mathbf{x}) - g(\mathbf{x})\nabla f(\mathbf{x})) \cdot \mathbf{n}(\mathbf{x})\ ds(\mathbf{x}) \qquad (1)$$

  where the boundary ∂D is oriented by the outward pointing unit normal $\mathbf{n}(\mathbf{x})$ at each point $\mathbf{x}$ in ∂D.

- The surface area of the unit sphere $\mathbb{S}^2$ is 4π:

$$\int_{\mathbb{S}^2} ds = 4\pi.$$

1. You do not need to provide any proofs on the various parts of this problem. Just give the answers.

(a) Consider a boundary value problem

$$\begin{cases} \mathcal{L}u(x) = f(x), \quad x \in (a,b) \\ \mathcal{B} \end{cases}$$

where $\mathcal{L}$ is a second order linear differential operator and $\mathcal{B}$ is a set of two boundary conditions. Give the definition of the adjoint operator $\mathcal{L}^*$ and adjoint boundary conditions $\mathcal{B}^*$.

[3 marks]

(b) Let H be the Heaviside function, and p be a differentiable function. What is the derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}(pH)?$$

Simplify as much as possible.

[2 marks]

(c) Give the free-space Green's function for the Laplacian in two dimensions.

[3 marks]

(d) Give the general form of a Fredholm integral equation of the second kind. Identify the kernel of the equation, and state what it means for the kernel to be degenerate.

[4 marks]

[Total 12 marks]

2. Consider the following boundary value problem:

$$\begin{cases} u''(x) - u'(x) - 2u(x) = f(x), \quad x \in (0, \ln(2)) \\ u(0) = 0, \quad u(\ln(2)) = 0. \end{cases}$$

Find the Green's function for this problem, or show that no Green's function exists.

[10 marks]

3. This problem concerns the Sturm-Liouville eigenvalue problem

$$\text{(BVP)} \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}\phi}{\mathrm{d}x}(x)\right) + \lambda x^2 \phi(x) = 0, \quad x \in (1,2) \\ \phi(1) = 0, \quad \phi(2) = 0. \end{cases}$$

(a) How many eigenvalues are there for this problem? [**NOTE: You DO NOT have to calculate the eigenvalues, or otherwise prove that your answer is correct.**]

[2 marks]

(b) Prove that all of the eigenvalues for this problem are real. You may use the fact that regular Sturm-Liouville boundary value problems are fully self-adjoint without proof.

[7 marks]

(c) Write down a boundary value problem associated to this eigenvalue problem. What condition on the eigenvalue problem will ensure that there is always a unique solution of this boundary value problem? When this condition is satisfied, give a formula for the Green's function in terms of the eigenvalues and eigenfunctions. [**NOTE: You DO NOT need to prove that your formula is correct.**]

[6 marks]

(d) Prove that zero is not an eigenvalue.

[5 marks]

[Total 20 marks]

4. This question concerns the Helmholtz equation modelling time harmonic waves on a string

$$u''(x) + k(x)^2 u(x) = f(x) \tag{2}$$

where we assume that for all $|x|$ sufficiently large $f(x) = 0$, and $k(x) = k_0 > 0$. We take the time dependent waves to be given by $U(x,t) = e^{-i\omega t}u(x)$, and the physical motion, i.e. the displacement of the string from a stretched equilibrium position, is the real part of this.

(a) The so-called radiation conditions are

$$\lim_{x\to\infty} u'(x) - ik(x)u(x) = 0, \qquad \lim_{x\to-\infty} u'(x) + ik(x)u(x) = 0.$$

Explain why these boundary conditions model the case in which no waves are "coming from infinity".

[5 marks]

(b) Combine the radiation conditions from part (a) with the ODE (2) to form a boundary value problem, and assume that $k(x) = k_0$ is constant for all x. Find the Green's function for this problem.

[5 marks]

(c) Now suppose that $k(x) = k_0$ except in the interval $[0,1]$, and suppose that u satisfying the ODE (2) with $f(x) = 0$ has the form

$$u(x) = e^{ik_0 x} + u_{sc}(x)$$

where u_{sc} satisfies the radiation conditions from part (a). Derive a Fredolm integral equation satisfied by u on the interval $[0,1]$.

[6 marks]

(d) What is the Neumann series solution for u from part (c)?

[3 marks]

(e) What is the Born approximation for u from part (c)? Write the answer as explicitly as possible.

[3 marks]

[Total 22 marks]

5. This question concerns the integral equation

$$u(x) = \frac{\lambda}{\ln(2^8) - 3} \int_1^2 x \ln(y) u(y) \, \mathrm{d}y + f(x). \tag{3}$$

(a) For what values of $\lambda \in \mathbb{C}$ is there a unique solution $u(x)$ to (3) for any continuous function f? [**HINT: You should see a nice cancellation if you've done the required integral properly**]

[5 marks]

(b) For the values of λ found in part (a), find a formula for the solution $u(x)$ of (3).

[5 marks]

(c) When λ is not one of the values found in part (a) find a condition on f so that there is a solution for (3). Also write a general solution for (3) in this case.

[4 marks]

[Total 14 marks]

6. The *free-space Green's function* for the Laplacian operator $\mathcal{L} = \nabla^2$ in three dimensions is

$$G_{3\infty}(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x} - \mathbf{x}_0|}$$

(a) Show that $\nabla^2_{\mathbf{x}} G_{3\infty}(\mathbf{x}, \mathbf{x}_0) = 0$ when $\mathbf{x} \neq \mathbf{x}_0$.

[6 marks]

(b) Let f be a function on $\mathbb{R}^3$ which is twice continuously differentiable, and such that $f(\mathbf{x}) = 0$ when $|\mathbf{x}|$ is sufficiently large. Also, suppose that for $\epsilon > 0$, $B_\epsilon(\mathbf{x}_0)$ is the ball of radius ϵ around $\mathbf{x}_0$. Show that

$$\int_{\mathbb{R}^3 \setminus B_\epsilon(\mathbf{x}_0)} G(\mathbf{x}, \mathbf{x}_0) \nabla^2 f(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \frac{1}{4\pi\epsilon^2} \int_{\partial B_\epsilon(\mathbf{x}_0)} f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0) \, ds(\mathbf{x}).$$

[11 marks]

(c) Take the limit $\epsilon \to 0^+$ in the formula from part (b) to show that

$$\int_{\mathbb{R}^3} G(\mathbf{x}, \mathbf{x}_0) \nabla^2 f(\mathbf{x}) \, \mathrm{d}\mathbf{x} = f(\mathbf{x}_0).$$

[HINT: You should change variables $\mathbf{r} = (\mathbf{x} - \mathbf{x_0})/\epsilon$.]

[5 marks]

[Total 22 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

WAVE MOTION

31 May 2017

14:00 – 16:00

Answer **ALL** six questions.

Electronic calculators are permitted, provided they cannot store text.

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1. A uniform elastic string is stretched between the fixed points $x = 0$, $x = l$ and its transverse displacement from the equilibrium position is denoted by $y(x, t)$. The motion of the string is governed by a modified wave equation

$$\frac{\partial^2 y}{\partial t^2} + 2\lambda \frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2} ,$$

where $c > 0$ and $0 \leq \lambda < \pi c/l$ are positive constants.

Using the method of separation of variables, show that a solution exists in the form

$$y(x, t) = e^{-\lambda t} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\omega_n t + \psi_n\right) ,$$

where the constants ω_n are to be found in terms of n, c, l, λ; the constants A_n and ψ_n are arbitrary.

[8 marks]

2. Consider a physical signal $f(x, y, t) = Ae^{i(kx+ly-\omega t)}$, in the usual notation, which satisfies the governing equation

$$\frac{\partial f}{\partial t} - R^2 \frac{\partial}{\partial t}\left(\nabla^2 f\right) - \beta R^2 \frac{\partial f}{\partial x} = 0 ,$$

where R and β are positive constants and ∇^2 is the two-dimensional Laplacian.

Obtain the dispersion relation.

Restricting attention to the case of $l = 1$, sketch ω as a function of k for $k > 0$. Determine the group velocity, c_g, and show that $c_g > 0$ for $k > k^*$, where k^* is to be determined.

[9 marks]

3. A two-dimensional small-amplitude surface-wave motion is taking place on a semi-infinite layer ($z > 0$) of incompressible, inviscid fluid. When surface tension T is introduced, the appropriate linearised conditions at the free surface ($z = 0$) are

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{and} \quad -\rho \frac{\partial \phi}{\partial t} = -\rho g \eta + T \frac{\partial^2 \eta}{\partial x^2} \,.$$

Here ρ is the constant fluid density, g is the local gravitational acceleration, $\phi(x, z, t)$ is the velocity potential and $z = \eta$ defines the free surface position.

(i) State the equation satisfied by $\phi(x, z, t)$ in the fluid region $0 < z < \infty$.

(ii) A travelling surface wave exists on this fluid layer with a velocity potential of the form

$$\phi(x, z, t) = \Phi(z) \cos(Kx - \omega t) \,,$$

where K and ω are positive constants. Determine $\Phi(z)$ such that Φ decays as $z \to \infty$. Hence obtain the dispersion relation for such a wave when surface tension effects are included.

(iii) If such a wave has a phase speed of U, obtain the possible values of the wavenumber K as a function of g, T, ρ and U.

[11 marks]

4. The wave equation for a spherically symmetric velocity potential $\phi(r, t)$ is

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{c^2}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) \,,$$

in the usual notation.

(i) By direct substitution verify that the monopole

$$\phi(r, t) = -\frac{1}{4\pi r} m(t - r/c) \,,$$

is a solution for any differentiable function m.

An inviscid, compressible fluid of sound speed c occupies the region $z \geq 0$, $-\infty < x < \infty$, $-\infty < y < \infty$. The fluid is bounded by an impermeable wall at $z = 0$.

(ii) Write down the boundary condition to be applied at $z = 0$.

(iii) If a monopole is placed at $x = y = 0$, $z = h$, use the method of images to find a solution in the form

$$\phi = \phi_1 + \phi_2 \,, \quad \text{with} \quad \phi_1 = -\frac{1}{4\pi r_1} m(t - r_1/c) \,,$$

where $r_1^2 = x^2 + y^2 + (z - h)^2$. Be sure to give the corresponding expression for ϕ_2 and demonstrate that ϕ satisfies the boundary condition on $z = 0$.

[12 marks]

5. Consider small disturbances to an inviscid, incompressible fluid of constant density ρ, rotating at constant angular frequency Ω. In a Cartesian coordinate system (x, y, z), a wave motion with velocity field $\underline{u} = (u, v, w)$ and associated pressure p satisfies

$$\frac{\partial \underline{u}}{\partial t} + 2\underline{\Omega} \times \underline{u} = -\frac{1}{\rho}\nabla p\,,$$

$$\operatorname{div} \underline{u} = 0\,,$$

where $\underline{\Omega} = (0, 0, \Omega)$.

(i) Using these equations, show that

$$2\Omega\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = -\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right)\,,$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) - 2\Omega\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0\,.$$

(ii) Hence show that the pressure p satisfies

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right) + 4\Omega^2\frac{\partial^2 p}{\partial z^2} = 0\,.$$

(iii) Verify that a wave solution exists in the form

$$p = Ae^{i(kx+ly+mz-\omega t)}\,,$$

in the usual notation and give the dispersion relation $\omega = \omega(k, l, m)$. Show that time-harmonic perturbations with a period shorter than π/Ω cannot exist in this form.

(iv) Determine the group velocity vector $\underline{c}_g$ and evaluate the dot product $\underline{c}_g \cdot \underline{K}$, where $\underline{K} = (k, l, m)$. What does this imply about energy propagation in this system?

[20 marks]

6. Consider small amplitude sound waves in an inviscid compressible fluid of sound speed c. You are given that a sound field that is spherically symmetric about the origin O must satisfy

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial r^2},$$

where $f(r,t) = r\phi(r,t)$, and $\phi(r,t)$ is the velocity potential.

(i) In terms of f, derive an expression for the volume flux through a sphere of radius R centred at the origin O.

Now consider such a spherically symmetric sound field contained within a conical tube, such that $f(r,t) = F(r)e^{i\omega t}$, where ω is a constant frequency. The tube exists between $r = \epsilon$ and $r = L$ as shown in figure 1.

(ii) If there is no volume flux through $r = \epsilon$, then explain why the appropriate boundary condition is $F(r = 0) = 0$ in the limit of $\epsilon \to 0$; be careful to identify any assumptions that you make.

(iii) If the boundary at $r = L$ is impermeable, state the condition to be applied there. Solve for the velocity potential (with $\epsilon \to 0$) and hence approximate the frequency ω when the time taken for a sound wave to propagate the length of the tube is much larger than the period of oscillation for the wave.

(iv) Suppose that the impermeability condition at $r = L$ is instead replaced by a requirement that the pressure perturbation associated with the sound wave vanishes there. What values of the frequency ω are possible in this case?

You may use: The pressure perturbation associated with the sound wave is

$$p(r,t) = -\rho_0 \frac{\partial \phi}{\partial t},$$

where ρ_0 is the undisturbed density of the fluid.

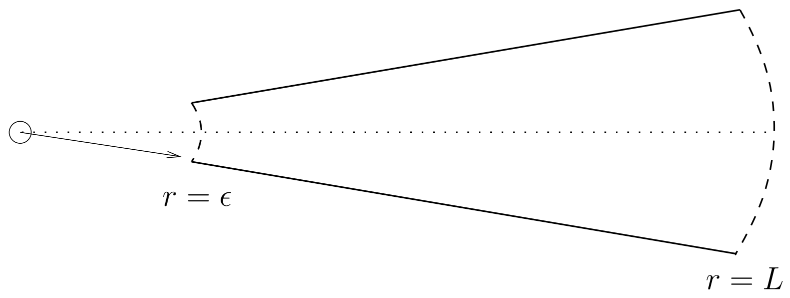


Figure 1: A cross section through the axis of the conical domain, for question 6.

[20 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL BIOLOGY

6 June 2017

9:45–11:45

Answer **ALL** THREE questions in Section A (30 marks in total).
Answer **TWO** of the THREE questions in Section B (50 marks in total).
If more than TWO questions from Section B are attempted,
then credit will be given for the best TWO answers.

University approved calculators may be used

SECTION A

Answer **ALL** 3 questions

A1. Large solitary animals such as pandas can, at low population densities, have trouble finding mates and so may, for small N, have population growth proportional to N^2. A model for such a population is

$$\frac{dN}{dt} = rN^2 \left(1 - \frac{N}{K}\right), \tag{A1.1}$$

where the parameters r and K are positive real numbers.

(a) Explain briefly what each of the parameters mean and what their units are.

(b) Show that by suitable choice of dimensionless variables one can reduce the system (A1.1) to the form

$$\frac{dx}{d\tau} = x^2 (1 - x)$$

Your answer should include expressions for x and τ in terms of N and t and the parameters r and K.

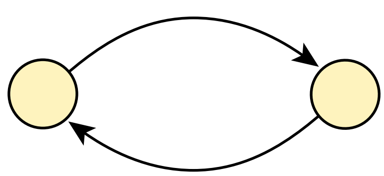
(c) If the initial population is nonzero, $x(0) = x_0 > 0$, find $\lim_{t\to\infty} x(t)$ and determine whether the limiting equilibrium is stable.

[10 marks]

A2. Explain what is meant by the following terms:

- a *motif* in a genetic regulatory network;
- a *coherent 3-node feed-forward loop*: your answer should include a sketch of such a loop.

In a randomly-assembled regulatory network with $N = 7$ nodes and $E = 3$ interactions, what is the probability of finding an example of the motif illustrated below?



[10 marks]

A3. The Lotka-Volterra predator-prey system is

$$\frac{dN}{dt} = N(a - bP) \qquad \frac{dP}{dt} = P(cN - d) \tag{A3.1}$$

where $N(t)$ is the population of prey, $P(t)$ is that of the predators and a, b, c and d are positive constants.

(a) Find an equilibrium point $(N_\star, P_\star)$ for which both $N_\star$ are $P_\star$ positive.

(b) Say what it means for a function $H(N, P)$ to be a *constant of the motion* for the system (A3.1), then show that

$$H(N, P) = \frac{e^{cN+bP}}{N^d P^a}$$

is a constant of the motion.

(c) Suppose that $(N(t), P(t))$ is a periodic solution to (A3.1) with period T. Define the time-average of the prey population $\overline{N}$ by

$$\overline{N} = \frac{1}{T}\int_0^T N(t)\,dt$$

and prove that $\overline{N} = N_\star$, where $N_\star$ is the equilibrium population found in part (a).

[10 marks]

SECTION B

Answer **2** of the 3 questions

B4. Experiments about the principle of competitive exclusion suggest that qualitative behaviour of an ecosystem can be altered by the regular removal of some animals. The ODEs below model such a system:

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \left(1 - m_1 - \frac{N_1 + a_1 N_2}{K_1}\right) \\ \frac{dN_2}{dt} &= r_2 N_2 \left(1 - m_2 - \frac{N_2 + a_2 N_1}{K_2}\right).\end{aligned} \tag{B4.1}$$

Here all the parameters—the r_j, the m_j, the a_j and the K_j—are positive real numbers.

(a) Explain the role of the m_j in (B4.1) and show that if both m_j exceed 1, the model's only biologically meaningful equilibrium has $N_1 = N_2 = 0$.

(b) Show that by suitable choice of dimensionless variables, it is possible to reduce the system (B4.1) above to

$$\frac{du}{d\tau} = f(u,v) = u\,(\gamma_1 - u - \alpha v) \qquad \text{and} \qquad \frac{dv}{d\tau} = g(u,v) = \rho v\,(\gamma_2 - v - \beta u)\,. \tag{B4.2}$$

Your answer should include formulae for $u, v, \tau, \alpha, \beta, \rho$ and the γ_j in terms of N_1, N_2, t and the parameters r_j, K_j, m_j and a_j of the original system.

(c) Sketch the null clines of (B4.2) for the case where $\alpha < \gamma_1 < 1/\beta < 1$ and $\gamma_2 = 1$, and then find all biologically meaningful equilibria.

(d) Linearise the system (B4.2) about an equilibrium point $(u_\star, v_\star)$ and show that if $u_\star, v_\star > 0$, then the community matrix simplifies to

$$A = \begin{bmatrix} \partial_u f & \partial_v f \\ \partial_u g & \partial_v g \end{bmatrix} = \begin{bmatrix} -u_\star & -\alpha u_\star \\ -\rho\beta v_\star & -\rho v_\star \end{bmatrix}.$$

(e) Classify the stabilities of the equilibria found in part (c).

(f) Discuss the ecological implications of your results from part (e).

[25 marks]

B5. Consider an enzyme whose active form is a dimer consisting of two monomeric subunits. Imagine that there are two variants of the monomer—called the α and β forms—so that the dimeric enzyme has three possible active forms whose assembly is governed by the following system of reactions

$$\begin{aligned} A + A &\xrightarrow{k_1} D_{\alpha\alpha}, \\ B + B &\xrightarrow{k_2} D_{\beta\beta}, \\ A + B &\underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} D_{\alpha\beta}. \end{aligned} \tag{B5.1}$$

Here the k_j are rate constants and A and B represent, respectively, the α- and β-monomers while $D_{\alpha\alpha}$, $D_{\alpha\beta}$ and $D_{\beta\beta}$ represent the various dimers.

(a) Construct a stoichiometric matrix N for the system of reactions (B5.1), explaining your work thoroughly.

(b) By performing Gaussian elimination on N, or by any other means, find two conserved quantities for the system (B5.1).

(c) Find the rank r of the stoichiometric matrix N and decompose N as a product $N = LN_R$ where N_R, the reduced stoichiometry matrix, has full rank and L is of the form

$$L = \begin{bmatrix} I_r \\ L_0 \end{bmatrix},$$

where I_r is the $r \times r$ identity matrix. Your answer should include expressions for L, L_0 and N_R.

(d) Using the Law of Mass Action, write down a system of ODEs that describes the temporal evolution of the concentrations of the chemical species in (B5.1).

Now imagine that we are interested in a very small, isolated volume in which there are so few molecules that the ODEs found in part (d) become unrealistic and we are obliged to model the reactions using the Gillespie algorithm.

(e) Suppose that at $t = 0$ there are 2 molecules of the α monomer, 2 molecules of the β monomer and no dimers. List the set of states the system can subsequently occupy.

(f) Draw a directed graph whose nodes are the states listed in part (e) and whose edges represent transitions allowed by the reactions (B5.1). Label the nodes of the graph using the notation j, k, l to indicate that the corresponding state has j molecules of the α monomer, k molecules of the β monomer and l molecules of the dimer $D_{\alpha\beta}$.

(g) Using the same labelling convention as in part (e), let $p_{jkl}(t)$ indicate the probability that the system is, at time t, in a state with j molecules of the α monomer, k molecules of the β monomer and l molecules of the dimer $D_{\alpha\beta}$. Find $\lim_{t\to\infty} p_{jkl}(t)$ for all the $p_{jkl}(t)$ associated with the states found in part (e).

[25 marks]

B6. The equation below models a population that is self-limiting, but has spatial structure

$$\partial_t u = D\partial_{xx} u + \mu u(1 - u^3). \tag{B6.1}$$

Here $u(x,t)$ is a population density, $x \in [0, L]$ is a spatial coordinate and $t \geq 0$ is the time and we will require $u(0,t) = u(L,t) = 0$.

(a) Find the ODE obeyed by a steady state solution to (B6.1).

(b) By introducing $v = du/dx$, reduce the ODE from part (a) to a system of two first order equations.

(c) Sketch the null clines of the system from part (b).

(d) Show that

$$H(u,v) = \frac{v^2}{2} + \frac{\mu}{2D}\left[u^2 - \frac{2u^5}{5}\right]$$

is a constant of the motion for the system of ODEs from part (b).

(e) For the case $\mu = D = 1$, find and classify (as local minima, local maxima or saddles) all the critical points of the function $H(u,v)$, then use these results to sketch a contour map of $H(u,v)$.

(f) Relate the contour map from part (e) to solutions of the corresponding ODE from part (a). In particular, if $(u(x), v(x))$ is a solution to the ODE from part (a), how does the curve traced by $(u(x), v(x))$ as x varies $0 \leq x \leq L$ appear on your contour map?

[25 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

SYMMETRY IN NATURE

01 June 2017

09:45 – 11:45

Answer **ALL FOUR** questions

Electronic calculators may be used, provided that they cannot store text, transmit or receive information, or display graphics

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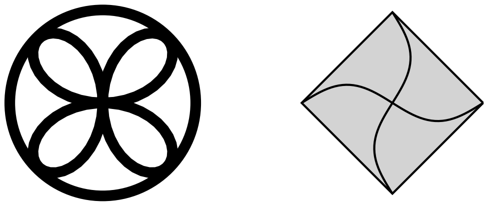
1.

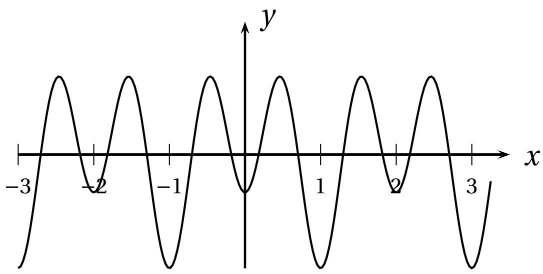
(a) Name the symmetry groups of the following two planar shapes, and state in each case the order of the group, generator(s) of the group and whether the group is Abelian.



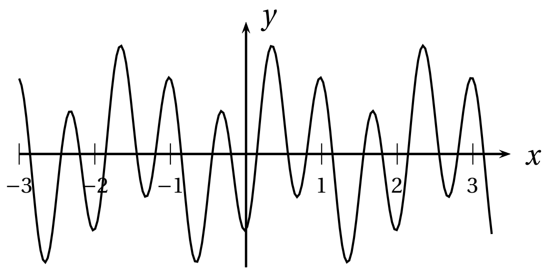
(i) (ii)

[6 marks]

(b) Consider the two functions $f(x) = \cos(\pi x) - 2\cos(2\pi x)$ and $g(x) = \sin(\pi x) - 2\cos(3\pi x)$. Their graphs are shown below.



$y = f(x)$



$y = g(x)$

Determine geometrically the full symmetry group of each of the functions f and g, including possible transformations of x that reverse the sign of the function, and verify these are symmetries using the expressions for the functions.

[8 marks]

2.

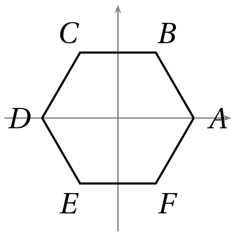
(a) Suppose a finite group G acts on a set X.

(i) Define the orbit and stabilizer of a point $x \in X$. Show that the stabilizer of a point is always a subgroup of G.

(ii) Let $y = k \cdot x$ for some $k \in G$. Show that the stabilizer subgroups G_x and G_y are conjugate subgroups of G.

[6 marks]

(b) Let D_6 be the usual dihedral group of order 12, and let $V = \{A, B, \ldots, F\}$ be the set of vertices of the hexagon as shown on the right. Denote by $\mathscr{P}(V)$ the power set of V, which is the set of all 2^6 subsets of V. Denote by $\mathscr{P}(V)_k$ the subset of $\mathscr{P}(V)$ consisting of those subsets of cardinality k ($k = 0, 1, \ldots, 6$).



There is a natural action of D_6 on $\mathscr{P}(V)$ defined by

$$g \cdot S = \{g \cdot x \mid x \in S\}, \qquad \text{for } S \subset V,\ g \in \mathsf{D}_6.$$

(i) State the orbit stabilizer theorem, and for $S_1 = \{B, C\}$ find the stabilizer and orbit of S_1 in $\mathscr{P}(V)_2$, and show they satisfy the theorem.

(ii) Find an element $S_2 \in \mathscr{P}(V)_2$ whose stabilizer has different order to that of S_1.

Consider in addition the action of $\mathbb{Z}_2 = \{e, \sigma\}$ on $\mathscr{P}(V)$ defined by

$$\sigma(S) = S'$$

where S' is the complement of S in V. Let $G = \mathsf{D}_6 \times \mathbb{Z}_2$.

(iii) Write down $\sigma(S_1)$. Use σ to show that the D_6 actions on $\mathscr{P}(V)_2$ and $\mathscr{P}(V)_4$ are isomorphic.

(iv) Consider $S_0 = \{A, C, E\}$ and find its stabilizer for each of the D_6 and the $\mathsf{D}_6 \times \mathbb{Z}_2$ actions on $\mathscr{P}(V)$. Show that the stabilizer for the $\mathsf{D}_6 \times \mathbb{Z}_2$ action is the graph of a homomorphism from D_6 to $\mathbb{Z}_2$. What is the relevance of the kernel of this homomorphism?

[16 marks]

3.

(a) Define the action on $\mathbb{R}^2$ of an element $(A \mid \mathbf{v}) \in \mathsf{E}(2)$, where $A \in \mathsf{O}(2)$ and $\mathbf{v} \in \mathbb{R}^2$, and deduce that multiplication in the group is given by

$$(A \mid \mathbf{v})(B \mid \mathbf{u}) = (AB \mid \mathbf{v} + A\mathbf{u}).$$

[4 marks]

(b) Consider the following lattice in the plane:

$$L = \left\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z},\ y \in \mathbb{Z},\ x + y \in 2\mathbb{Z}\right\}.$$

(i) Show that the two vectors $\mathbf{u}_1 = (2, 0)$ and $\mathbf{u}_2 = (0, 2)$ both belong to L, and sketch a diagram showing all the points of L that lie in the rectangle $0 \le x \le 6$ and $0 \le y \le 6$. Deduce that $L \ne \mathbb{Z}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(ii) Find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$, proving carefully that this is the case.

(iii) Define the point group of a lattice. Show that the point group J_L of the lattice L has order 8 by finding appropriate elements of its symmetry group $\mathscr{W}_L < \mathsf{E}(2)$, expressed in the form $(A \mid \mathbf{v})$.

(iv) Now consider the wallpaper pattern produced by placing the symbol $\bigcup$ at those points of L where $y \in 2\mathbb{Z}$, and $\bigcap$ at the points of L where $y \notin 2\mathbb{Z}$. Let $\mathscr{W}$ be the wallpaper group of this pattern, with point group J. Draw the modified version of the diagram from (i) (with just $0 \le x \le 4$ and $0 \le y \le 4$), and show that $J = \mathsf{D}_2$. Explain how r_0 is an element of the point group, again by finding an element $(r_0 \mid \mathbf{v}) \in \mathscr{W}$. Describe this last transformation.

[20 marks]

4.

(a) Let $\mathbb{Z}_2 = \{e, \beta\}$ act on $\mathbb{R}$ by $\beta \cdot x = -x$. Show that the differential equation $\dot{x} = x\cos(x)$ has symmetry group $\mathbb{Z}_2$. Let $x(t)$ be the solution with initial value $x(0) = 1$, and let $u(t)$ be the solution with initial value $u(0) = -1$. How are $x(t)$ and $u(t)$ related? [4 marks]

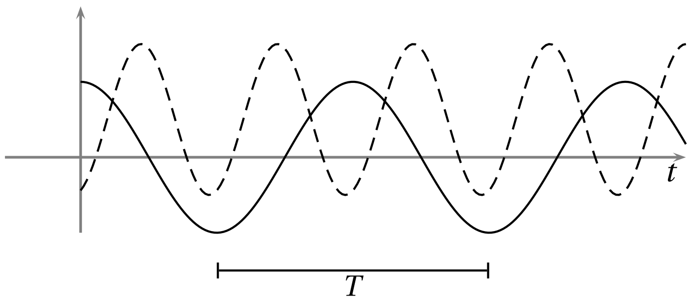
(b) Let D_4 be the usual dihedral subgroup of order 8 of $\mathsf{O}(2)$, generated by r_0 and $R_{\pi/2}$. List all axial subgroups of D_4. Show that the smooth function $f(x, y) = x^2 + y^2 - x^2y^2$ is invariant under the D_4 action. Now consider the system of differential equations

$$\dot{x} = \frac{\partial f}{\partial x}, \qquad \dot{y} = \frac{\partial f}{\partial y}.$$

Find all equilibria of this system with axial symmetry, and find the solution with initial condition $(x, y) = (1, 0)$. [8 marks]

(c) Suppose $\gamma(t)$ is a periodic solution with fundamental period T of a differential equation with symmetry group G. Define the symmetry group of the periodic solution.

Suppose a mechanical system in the plane has potential energy $V(x, y)$, and that the system has D_4 symmetry. Let γ be a periodic orbit with period T and symmetry group $\widetilde{\mathsf{C}_4}$, and let $\delta(t)$ be a periodic solution of the same period and with symmetry $\widetilde{D_1}$. Write down in each case what the symmetry group tells us about the periodic orbit. Below is shown numerical data taken from an experiment with this symmetry showing both $x(t)$ and $y(t)$. Determine whether the solution has symmetry $\widetilde{D_1}$ or $\widetilde{C_4}$, explaining which graph is $x(t)$ and which is $y(t)$.



[8 marks]

END OF EXAMINATION PAPER

Two Hours

UNIVERSITY OF MANCHESTER

NUMERICAL ANALYSIS 2

24 May 2017
1400 - 1600

Answer **ALL** five questions in **SECTION A** (40 marks in total)
and
Answer **TWO** of the three questions in **SECTION B** (20 marks each).
If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

University approved calculators may be used.

SECTION A

Answer **ALL** five questions

A1. Let $T_3(x)$ denote the Chebyshev polynomial of degree 3.

(a) State the leading term (the term with the highest power of x) of $T_3(x)$ and list all the values of x in $[-1, 1]$ such that $T_3(x) = \pm 1$.

(b) State the Chebyshev Equioscillation Theorem concerning the best L_∞ approximation $p_n(x)$ (from the set of polynomials of degree $\leq n$) to a given function $f \in C[-1, 1]$.

(c) Write down the best L_∞ approximation to $f(x) = 1 - 2x^2 + 3x^3$ on $[-1, 1]$ from the set of polynomials of degree less than or equal to two. Use parts (a) and (b) to justify your answer.

[10 marks]

A2. Let

$$r_{km}(x) = p_{km}(x)/q_{km}(x)$$

be a rational function with p_{km} a polynomial of degree at most k, q_{km} a polynomial of degree at most m, and $q_{km}(0) = 1$.

(a) State the condition that r_{km} must satisfy if it is to be a $[k/m]$ Padé approximant of a given function $f(x)$.

(b) Write the expression

$$r_3(x) = \frac{1}{1 + \frac{x/2}{1+x/6}}$$

as a ratio of two polynomials and then show that $r_3(x)$ is the $[2/1]$ Padé approximant to $f(x) = \log(1 + x)$.

[9 marks]

A3. Consider the integral

$$I(f) = \int_a^b w(x) f(x)\, dx,$$

where $w(x)$ is a continuous and non-negative weight function on $[a, b]$.

(a) Using the idea of interpolatory quadrature, derive an expression for the weights w_i in the n-point quadrature rule

$$G_n(f) = \sum_{i=1}^{n} w_i f(x_i)$$

that ensures the rule is exact for all polynomials of degree $\leq n-1$.

(b) Explain how we can choose the nodes $x_1, \ldots, x_n$ such that the rule $G_n(f)$ is exact for polynomials of degree $\leq 2n-1$.

[6 marks]

A4. Derive the two point Gauss–Legendre rule

$$\int_0^1 f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2),$$

using the method of undetermined coefficients. That is, determine the choice of nodes x_1, x_2 and weights w_1, w_2 to maximize the degree of precision. HINT: You may find it useful to define a quadratic polynomial $\Phi(x) = x^2 + ax + b$ whose roots coincide with x_1 and x_2, and find a and b.

[10 marks]

A5. The ℓ-step Adams linear multistep methods for the initial value problem $y' = f(x, y)$, $x \geq x_0$, $y(x_0) = y_0$ have the form

$$y_{n+1} = \sum_{i=1}^{\ell} a_i y_{n+1-i} + h \sum_{i=0}^{\ell} b_i f_{n+1-i},$$

where a_i, b_i are given constants, h is the step size, and $f_n = f(x_n, y_n)$.

Explain how we can derive the ℓ-step Adams–Bashforth method and the ℓ-step Adams–Moulton method by replacing the integrand in

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))\,dx$$

with a suitable polynomial. Briefly comment on the accuracy of the two methods and the computational cost of implementing them.

[5 marks]

SECTION B

Answer **TWO** of the three questions

B6.

(a) Let $w(x)$ be a weight function. Define the $L_{2,w}$ norm of $f \in C[a, b]$ (denoted $\| f \|_{2,w}$), taking care to state any properties that $w(x)$ must satisfy. Explain what is meant by the phrase g is a "best $L_{2,w}$ approximation" to f from a subspace $G \subset C[a, b]$.

[3 marks]

(b) Consider the set of polynomials defined by $\phi_0(x) = 1$, $\phi_1(x) = x - \alpha_0$, and

$$\phi_{n+1}(x) = (x - \alpha_n)\phi_n(x) - \beta_n \phi_{n-1}(x), \qquad n \geq 1,$$

where

$$\alpha_n := \frac{\langle x\phi_n, \phi_n \rangle_w}{\| \phi_n \|_{2,w}^2}, \qquad \beta_n := \frac{\langle x\phi_n, \phi_{n-1} \rangle_w}{\| \phi_{n-1} \|_{2,w}^2},$$

and the inner-product $\langle \cdot, \cdot \rangle_w$ is defined by

$$\langle u, v \rangle_w := \int_a^b w(x)u(x)v(x)dx.$$

Show that the resulting set of polynomials is orthogonal. HINT: First, show that $\langle \phi_1, \phi_0 \rangle_w = 0$. Then, using induction, show that for any $n \geq 1$,

$$\langle \phi_n, \phi_i \rangle_w = 0 \qquad \text{for } i = 0, 1 \ldots, n - 1.$$

[11 marks]

(c) Suppose we want to find an approximation to $f \in C[a, b]$ of the form

$$g(x) = \sum_{i=0}^{n} c_i \phi_i(x).$$

Define the normal equations that can be solved to find the best $L_{2,w}$ approximation that has this form. Briefly explain why it is advantageous to use orthogonal polynomials $\phi_i(x)$.

[4 marks]

(d) Give two examples of weight functions $w(x)$ and the names of the corresponding families of orthogonal polynomials $\{\phi_i(x)\}$.

[2 marks]

B7. Consider the integral

$$I(f) = \int_0^1 f(x)\,dx.$$

(a) Divide $[0,1]$ into n equal subintervals $[x_i, x_{i+1}]$ of width h, where $x_i = ih$, for $i = 0, 1, \ldots, n$. Let $f(x)$ be twice continuously differentiable on $[0,1]$ and consider the repeated trapezium rule

$$T(h) := \sum_{i=0}^{n-1} \frac{h}{2}\big(f(x_i) + f(x_{i+1})\big).$$

Given that the error in applying the trapezium rule on a single interval $[x_i, x_{i+1}]$ satisfies

$$\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{h}{2}\big(f(x_i)+f(x_{i+1})\big) = -\frac{h^3}{12} f''(\theta_i), \qquad \text{for some } \theta_i \in (x_i, x_{i+1}),$$

derive an expression for the error $E_{RT}(f) = I(f) - T(h)$. You may use any version of the Mean Value Theorem, without proof.

[7 marks]

(b) The Euler–Maclaurin summation formula gives an alternative asymptotic expansion for the error of the form

$$I(f) - T(h) = a_2 h^2 + a_4 h^4 + a_6 h^6 + \ldots$$

for some real constants $a_2, a_4, a_6, \ldots$. Use this formula to determine what happens to the error when we double the number of intervals $[x_i, x_{i+1}]$. By taking an appropriate linear combination of the rules $T(h)$ and $T(\frac{h}{2})$, derive a new quadrature scheme that gives an error of $\mathcal{O}(h^4)$.

[7 marks]

(c) Define $T_{k,1} = T(2^{-(k-1)})$ for $k \geq 1$ and

$$T_{i,j} = \frac{4^{j-1} T_{i,j-1} - T_{i-1,j-1}}{4^{j-1} - 1}, \quad i, j \geq 2.$$

Starting from the approximation $T_{1,1}$, describe the Romberg Scheme for generating a sequence of increasingly accurate approximations $T_{i,j}$ to the integral $I(f)$.

[6 marks]

B8. Consider the initial value problem $y'(x) = f(x, y(x))$, $x \geq x_0$, $y(x_0) = y_0$. Recall that a two-stage Runge–Kutta scheme has the general form

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + c_2 h, y_n + h a_{21} k_1) \\ y_{n+1} &= y_n + h(b_1 k_1 + b_2 k_2) \end{aligned}$$

where b_1, b_2, c_2 and a_{21} are constants, h denotes the step size and y_n is the approximation to $y(x_n)$ at the grid point x_n.

(a) Show that the truncation error satisfies

$$\tau(h) = y(x_{n+1}) - y(x_n) - hb_1 y'(x_n) - hb_2 f(x_n + c_2 h, y(x_n) + ha_{21} y'(x_n)).$$

[3 marks]

(b) By examining the terms in h^0, h^1 and h^2 in the truncation error $\tau(h)$, derive conditions on the coefficients b_1, b_2, c_2 and a_{21} that guarantee the method has order two.

HINT: Recall that if $y'(x) = f(x, y(x))$ then

$$y''(x) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f.$$

[10 marks]

(c) Suppose we choose $a_{21} = c_1 = b_2 = 1$ and $b_1 = 0$. Briefly explain how we can interpret the method as a predictor-corrector method.

[3 marks]

(d) By examining the test problem $y'(x) = \lambda y(x)$ where $\lambda \in \mathbb{C}$, derive a quadratic polynomial $p(\lambda h)$ such that the condition $|p(\lambda h)| < 1$ provides the region of absolute stability for the method in part (c).

[4 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

MARKOV PROCESSES

30 May 2017

9.45 - 11.45

Answer 3 of the 4 questions. If more than 3 questions are attempted, credit will be given for the best 3 answers. Each question is worth 20 marks.

Electronic calculators may be used, provided that they cannot store text.

Answer 3 questions

1.
a)

(i) Define the terms "recurrent state" and "transient state" in the context of discrete time Markov chains.

[2 marks]

(ii) Define the stationary distribution of a discrete time Markov chain.

[2 marks]

(iii) Give an example of a Markov chain which has more than one stationary distribution.

[2 marks]

(iv) Prove that if a Markov chain has more than one stationary distribution, it has an infinite number of stationary distributions.

[2 marks]

b) Consider the discrete time Markov chain $\{X_n, n \geq 0\}$ with state space $\{0, 1, 2, ..., m\}$ and transition probabilities

$$\begin{aligned} p_{0,1} &= 1, \quad p_{m,m-1} = 1 \\ p_{i,i+1} &= \frac{m-i}{m}, \quad 0 < i \leq m-1 \\ p_{i,i-1} &= \frac{i}{m}, \quad 1 \leq i < m. \end{aligned}$$

(i) Determine the stationary distribution of the Markov chain and write down its variance.

[5 marks]

(ii) Explain why $\lim_{n\to\infty} p_{0,0}(n)$ does not exist.

[1 marks]

c)

(i) Define the period of a state of a discrete time Markov chain.

[2 marks]

(ii) Prove that all states of an irreducible Markov chain such that $p_{i,i} > 0$ for some i, have period 1.

[4 marks]

2. Let $\{X_n, n \geq 0\}$ be a Markov chain, with state space the set of non-negative integers and transition probabilities given by

$$\begin{aligned} p_{0,0} &= p \\ p_{i,i+1} &= q, \quad 0 \leq i \\ p_{i,i-1} &= p, \quad 1 \leq i. \end{aligned}$$

where $0 < p < 1$ and $q = 1 - p$.

(i) Determine a stationary distribution for the Markov chain, stating the range of p for which a stationary distribution exists.

[8 marks]

(ii) Giving a reason, state the range of p for which the Markov chain is positive recurrent. For p in this range, write down the mean recurrence time of state 0.

[4 marks]

(iii) Let a denote the expected time of first visit to state 3, starting from state 4. Show that, when a is finite, it satisfies $a = 1 + 2aq$.

[4 marks]

(iv) By considering the simple random walk on the integers, or otherwise, determine the range of p for which $\{X_n, n \geq 0\}$ is transient.

[4 marks]

3. a) A population comprises m individuals at time 0. Lifetimes of individuals are independent and each exponentially distributed with parameter μ. Let $\{X(t), t \geq 0\}$ denote the number of individuals alive in the population at time t.

i) Determine the matrix Q for this Markov chain and show that the forward equations for the process $\{X(t), t \geq 0\}$, include

$$\begin{aligned} p'_{m,i}(t) &= \mu(i+1)p_{m,i+1}(t) - \mu i p_{m,i}(t) \qquad 0 \leq i < m; \\ p'_{m,m}(t) &= -\mu m p_{m,m}(t). \end{aligned}$$

[4 marks]

ii) Hence, or otherwise, show that, for $t \geq 0$,

$$p_{m,m-1}(t) = m\{\exp(-\mu t)\}^{m-1}\{1 - \exp(-\mu t)\}.$$

[4 marks]

iii) By considering the probability of any given individual from the initial m living beyond time t, or otherwise, derive an expression for $p_{m,i}(t)$ for each $0 \leq i \leq m$. Hence write down the expected number of individuals alive at time t.

[3 marks]

b) Consider the continuous time Markov chain $\{X(t), t \geq 0\}$ with state space $\{0, 1, 2\}$, where the holding times for states 0,1 and 2 are independent exponential random variables with parameters λ, μ and α respectively. Upon leaving a state, the process is equally likely to enter either of the other two.

(i) Determine the matrix Q for this Markov chain.

[3 marks]

From here onward, suppose $\mu = \alpha$

(ii) With the initial condition $p_{00}(0) = 1$, use the forward equations $P'(t) = P(t)Q$ to show that, for $t \geq 0$,

$$p_{00}(t) = \frac{1}{2\lambda + \mu}\left\{\mu + 2\lambda \exp\left(-\left[\lambda + \frac{\mu}{2}\right]t\right)\right\}.$$

[6 marks]

4. a) Cars arrive at a filling station according to a Poisson process with an average 10 cars per hour. A car enters the filling station only if there are three or fewer other cars present. The filling station has just one pump. The amount of time required to serve a car has an exponential distribution with a mean of four minutes. Let $X(t), t \geq 0$ denote the number of cars in the filling station at time t.

(i) Find the stationary distribution for the process $\{X(t), t \geq 0\}$.

[6 marks]

(ii) Determine the long-run average number of customers per hour lost.

[3 marks]

b) From here onward, consider the model above modified such that all arriving cars enter the filling station, with no change in the arrival process and service time distribution.

(i) Determine the long-run average number of cars in the filling station.

[3 marks]

(ii) Determine the long-run average waiting time until completion of service, for an arriving customer. [Hint: PASTA property]

[4 marks]

c) The pump attendant now decides to work harder. When the process enters state $i \neq 0$, the current service time assumes the exponential distribution, with mean $4/i$ minutes.

Determine the new long-run proportion of time for which the attendant is occupied.

[4 marks]

END OF EXAMINATION PAPER

Statistical Tables to be provided

Two hours

THE UNIVERSITY OF MANCHESTER

Time Series Analysis

05 June 2017
09.45 – 11.45

Electronic calculators are permitted, provided they cannot store text.

Answer **ALL** Four questions in Section A (32 marks in all)

and

Two of the Three questions in Section B (24 marks each).

If more than Two questions from Section B are attempted, then credit will be given for the best Two answers.

SECTION A
Answer ALL four questions

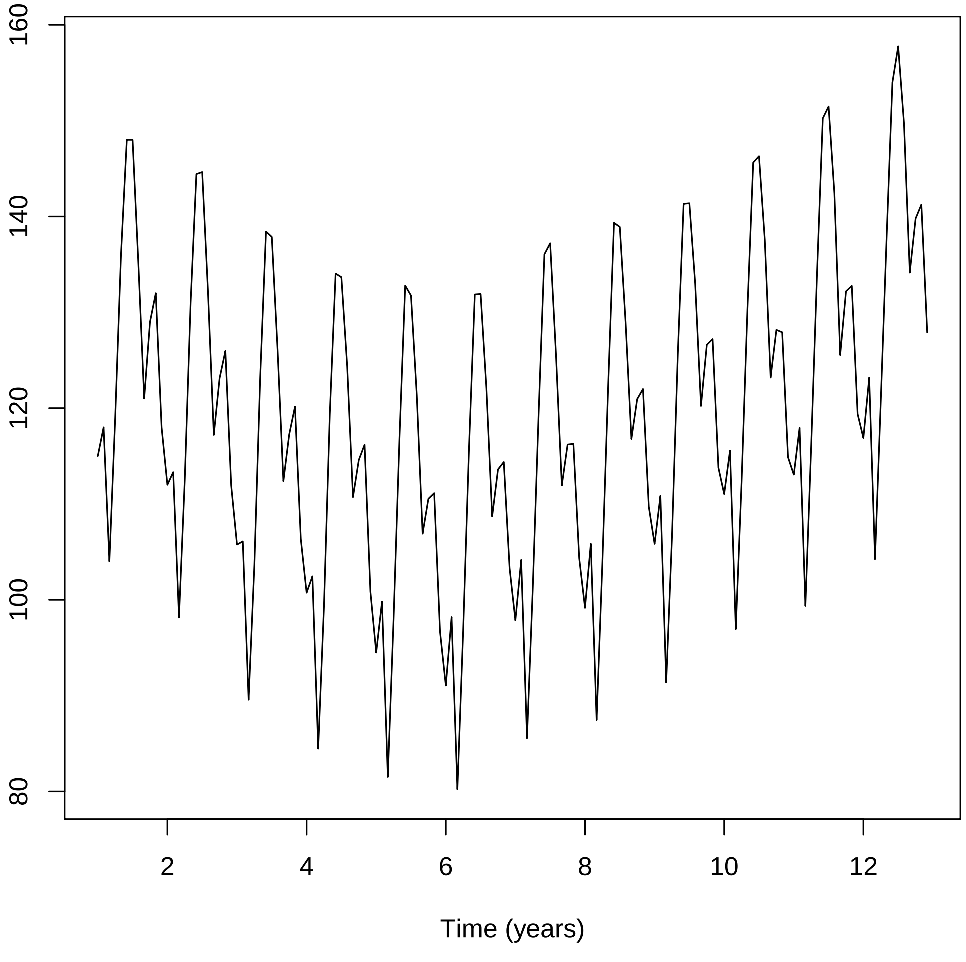
A1.
A graph of a monthly time series, $\{x_t\}$, is shown below. The series spans a period of 14 years. Give concise answers to the following questions.

a) (2 marks) What major features of the data can be inferred from this plot?

b) (2 marks) What might be the effect of differencing $\{x_t\}$?

c) (2 marks) What might be the effect of a seasonal differencing, $(1 - \mathbf{B}^{12})$, of $\{x_t\}$?

d) (2 marks) What feature(s) in a time series plot might prompt you to consider a log-transformation? Is such a transformation appropriate for the time series $\{x_t\}$?



[8 marks]

A2. Let $\{\varepsilon_t\}$ be a white noise process with variance σ^2, i.e. $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$.

a) (2 marks) Write the following model for a process $\{X_t\}$ in operator form.

$$X_t = -0.1X_{t-1} + 0.72X_{t-2} + \varepsilon_t - 0.5\varepsilon_{t-12}.$$

b) (2 marks) Let $\{X_t\}$ be a stationary process, such that

$$(1 - 0.3\mathbf{B})X_t = 4.2 + (1 + 0.7\mathbf{B})\varepsilon_t.$$

Find the mean corrected representation of this model. In other words, write this model in terms of the mean corrected process $X_t - \mu$, where $\mu = \mathrm{E}\, X_t$.
Give your answer in: (i) operator form, and (ii) difference equation form.

c) (2 marks) Suppose that the process $\{X_t\}$ is such that

$$(1 - \mathbf{B})(1 - \mathbf{B}^{12})X_t = (1 - 0.5\mathbf{B})(1 + 0.1\mathbf{B}^{12})\varepsilon_t.$$

Let $Y_t = (1 - \mathbf{B})X_t$. Write down the model for $\{Y_t\}$.

d) (2 marks) Give two different multiplicative forms of the filter $1 - \mathbf{B}^4$.

[8 marks]

A3.
A time series is analysed with R and a model is fitted with the `arima` function. Here is a summary of the fitted model.

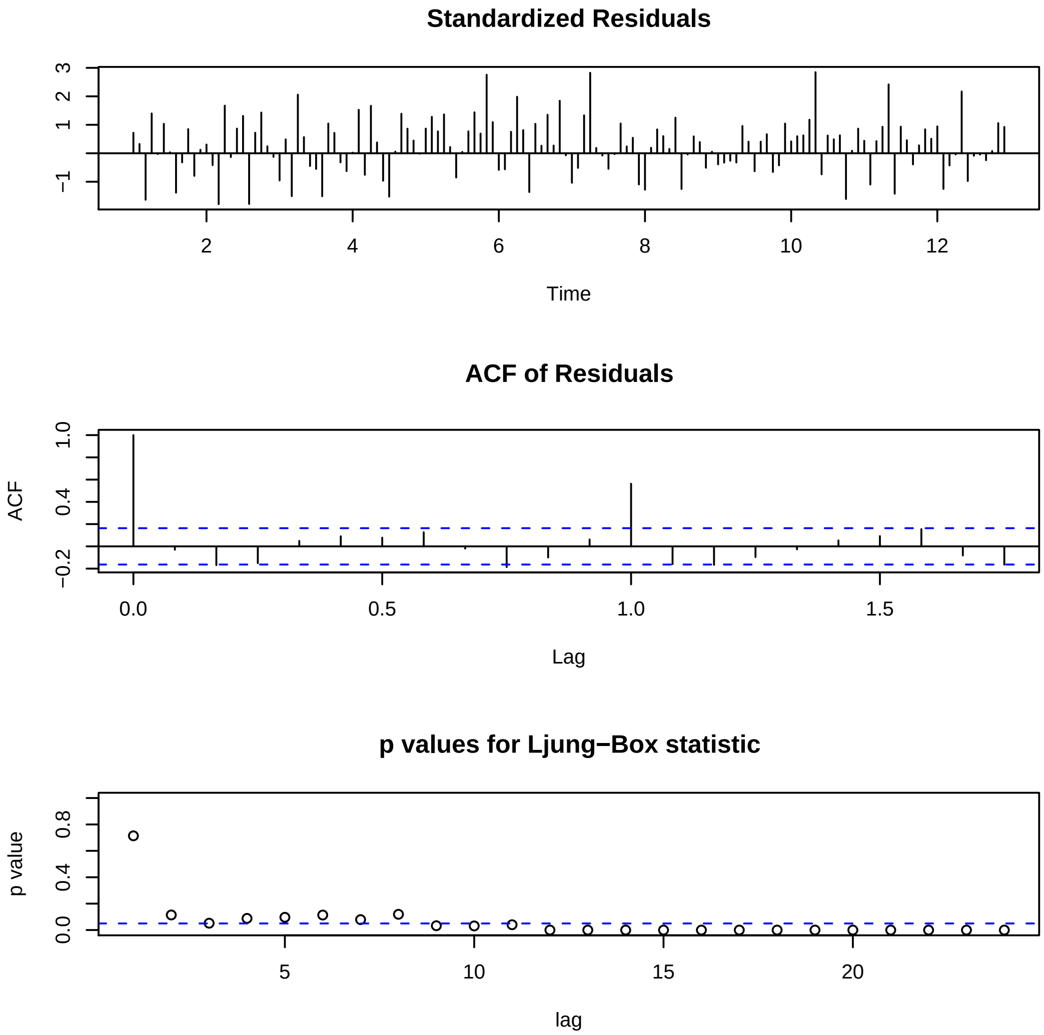
```
Call:
arima(x = tsqu3, order = c(1, 0, 0), seasonal = list(order = c(0, 0, 0), period = 12),
    include.mean = FALSE)

Coefficients:
         ar1
      0.4016
s.e.  0.0765

sigma^2 estimated as 1.798:  log likelihood = -246.67,  aic = 497.34
```

a) (1 mark) Write down the fitted model in a mathematical form (e.g. operator form or difference equation form).

b) (0.5 marks) Give the estimated variance of the innovations.

c) (1 mark) Comment on the statistical significance of the autoregressive parameter.

d) (3.5 marks) The following diagnostic plots (see next page) for the fitted model were obtained.



Explain what the autocorrelations of the residuals and the p-values of the Ljung-Box statistics tell you about the quality of the model's fit.

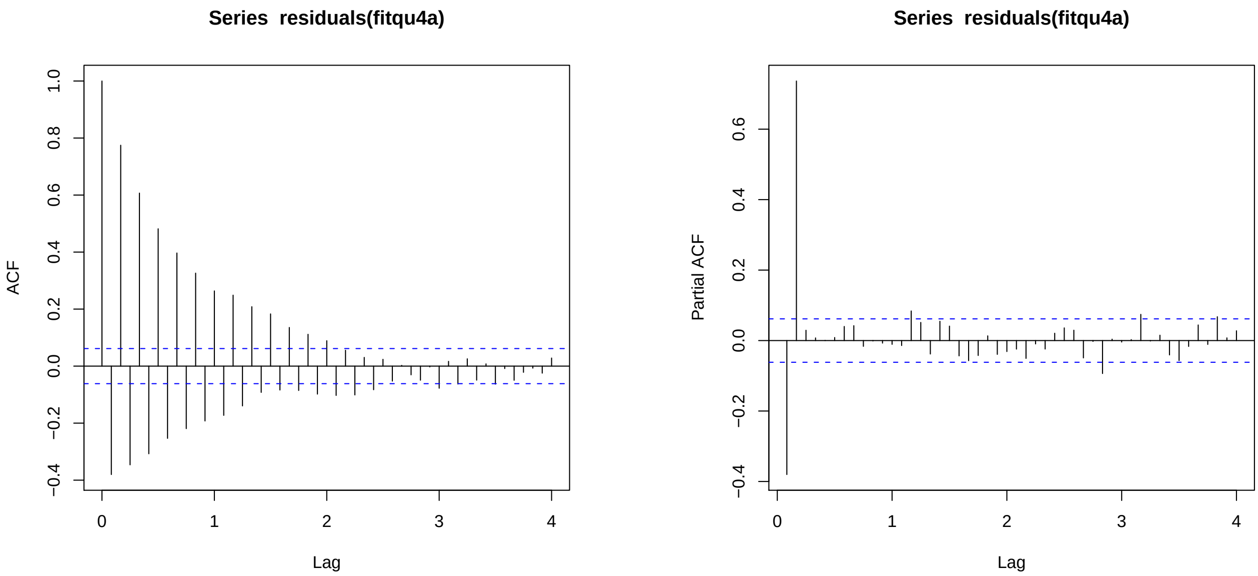
e) (2 marks) Taking into account the plots above, indicate how you would go about trying to improve the fit of the model.

[8 marks]

A4.
A time series, $\{x_t\}$, is analysed with R and a model is fitted with the `arima` function with the following results.

```
Call:
arima(x = tsqu4, order = c(0, 1, 0), seasonal = list(order = c(0, 1, 1), period = 12))

Coefficients:
         sma1
       0.8784
s.e.   0.0128

sigma^2 estimated as 2.499:  log likelihood = -1885.84,  aic = 3775.68
```

a) (4 marks) Here are plots of the autocorrelations and partial autocorrelations of the residuals from the above fit.



Explain what these plots tell you about the quality of the model's fit.

b) (2 marks) On the basis of the above analysis it was decided to fit an ARIMA$(2,1,0)(0,1,1)_{12}$ model to $\{x_t\}$. Explain why this is a sensible decision.

c) (2 marks) Several additional models were fitted to the observed series. For two of them the residual diagnostics did not show evidence for a bad fit. Numerical results for the two fitted models are shown on the next page. What inferences would you draw from these results? Which model would be your choice and why?

Model 1

```
Call:
arima(x = tsqu4, order = c(2, 1, 0), seasonal = list(order = c(0, 1, 1), period = 12))

Coefficients:
          ar1     ar2    sma1
      -0.0960  0.7472  0.8220
s.e.   0.0209  0.0210  0.0203

sigma^2 estimated as 0.9605:  log likelihood = -1406.77,  aic = 2821.54
```

Model 2

```
Call:
arima(x = tsqu4, order = c(3, 1, 1), seasonal = list(order = c(0, 1, 1), period = 12))

Coefficients:
         ar1     ar2      ar3      ma1    sma1
      0.2037  0.7836  -0.2157  -0.3222  0.8215
s.e.  0.4976  0.0546   0.3720   0.5055  0.0204

sigma^2 estimated as 0.9599:  log likelihood = -1406.45,  aic = 2824.91
```

[8 marks]

SECTION B
Answer 2 of the 3 questions

B5. Consider the seasonal model

$$(1 - 0.8\mathbf{B}^{12})X_t = (1 + 0.5\mathbf{B})\varepsilon_t,$$

where $\{\varepsilon_t\}$ is a white noise with variance 1, i.e. $\{\varepsilon_t\} \sim \mathrm{WN}(0, 1)$.

a) (2 marks) Write the model in difference equation form.

b) (2 marks) This is an ARIMA$(p, d, q)(p_s, d_s, q_s)_s$ model. Give the values of s, p, d, q, p_s, d_s, and q_s.

c) (2 marks) Show that the model is invertible.

d) (4 marks) Show that the infinite moving average representation of $\{X_t\}$ is

$$X_t = \sum_{k=0}^{\infty} 0.8^k \mathbf{B}^{12k} \varepsilon_t + \sum_{k=0}^{\infty} 0.5 \times 0.8^k \mathbf{B}^{12k+1} \varepsilon_t,$$

e) (4 marks) Write down the coefficients of ε_t, ε_{t-1}, and ε_{t-12}, in the above infinite moving average representation. What are the values of the coefficients of ε_{t-i} for $i = 2, 3, \ldots, 11$?

f) (10 marks) Assuming that at time t the following information is available: ε_t, X_t, and X_{t-i} for $i = 1, 2, \ldots, 11$,

  i) write down the predictors of X_{t+h} at time t for horizons $h = 1, 12, 13$;

  ii) give the variances of the corresponding prediction errors.

[24 marks]

B6. Let $\{X_t\}$ be a stationary time series with $\mathrm{E}\,X_t = 0$.

a) (12 marks) For any integer $j \geq 1$ consider the best linear predictor of X_t in terms of $X_{t-1}, \dots, X_{t-j}$. Denote by v_j the variance of the corresponding prediction error, and by $\phi_1^{(j)}, \dots, \phi_j^{(j)}$ the coefficients of $X_{t-1}, \dots, X_{t-j}$, respectively.

i) Derive the Yule-Walker equations relating the coefficients $\phi_1^{(j)}, \dots, \phi_j^{(j)}$ and v_j to the autocovariances $\gamma_0, \gamma_1, \dots, \gamma_j$.

ii) Write down the Yule-Walker systems for the particular cases $j = 1$ and $j = 2$.

iii) What name is usually used to refer to the coefficients $\phi_1^{(1)}, \phi_2^{(2)}, \dots, \phi_j^{(j)}, \dots$?

b) (12 marks) Assume further that the first two theoretical partial autocorrelations of $\{X_t\}$ are $\beta_1 = 0.8$ and $\beta_2 = -0.25$, respectively.

For each model below state, giving your reasons, whether or not it is consistent with the provided information about $\{X_t\}$. *Hint: Aim to avoid unnecessary calculations, when possible, by referring to properties you have studied.*

i) $X_t = 0.8X_{t-1} + \varepsilon_t$

ii) $X_t = -0.8X_{t-1} + \varepsilon_t$

iii) $X_t = \varepsilon_t + 0.8\varepsilon_{t-1}$

iv) $X_t = \varepsilon_t - 0.8\varepsilon_{t-1}$

v) $X_t = X_{t-1} - 0.5X_{t-2} + \varepsilon_t$

vi) $X_t = X_{t-1} - 0.25X_{t-2} + \varepsilon_t$

vii) $\{X_t\}$ is a random walk.

[24 marks]

B7. Let $\{X_t\}$ be an ARMA(2,1) process specified by the equation

$$X_t = X_{t-1} - \tfrac{1}{2}X_{t-2} + \varepsilon_t + \tfrac{1}{2}\varepsilon_{t-1}, \tag{1}$$

where $\{\varepsilon_t\}$ is $\mathrm{WN}(0,1)$. For any integer k, let $R_{X\varepsilon}(k) = \mathrm{E}(X_t\varepsilon_{t-k})$ and $\gamma_k = \mathrm{E}(X_tX_{t-k})$.

a) (4 marks) Show that the model given by Equation (1) is invertible.

b) (8 marks) Obtain a linear system of equations with unknowns $\gamma_0, \gamma_1, \gamma_2$ and $R_{X\varepsilon}(1)$.

c) (8 marks) Given that $\gamma_0 = \frac{23}{5}$, $\gamma_1 = \frac{17}{5}$, $\gamma_2 = \frac{11}{10}$, find $R_{x\varepsilon}(k)$ for $k = 0, 1, 2$, and all negative k. Give a formula that may be used to compute $R_{X\varepsilon}(k)$ for $k > 2$.

d) (4 marks) An ARMA(2,1) model was fitted to a series of length 100 with the following results.

```
> ts.sim.fit <- arima(ts.sim,order=c(2,0,1))
> ts.sim.fit

Call:
arima(x = ts.sim, order = c(2, 0, 1))

Coefficients:
         ar1      ar2     ma1  intercept
      0.9272  -0.3879  0.6286     4.3474
s.e.  0.1211   0.1160  0.1177     0.3387

sigma^2 estimated as 0.9301:  log likelihood = -139.53,  aic = 289.07
```

Using the given estimates of the parameters, obtain an estimate for the mean of the time series.

[24 marks]

END OF EXAMINATION PAPER

Two Hours

Mathematical formula books and statistical tables are to be provided

THE UNIVERSITY OF MANCHESTER

GENERALISED LINEAR MODELS

31 May 2017

9:45 – 11:45

Answer ALL TWO questions in Section A.
Answer TWO of the THREE questions in Section B.

If more than TWO questions from section B are attempted
then credit will be given to the best TWO answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** of the two questions

A1.

(a) For a distribution in the exponential family, the probability density/mass function can be written as

$$f(y;\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\},$$

where $a(\cdot)$, $b(\cdot)$ and $c(\cdot,\cdot)$ are given functions, θ and ϕ are parameters. State without proof Property 1 which relates the mean and variance of the distribution to the first and second derivatives of $b(\theta)$.

[2 marks]

(b) Show that the geometric distribution with probability mass function

$$P(Y = y) = (1-p)^y p, \ y = 0, 1, 2, \ldots$$

belongs to the exponential family, where $p \in [0, 1]$ is an unknown probability value.

[4 marks]

(c) Derive the mean and variance of the distribution in (b) using your answers in (a).

[4 marks]

(d) State the three components of a generalised linear model and give the definition of the canonical link.

[4 marks]

(e) Derive the canonical link for a generalised linear model when the response variable follows the distribution in (b).

[2 marks]

(f) By differentiating the log-likelihood function of $y_1, \ldots, y_n$ following a generalised linear model, show that the Fisher scores are

$$\frac{\partial \ell}{\partial \beta_j} = \frac{1}{a(\phi)} \sum_{i=1}^{n} (y_i - \mu_i) x_{ij}, \ j = 1, \ldots, p$$

when the canonical link is used, where $x_{i1}, \ldots, x_{ip}$ are values of the predictors corresponding to y_i, μ_i is the mean response, $i = 1, \ldots, n$.

[4 marks]

[20 marks in total for this question.]

A2.

(a) Write down a generalised linear model with probit link for binomial responses $y_i \sim B(n_i, \pi_i)$ on a single explanatory variable x with values x_i, $i = 1, \ldots, n$, and state any further assumptions on $y_1, \ldots, y_n$. An intercept or constant term should be included in the model.

[3 marks]

(b) The Fisher information matrix $I(\underset{\sim}{\beta})$ based on $y_1, \ldots, y_n$ in (a) is given by

$$I(\underset{\sim}{\beta}) = X^T W X.$$

where $\underset{\sim}{\beta}$ is the vector of regression coefficients and X is the data matrix (X^T being its transpose).

(i) Describe the matrix W and show that its diagonal elements are $n_i\pi_i(1 - \pi_i)$, $i = 1, \ldots, n$.

[3 marks]

(ii) Show that

$$I(\underset{\sim}{\beta}) = \sum_{i=1}^{n} n_i\pi_i(1 - \pi_i) \begin{pmatrix} 1 & x_i \\ x_i & x_i^2 \end{pmatrix}.$$

You may use the expression of $I(\underset{\sim}{\beta})$ in terms of X and W without proof.

[4 marks]

(iii) What can $I(\underset{\sim}{\beta})$ be used for and how to use it?

[3 marks]

(c) Let

$$\underset{\sim}{\xi} = X\underset{\sim}{\beta} + W^{-1}(\underset{\sim}{y} - \underset{\sim}{\mu}),$$

where $\underset{\sim}{y} = (y_1, \ldots, y_n)^T$ and $\underset{\sim}{\mu}$ is the corresponding mean vector.

(i) Write down the mean vector and variance-covariance matrix of $\underset{\sim}{\xi}$.

[3 marks]

(ii) Write down the updating equation for the iteratively re-weighted least squares (IRLS) algorithm and explain the role of $\underset{\sim}{\xi}$ in updating the values of $\underset{\sim}{\beta}$.

[4 marks]

[20 marks in total for this question.]

SECTION B

Answer **TWO** of the three questions

B1. The Gamma distribution is given by the probability density function

$$f(y;\ \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y},\ y > 0,$$

where α, $\lambda > 0$ are parameters and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \mathrm{d}x$.

(a) Express the Gamma density in exponential family form, putting $1/\alpha$ on the denominator. Identify the natural parameter θ.

[3 marks]

(b) Find the mean μ in terms of α and λ and find the variance function $V(\mu)$.

[4 marks]

(c) A generalised linear model with Gamma response, linear predictor $\eta = \beta_0 + \beta_1 x + \beta_2 x^2$ and reciprocal link was fitted to $n = 35$ observations on (x, y). The deviance was 44.28 for the fitted model.

(i) Give a reason for the reciprocal link to be used.

[2 marks]

(ii) The parameter α is assumed to be unknown but identical for all the observations 1 to 35. Estimate α using the deviance.

[4 marks]

(iii) Can the deviance be used to check the adequacy of the fitted model? Give reasons for your answer.

[2 marks]

(iv) The deviance was 48.72 for the reduced model with $\eta = \beta_0 + \beta_1 x$. Compare the two models by testing appropriate hypotheses at the 5% level.

[5 marks]

[20 marks in total for this question.]

B2. For each of three insecticides A, B and C, batches of fifty insects were exposed to six different dosages and the numbers of insects killed after exposure for a week were recorded. Two generalised linear models were fitted to the data, with the natural logarithm of dosage (Ldosage) as a covariate and the insecticide as a factor.

```
> fit1<-glm(y ~ Insecticide * Ldosage, family=binomial)
> anova(fit1,test="Chisq")

  Analysis of Deviance Table
  Terms added sequentially (first to last)

                       Df  Deviance Resid.  Df   Resid. Dev  P(>|Chi|)
  NULL                                      17     407.13
  Insecticide           2     169.53        15     237.60    1.541e-37
  Ldosage               1     214.92        14      22.68    1.161e-48
  Insecticide:Ldosage   2       3.39        12      19.29    0.18

> fit2<-glm(y ~ Insecticide + Ldosage, family=binomial)
> summary(fit2)

  Coefficients:
                 Estimate  Std. Error  z value  Pr(>|z|)
  (Intercept)    -4.4541     0.3575    -12.460  < 2e-16
   InsecticideB   0.6144     0.1999      3.074  0.00211
   InsecticideC   3.0314     0.2521     12.022  < 2e-16
   Ldosage        2.6938     0.2146     12.551  < 2e-16
```

(a) Is fit1 the full or saturated model? Write down the saturated model in mathematical/statistical terms and specify any constraints on the parameters.

[2+4 marks]

(b) Does the model fit1 provide adequate fit for the data? Use the 5% level for your test.

[4 marks]

(c) Can the model fit1 be simplified without causing a significant change in deviance? Test appropriate hypotheses at the 5% level.

[4 marks]

(d) Based on the summary of fit2, for each insecticide compute the fitted probability for an insect to be killed by a 3mg dosage (you are given that $\log 3 = 1.0986$).

[3 marks]

(e) Based on the summary of fit2, give the median lethal dose (LD50) for each insecticide.

[3 marks]

[20 marks in total for this question.]

B3.

(a) Define the multinomial distribution for the cell counts of a 2-way contingency table where the cell probabilities are π_{ij}, $i = 1, \ldots, I$, $j = 1, \ldots, J$, and the total is n.

[3 marks]

(b) Show that apart from an additive constant, the log-likelihood of the data in (a) is the same as that of the same data (so total $= n$) under a Poisson model with parameters $\lambda_{ij} = n\pi_{ij}$, $i = 1, \ldots, I$, $j = 1, \ldots, J$, and independence between observations.

[5 marks]

(c) A random sample of 100 motor vehicles stopped at roadside were classified according to type and mechanical condition, with the following results:

Mechanical	Vehicle Type		
Condition	Car	Lorry	Van
Good	17	8	7
Satisfactory	19	17	5
Poor	12	4	11

(i) Use the summary below to test independence or lack of association between vehicle type and mechanical condition at the 5% level.

```
> summary(fit)
Call: glm(formula = y ~ type + condition, family = poisson)

Coefficients:
                       Estimate Std. Error z value Pr(>|z|)
(Intercept)              2.7318     0.2051  13.316  < 2e-16 ***
typelorry               -0.5039     0.2352  -2.143  0.03215 *
typevan                 -0.7357     0.2536  -2.901  0.00372 **
conditionpoor           -0.1699     0.2613  -0.650  0.51559
conditionsatisfactory    0.2478     0.2359   1.051  0.29340

    Null deviance: 23.048  on 8  degrees of freedom
Residual deviance: 10.205  on 4  degrees of freedom
```

[4 marks]

(ii) Can the model in (i) be simplified by removing a nonsignificant variable? Give reasons for your answer.

[4 marks]

(iii) When the mean age, x_{ij}, for vehicle type i and mechanical condition j was added to the model, the deviance decreased by 4.821. Is this sufficient improvement for the new model to provide adequate fit?

[4 marks]

[20 marks in total for this question.]

END OF EXAMINATION PAPER

Two Hours

Statistical tables are attached

UNIVERSITY OF MANCHESTER

MEDICAL STATISTICS

23 May 2017

9:45-11:45

Answer **ALL** five questions in **SECTION A** (40 marks in total).

Answer **TWO** of the three questions in **SECTION B** (40 marks in total). If more than two questions from Section B are attempted, then credit will be given for the two best answers.

The total number of marks for the paper is 80.

Electronic calculators may be used provided that they cannot store text/transmit or receive information/display graphics

---

A1.

(i) In the context of randomised controlled trials, what is meant by the term bias?

(ii) Describe two possible sources of bias in clinical trials.

(iii) Explain what is meant by the term double-blind.

(iv) Describe two ways in which a trial being double blind might reduce bias?

[8 marks]

A2.

A randomised controlled trial is being planned to compare a new treatment (T) and a control treatment (C). Suppose the primary outcome measure is continuous and normally distributed. The power to demonstrate a treatment effect τ with a two-sided two sample t-test is given by the expression $1-\Phi\left(z_{\alpha/2}-\frac{\tau\sqrt{n}}{\sigma\sqrt{2}}\right)$ where σ is the known within-treatment group standard deviation, n is the sample size of each of two equal size groups, α is the significance level, and Φ is the cumulative density function of a standardised normal distribution.

(i) Suppose one wishes to detect a treatment effect of 5 units, and the within treatment group standard deviation has been estimated to be 20 units. Estimate the power of a trial with 160 subjects in each treatment group, assuming a 5% significance level.

(ii) What are the implications of this result for the proposed trial?

[6 marks]

A3.

(i) Illustrate how you might prepare a randomisation list for the first twenty patients in a trial with two treatments using *Block randomisation* with a block size of 4.

(ii) How might you use block randomisation to guarantee balance between two randomly allocated treatment groups for a dichotomous prognostic factor?

[6 marks]

A4.

(i) Explain what is meant by an equivalence trial.

(ii) Outline the statistical analysis one could use in a parallel group trial to establish whether a new treatment T is equivalent to a control treatment C for a continuous normally distributed outcome measure Y.

(iii) A randomised controlled equivalence trial is carried out to test whether a new *generic drug* is as effective as a current *standard drug* for controlling pain. At follow-up this is measured by a 100 mm analogue scale with higher scores representing greater pain. Forty-two patients are randomised to the standard treatment and forty-one to the new generic treatment. The statistical computer package output is given below. A difference of 5 mm was considered by researchers to be the minimum that was clinically important. Using the results in the output, test whether the new generic drug is equivalent to the current standard drug specifying the significance level of the test.

```
Two-sample t test with equal variances
------------------------------------------------------------------------------
         |     Obs        Mean    Std. Err.   Std. Dev.   [90% Conf. Interval]
---------+--------------------------------------------------------------------
Standard |      42        35.2     2.79289        18.1    30.49991    39.90009
Generic  |      41        34.1     2.79551        17.9    29.39278    38.80722
---------+--------------------------------------------------------------------
    diff |                 1.1    3.952132               -5.475889    7.675889
------------------------------------------------------------------------------
   diff = mean(Standard) - mean(Generic)                        t =   0.2783
Ho: diff = 0                                     degrees of freedom =       81

    Ha: diff < 0                 Ha: diff != 0                 Ha: diff > 0
 Pr(T < t) = 0.6093         Pr(|T| > |t|) = 0.7815          Pr(T > t) = 0.3907
```

[10 marks]

A5.

In meta-analysis suppose $\hat{\theta}_i$ is an estimate of the treatment effect for the i^{th} study and let $Var\left[\hat{\theta}_i\right]$ be its sampling variance.

(i) For the weighted estimate of the overall effect, defined by $\hat{\theta} = \dfrac{\sum_i^k w_i\hat{\theta}_i}{\sum_i^k w_i}$ where w_i are weights,

show that $Var\left[\hat{\theta}\right] = \dfrac{\sum_i^k w_i^2 . Var\left[\hat{\theta}_i\right]}{\left(\sum_i^k w_i\right)^2}$.

The table below summarizes the outcome of three randomised trials of a *new antipsychotic drug* compared to a *standard antipsychotic drug* for people with schizophrenia. The treatment effect for each study ($\hat{\theta}_i$, i = 1,2,3) is the difference in mean symptom scores for the two treatments, with lower values representing a benefit of treatment. $\hat{V}ar\left[\hat{\theta}_i\right]$ is the sample variance estimate of the i^{th} study.

Study	*Mean difference in symptoms*,* $\hat{\theta}_i$	$\hat{V}ar\left[\hat{\theta}_i\right]$
Holmes (2000)	2.4	0.5
Bourne (2002)	-2.8	2.5
Bond (2006)	3.0	2.0

* *new drug – standard drug.*

(ii) The minimum variance estimator, $\hat{\theta}_{MV}$, is obtained when $w_i \propto 1/Var\left[\hat{\theta}_i\right]$. For the data in the table above, compute the minimum variance estimate of the overall treatment effect.

(iii) Illustrate the three studies and the overall treatment effect using a sketch of a forest plot.

[10 marks]

B1.

For a parallel group randomised controlled trial comparing a control treatment (C) with a new treatment (T), suppose Y is a continuous normally distributed outcome variable and X is the value of the same variable recorded at the start of the study prior to randomisation. Suppose that τ is the treatment effect such that:

$$Y = \mu_y + \varepsilon_y \quad \text{and} \quad X = \mu_x + \varepsilon_x \quad \text{for} \quad \text{treatment C}$$

$$Y = \mu_y + \tau + \varepsilon_y \quad \text{and} \quad X = \mu_x + \varepsilon_x \quad \text{for} \quad \text{treatment T}$$

with $E\left[\varepsilon_x\right] = E\left[\varepsilon_y\right] = 0$, $Var\left[\varepsilon_y\right] = \sigma_y^2$, $Var\left[\varepsilon_x\right] = \sigma_x^2$, and $Cov\left[\varepsilon_x, \varepsilon_y\right] = \sigma_{xy}$.

Define $D = Y - X$ and suppose that $\overline{x}_T$, $\overline{x}_C$, $\overline{y}_T$, $\overline{y}_C$, $\overline{d}_C$ and $\overline{d}_T$ are the sample means of X, Y and D for each treatment respectively. For any real number theta, define $\hat{\tau}(\theta) = (\overline{y}_T - \theta\overline{x}_T) - (\overline{y}_C - \theta\overline{x}_C)$.

(i) Show that $E\left[\hat{\tau}(\theta)\right] = \tau$ for all theta. [4 marks]

(ii) Show that $Var\left[\hat{\tau}(\theta)\right] = \lambda^2\left(\sigma_y^2 + \theta^2\sigma_x^2 - 2\theta\sigma_{xy}\right)$ where $\lambda = \sqrt{\frac{1}{n_T} + \frac{1}{n_C}}$, n_T is the number of patients allocated to the new treatment and n_C is the number allocated to the control treatment. [7 marks]

(iii) Show that $Var\left[\hat{\tau}(\theta)\right]$ has a minimum when $\theta = \sigma_{xy}/\sigma_x^2$, which is the regression coefficient of Y on X.

[4 marks]

(iv) Three statistical analyses might be used to estimate and test the treatment effect:

a) an unadjusted analysis using just the outcome variable Y
b) an analysis based on the change score Y-X
c) a linear model of the outcome variable Y with treatment group and X as covariates.

What are the implications of the results in (i) and (iii) for the choice between the three analyses in terms of bias and precision of the treatment effect? [3 marks]

(v) Why is it important for randomised controlled trials to have a statistical analysis plan?

[2 marks]

[Total 20 marks]

B2.

For an *AB*/*BA* crossover trial a standard model for a continuous outcome y_{ij} for the i^{th} patient in the j^{th} period is:

$y_{i1} = \mu + \tau + \xi_i + \varepsilon_{i1}$	for a patient in sequence *AB* in period 1
$y_{i2} = \mu + \phi + \xi_i + \varepsilon_{i2}$	for a patient in sequence *AB* in period 2
$y_{i1} = \mu + \xi_i + \varepsilon_{i1}$	for a patient in sequence *BA* in period 1
$y_{i2} = \mu + \tau + \phi + \xi_i + \varepsilon_{i2}$	for a patient in sequence *BA* in period 2

where μ is the mean for the sequence *BA* in period 1, τ is the treatment effect of A compared to B, and ϕ is the period effect, with ξ_i and ε_{ij} being two independent random variables with $\xi_i \sim N\left[0,\sigma_B^2\right]$ and $\varepsilon_{ij} \sim N\left[0,\sigma_\varepsilon^2\right]$. Defining $d_i = y_{i2} - y_{i1}$ let $\bar{d}_{AB}$, $\bar{d}_{BA}$, μ_{AB}^d, μ_{BA}^d be the sample and population means of d_i for sequences *AB* and *BA* respectively.

(i) Show that a test of the null hypothesis $H_0 : \mu_{AB}^d = \mu_{BA}^d$ is the same as a test of no treatment effect, $H_0 : \tau = 0$. [3 marks]

(ii) Two anti-cholesterol lowering drugs were compared in a randomised controlled crossover trial. Ten patients were randomly allocated to sequence drug A then drug B, and eight patients were randomly allocated to sequence drug B then drug A. The table below summarizes the sample mean and standard deviation in each period for both of these sequences. Test the hypothesis $H_0 : \tau = 0$ vs. $H_1 : \tau \neq 0$ using a 5% significance level, and compute the 95% confidence interval.

	Period 1		*Period 2*		*Period 2 – Period 1*		
Sequence	*mean*	*s.d.*	*mean*	*s.d.*	*Mean*	*s.d*	*N*
AB	6.42	0.81	6.01	0.72	-0.41	0.52	10
BA	6.23	0.63	6.11	0.71	-0.12	0.43	8

[7 marks]

[continued]

(iii) Define $c_i = y_{i1} - y_{i2}$ for sequence AB and $c_i = y_{i2} - y_{i1}$ for sequence BA. Let μ^c_{AB} μ^c_{BA}, $\bar{c}_{AB}$ and $\bar{c}_{BA}$ be the population and sample means of these for sequences AB and BA respectively. Show that a test of the null hypothesis $H_0 : \mu^c_{AB} = \mu^c_{BA}$ is the same as a test of the period effect, $H_0 : \phi = 0$.

[3 marks]

(iv) From the data in the table above, test the null hypothesis $H_0 : \phi = 0$ vs. $H_1 : \phi \neq 0$ using a 5% significance level.

[5 marks]

(v) Briefly comment on the result of the trial [2 marks]

[Total marks 20]

B3.

Consider a randomised controlled trial. Suppose the patient population can be divided into three sub-groups as follows:

Compliers:	patients who will comply with the allocated treatment
Always control treatment:	patients who will receive control treatment regardless of allocation
Always new treatment:	patients who will receive the new treatment regardless of allocation.

This division assumes that there are no defiers, that is patients who will always receive the opposite of the treatment to which they are randomised. Assume that the proportion and characteristics of *Compliers*, *Always control treatment*, *Always new treatment* is the same in both arms and that randomization can only affect the outcome through the receipt of treatment.

(i) Show that an *Intention-To-Treat* estimate of the treatment effect is biased towards the null hypothesis of no treatment effect. [4 marks]

(ii) Show that an *As-Treated* estimate of the treatment effect may be biased either towards or away from the null hypothesis of no treatment effect. [5 marks]

The table below summarizes the outcome of patients from randomised controlled trial comparing two treatments according to randomized group and treatment received. Some patients allocated to the *New* treatment received the *Control* treatment and some patients allocated to *Control* treatment received the *New* treatment.

	Randomised Group			
	New Treatment		*Control*	
Recovered after 12 weeks	*Received New*	*Received Control*	*Received New*	*Received Control*
Yes	360	72	48	360
No	120	48	12	180
Total	480	120	60	540

[continued]

(iii) For the outcome measure *Recovered after 12 weeks*, calculate the point estimate of the difference in proportions between the *New* treatment compared to the *Control* treatment assuming

(a) an *Intention-To-Treat* analysis

(b) an *As-Treated analysis*. [3 marks]

(iv) Explain why an *Intention-To-Treat* estimate is preferable to an *As-Treated* estimate in a superiority trial. [3 marks]

(v) Calculate the point estimate of the C*omplier Average Causal Effect* of *New* treatment compared to *Control*. [5 marks]

[Total 20 marks]

END OF EXAMINATION PAPER

2 hours

Tables of the cumulative distribution function are provided.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL MODELLING IN FINANCE

18 May, 2017

2pm – 4pm

Answer **all** 4 questions in **Section A** (60 marks in all) and **2** of the 3 questions in **Section B** (20 marks each). If all three questions from Section B are attempted then credit will be given for the two best answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** 4 questions

A1. Consider a portfolio comprising European call options on the same underlying asset and same expiration date. In particular, this comprises long two options C_{20} with exercise price \$20, short three options C_{30} with exercise price \$30 and long one option C_{50} with strike \$50. In addition, consider P_{50}, the corresponding put option with exercise price \$50. Show that for all times prior to expiry

$$P_{50} \geq 2C_{20} - 3C_{30} + C_{50}.$$

[15 marks]

A2. This question concerns the calculation of delivery prices:

(i) What is the delivery price for a futures contract on a stock S, currently with value \$10, for delivery in 9 months time; the (constant) risk-free rate is 5% p.a.? Justify your answer.

(ii) What is the delivery price for a futures contract on a stock S, currently with value \$25, for delivery in 1 year's time; the (constant) risk-free rate is 2.5% p.a. for the first 6 months, and 4% p.a. thereafter? Justify your answer.

(iii) What is the delivery price for a futures contract on 1 kilo of gold, currently with value £31500, for delivery in 6 months time; the (constant) risk-free rate is 4% p.a. and the cost of storing/securing (paid by the owner of the gold) the metal is 10% p.a. of its (current) value? Justify your answer.

[15 marks]

A3. Draw the expiry payoff diagrams for each of the following portfolios:

(i) Long two shares, short one put with an exercise price X.

(ii) Long one share, short two puts both with exercise price X_1, long one call with exercise price X_2. Consider each of the cases $X_1 > X_2$, $X_1 = X_2$, $X_1 < X_2$.

[15 marks]

A4. Consider the Black-Scholes equation for an option, V, on a stock S which pays no dividends, namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

in the usual notation, where r and σ are both constants.

(i) By neglecting the time derivative in the above equation, seek solutions of the form

$$V(S) = A_1 S^{\alpha_1} + A_2 S^{\alpha_2},$$

where A_1 and A_2 are constants, and $\alpha_1 > \alpha_2$. Determine the values of α_1 and α_2.

(ii) Consider a perpetual American option $V(S)$, i.e. an option with no expiry date but where it is possible at any point in time to exercise, which has a (non-standard) exercise value of $e^{-\lambda S}$, where λ is a constant. This can be valued using the time independent solutions in (i) above. The exercise boundary is denoted by S_f. State and justify the two conditions at $S = S_f$ and the other appropriate boundary condition.

(iii) Determine S_f.

[15 marks]

SECTION B

Answer **2** of the 3 questions

B5. The change in price of an asset S satisfies the stochastic differential equation

$$dS = a(S,t)dt + b(S,t)dX$$

where a and b are given functions of S and t and dX is sampled from a Brownian motion of mean zero and variance dt.

(i) Using Ito's Lemma and delta hedging, derive the equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}b^2(S,t)\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

for the fair price of an option $V(S,t)$, where r is a constant risk-free interest rate.

(ii) If $b = \sigma(t)S$ where $\sigma(t)$ is a known function of time, find $V(S,t)$ for $t < T$ for a payoff $V(S,T) = S^2$. You may assume $V(S,t) = S^2 f(t)$, and you may leave your answer in terms of an integral of $\sigma(t)$.

(iii) If $b = \sigma_1$ where σ_1 is a constant, find $V(S,t)$ for $t < T$ for a payoff $V(S,T) = S^2$. You may assume $V(S,t) = S^2 f_1(t) + f_2(t)$ (where $f_1(t)$ and $f_2(t)$ are to be fully determined).

[20 marks]

B6.

(i) Suppose that an asset of value S pays out a dividend $DSdt$ in time dt (i.e. a constant dividend yield D). Given that an option $V(S,t)$ on this asset satisfies the modified Black-Scholes equation (in the usual notation)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0,$$

show that the substitution

$$V(S,t) = e^{-D(T-t)}V_1(S,t)$$

results in the standard Black-Scholes equation, but with a modified interest rate (which you are to determine).

(ii) You are given that the solution for a vanilla (non-dividend paying) European call option is

$$C(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log(S/X) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\log(S/X) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and X is the exercise price and $N(x)$ is the cumulative distribution function. Write down the corresponding result for a call option with a dividend yield of D.

(iii) Consider a stock whose price today (9 months from the expiration of an option) is £50, the exercise price of a European call option on this stock is £51, the risk-free interest rate is 1.5% per annum (fixed) and the volatility (constant) is 20% per $(\text{annum})^{\frac{1}{2}}$. The stock pays a continuous dividend at a rate 5% of the value of the stock (per annum).

What is today's value of a European call option on this asset? Values of the cumulative distribution function may be determined by interpolation of the provided tables.

(iv) Given that the put-call parity relationship for non-dividend, vanilla European options is

$$C + Xe^{-r(T-t)} - P - S = 0,$$

write down the corresponding relationship between call and put (C_d and P_d, respectively) options on a continuous dividend paying asset, with a yield D.

[20 marks]

B7. Consider an option $V(S_1, S_2, t)$ where the two underlyings S_1 and S_2 have constant volatilities σ_1 and σ_2 respectively, and drifts μ_1, μ_2 respectively; the risk-free interest rate is r.

(i) Justify the use of the following stochastic processes to model the movement of stocks/shares:

$$dS_i = \mu_i S_i dt + \sigma_i S_i dW_i$$

where the dW_i are Wiener processes, such that

$$E[dW_i^2] = dt, \quad E[dW_1 dW_2] = \rho dt.$$

(ii) By considering a portfolio (using the dS_i in (i) above)

$$\Pi = V(S_1, S_2, t) - \Delta_1 S_1 - \Delta_2 S_2,$$

find the choices of Δ_1 and Δ_2 for which the portfolio is perfectly hedged.

(iii) Equating the hedged portfolio in (ii) above to the risk-free return on the portfolio, show that the option value is determined from

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV = 0.$$

(iv) Consider now the case of an option whose payoff at expiry ($t = T$) is

$$V(S_1, S_2, t = T) = \max(q_1 S_1 - q_2 S_2, 0),$$

where q_1 and q_2 are constants. Assuming a solution of the form

$$V(S_1, S_2, t) = q_1 S_2 H(\xi, t),$$

where $\xi = \frac{S_1}{S_2}$, write down all the boundary conditions for $H(\xi, t)$.

(v) If $\rho = 0$, show that H satisfies the PDE

$$\frac{\partial H}{\partial t} + \frac{1}{2}\bar{\sigma}^2 \xi^2 \frac{\partial^2 H}{\partial \xi^2} = 0,$$

where $\bar{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2}$.

[20 marks]

END OF EXAMINATION PAPER

Two hours

Statistical Tables provided

THE UNIVERSITY OF MANCHESTER

ACTUARIAL MODELS 2

26 May 2017
0945 - 1145

Answer ALL FOUR questions.
The total number of marks in the paper is 90.

University approved calculators may be used.

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1. Consider two homogeneous groups of individuals, group a and group b. The hazard function of the survival time of an individual from group a, respectively b, is denoted by $\mu_a(t)$, respectively $\mu_b(t)$. Assume that all the individuals are independent and that there is independent (right-)censoring and no truncation. Suppose we have the following survival data for the two groups, where + denotes a censored survival time:

Group a	16+	28	30+	40	42+	43+
Group b	9	12+	20	25	32	

(a) Explain why the sample mean of the survival data corresponding to group a (i.e. $\frac{16+28+30+40+42+43}{6}$) is not a sensible estimate for the mean of the survival time of a member from group a.

[3 marks]

(b) Assume that $\mu_a(t) = \mu_b(t)$ for all $t \geq 0$ and consider all the individuals from both groups together as one aggregate group of 11 individuals. Use the data of all individuals combined to calculate the Kaplan-Meier estimate of the survival function of the survival time of an arbitrary individual from this aggregate group and plot your estimate (you should show the details of your calculations).

[9 marks]

(c) Assuming the same setting as in part (b), estimate, by using the Kaplan-Meier estimate, the median of the survival time of an arbitrary individual from the aggregate group.

[4 marks]

(d) Assuming the same setting as in part (b), estimate, by using the Kaplan-Meier estimate, the probability that an arbitrary individual from the aggregate group dies between times 27 and 35 given that he/she is still alive at time 15.

[4 marks]

(e) We now want to check whether the assumption that $\mu_a(t) = \mu_b(t)$ for all t in a suitable time region, is appropriate. To this end, we want to test the null hypothesis, for a given t_0,

$$H_0 : \mu_a(t) = \mu_b(t) \quad \text{for all } t \in [0, t_0],$$

versus $H_A : \mu_a(t) \neq \mu_b(t)$ for some $t \in [0, t_0]$. The Gehan-Breslow test says to reject H_0 at significance level α if $|Z_{t_0}/\sqrt{V_{t_0}}| > z_{\alpha/2}$, where

- $Z_{t_0} = \int_0^{t_0} R_s^a R_s^b \mathrm{d}\left(\widehat{A}_a(s) - \widehat{A}_b(s)\right)$ with R_t^j denoting the number of individuals at risk just before time t in group j and $\widehat{A}_j(t)$ denoting the Nelson-Aalen estimator at time t of the cumulative hazard function of an individual from group j;
- $V_{t_0} = \int_0^{t_0} R_s^a R_s^b \mathrm{d}\left(N_s^a + N_s^b\right)$ with N_t^j denoting the number of individuals in group j that are observed to have failed before or at time t;
- $z_\alpha > 0$ is such that $\Phi(z_\alpha) = 1 - \alpha$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Given the survival data at the beginning of the question, carry out the Gehan-Breslow test with $t_0 = 43$ at significance level 0.05 and report your conclusions.

[9 marks]

(f) With the notation as in part (e), show that under the null hypothesis H_0, $\mathbb{E}[Z_{t_0}] = 0$ for $t_0 = 43$. You can use here without proof that for $j \in \{a, b\}$, the process $\{M_t^j : t \geq 0\}$ given by $M_t^j = N_t^j - \int_0^t R_s^j \mu_j(s) \mathrm{d}s$, is a martingale (w.r.t. the filtration $\{\mathcal{F}_t : t \geq 0\}$, where $\mathcal{F}_t$ consists of the information on the survival and censoring times of all individuals in both groups up to time t).

[5 marks]

[Total 34 marks]

2. Consider the following observed values of the survival times of four independent homogeneous individuals, where + denotes a censored value:

$$4 \quad 8 \quad 10+ \quad 10+$$

Assume the following parametric form of the common hazard function of the individuals:

$$\mu(t) = \mu(t; \gamma, \beta) = \beta \mathrm{e}^{\gamma t}, \tag{1}$$

with $\beta > 0$ and $\gamma \in \mathbb{R}$. Assume further that the times at which individuals are censored are all deterministic quantities and do not depend on the parameters β and γ.

(a) Derive, from first principles, an explicit expression (in term of β and γ) for the log-likelihood function of the parameters β and γ given the observations. (Note that directly applying the (log-)likelihood formula from the notes is not sufficient for getting full marks.) Here any constants that do not effect the maximum likelihood estimation can be ignored.

[8 marks]

(b) Describe how one can graphically check whether the choice of the parametric form of the hazard function as given in (1) is an appropriate one. (Note that you do not have to carry out this test.)

[5 marks]

[Total 13 marks]

3. A statistician wants to compare two different football training methods to see which method is better for preventing injuries. To this end a number of players who train either under method A or under method B and are currently without injury, are observed for a period during the football season. The survival time of interest is the number of matches played until the occurrence of an injury. The table below contains the survival times of eight players, four who trained under method A and four who trained under method B. Here some of the survival times are left-truncated and some of them are right-censored; the latter ones are indicated by a +.

Training method A	$(0, 8]$	$(10, 18+]$	$(6, 20+]$	$(2, 25]$
Training method B	$(0, 4]$	$(6, 10]$	$(5, 12+]$	$(8, 14]$

(a) Explain, in the context of this question, what it means if the survival time of a player is left-truncated.

[3 marks]

(b) Consider a Cox proportional hazards model for comparing the number of matches played until injury of the two groups of players. Assume here that the hazard function for the players with training method B divided by the hazard function for the players with training method A, equals e^{β} at any time, where $\beta \in \mathbb{R}$.

Derive an explicit expression in terms of β for the partial likelihood in this model.

[9 marks]

(c) Suppose you want to test $H_0 : \beta = 0$ versus $H_A : \beta \neq 0$, where β is as in part (b). Explain what this test means in the context of this question. (Note that you do not have to carry out this test.)

[3 marks]

In Figures 1-3 below a plot of respectively the partial likelihood L_P, the partial log-likelihood ℓ_P and the derivative of the latter is given. Use these plots for the next two questions.

(d) Estimate the relative risk of a player with training method B in comparison to a player with training method A.

[3 marks]

(e) Carry out the Wald test at significance level 0.05 to test whether or not training method A is better than method B for preventing/delaying injuries. Report your conclusions.

[5 marks]

[Total 23 marks]

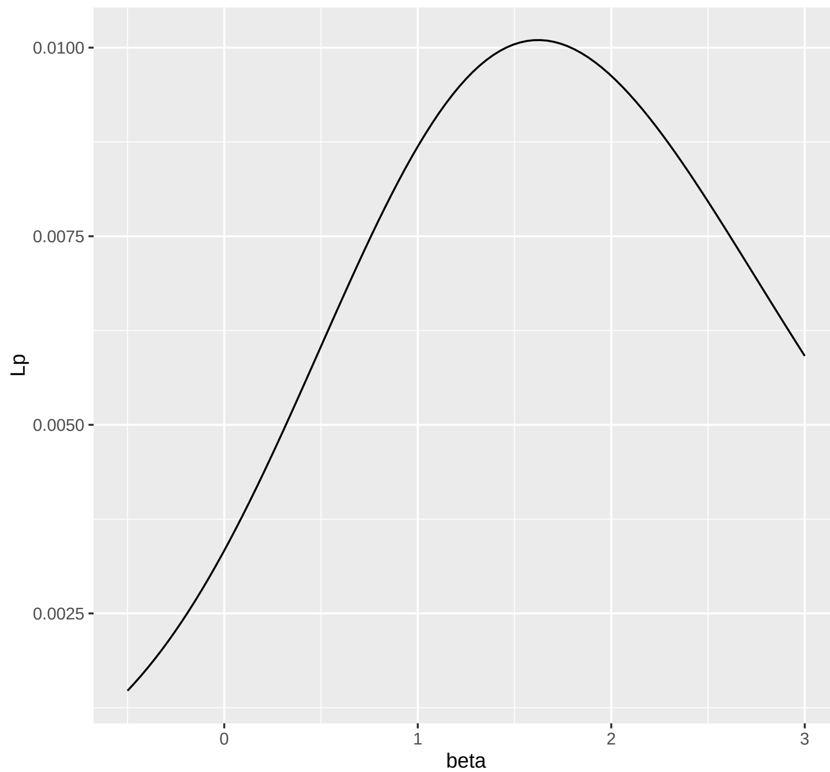

Figure 1: Plot of $L_P(\beta)$ against β corresponding to Question 3.

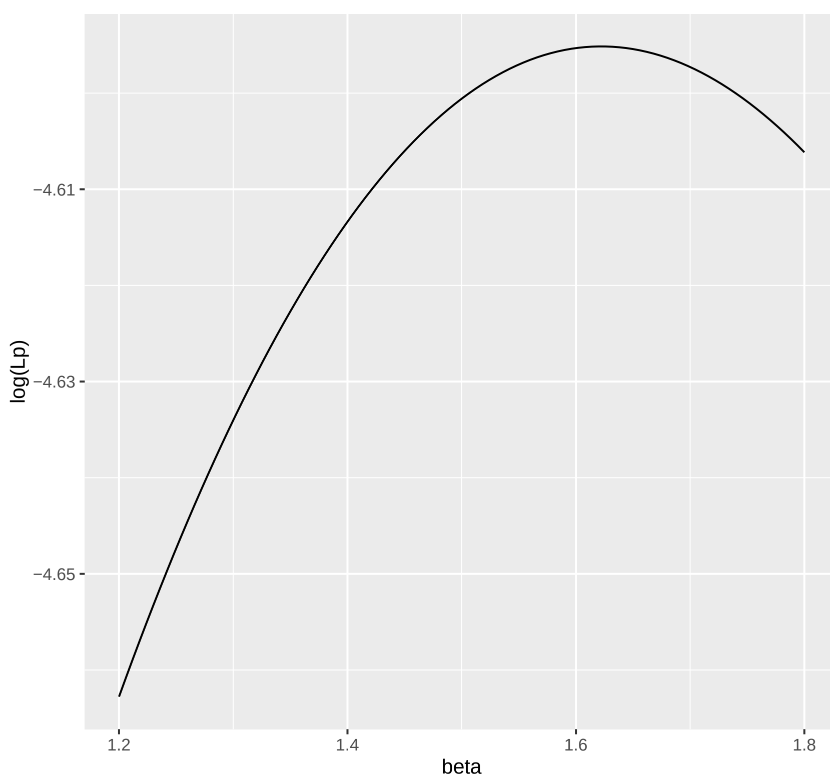

Figure 2: Plot of $\ell_P(\beta)$ against β corresponding to Question 3.

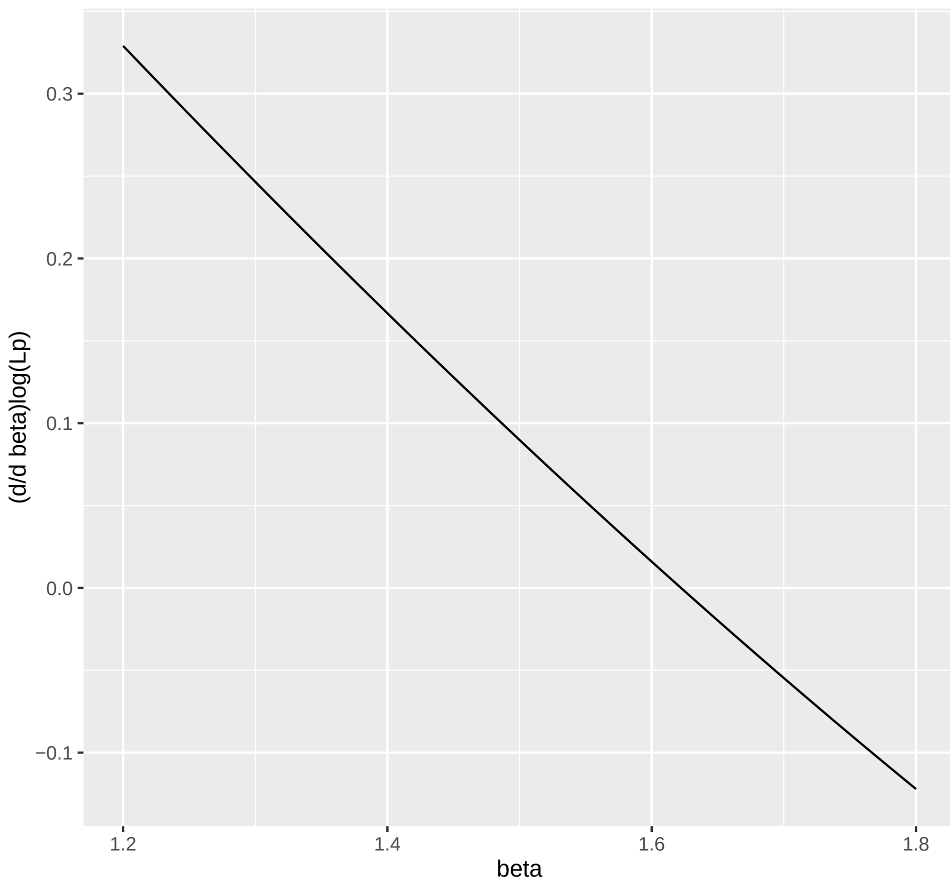

Figure 3: Plot of $\frac{\mathrm{d}}{\mathrm{d}\beta}\ell_P(\beta)$ against β corresponding to Question 3.

4. Assume a proportional frailty model for the mortality of a certain population where the individual hazard function is given by $\mu_1(t|Z) = 2tZ$ and the frailty random variable Z has a probability density distribution given by

$$f_Z(z) = 3\left(\mathrm{e}^{-1.5z} - \mathrm{e}^{-3z}\right), \quad z > 0.$$

(a) Show that the population hazard function is given by

$$\mu_1(t) = 2t\frac{2t^2 + 4.5}{(t^2 + 1.5)(t^2 + 3)}, \quad t \geq 0.$$

[7 marks]

(b) Explain what is meant by a "crossover phenomenon" when comparing the hazard functions of two populations.

[3 marks]

(c) Consider a second population whose mortality is also modelled by a proportional frailty model but where now the individual hazard function is given by $\mu_2(t|Y) = 4tY$ and the frailty random variable Y is exponentially distributed with parameter 1.

Show that there is a crossover phenomenon when comparing these two populations. (Hint: show first that the population hazard of the second population is given by $\mu_2(t) = \frac{4t}{2t^2+1}$, $t \geq 0$.)

[7 marks]

(d) Explain why this example forms an illustration of Simpson's paradox.

[3 marks]

[Total 20 marks]

END OF EXAMINATION PAPER

2 Hours

UNIVERSITY OF MANCHESTER

CONTINGENCIES 2

2 June 2017

1400 - 1600

Answer ALL NINE questions
The total number of marks in the paper is 100.

University approved calculators may be used.

Please note the following before you get started.

- You can find a formula sheet on the last page of this exam paper.
- The marking is based on your answer booklet only, so make sure to write down your working there.

1. In PFA92C20, express the probabilities below in terms of single life probabilities and then calculate their value (in all cases both lives are female):

a) $p_{65:60}$

[2 marks]

b) ${}_5q_{60:55}$

[3 marks]

c) ${}_5p_{\overline{60:55}}$

[3 marks]

d) ${}_{5|}q_{\overline{60:55}}$

[5 marks]

[Total 13 marks]

2. An insurance company issues a policy to two male lives both aged 30 exact. We will refer to the two people as *person 1* and *person 2* to distinguish between them. You have been asked to calculate the net premium for two possible policies.

In both cases, assume interest at 4% p.a. and mortality of PMA92C20 for both lives. Also use the standard approximation $\bar{A}_u \approx (1+i)^{0.5} \cdot A_u$ to convert from an assurance payable at the end of the year in which the status u fails to an assurance that pays out immediately on the failure of status u.

a) Assume the policy pays out £20,000 immediately on the second death of the two lives. Premiums are payable annually in advance until the first death occurs.

Calculate the net premium.

[8 marks]

b) In the second case assume the policy pays out a sum of £15,000 immediately on the death of *person 1* but only if he dies first, or a payment of £5,000 immediately on the death of *person 1* if he dies second. Premiums continue to be paid annually in advance until the first death occurs.

Calculate the net premium.

[4 marks]

[Total 12 marks]

3. Consider a 3 state state sickness model with states a, able, i, ill and d, dead. Transition intensities are:

- from a to i at age x : μ_x^{ai},
- from a to d at age x : μ_x^{ad},
- from i to a at age x : μ_x^{ia},
- from i to d at age x : μ_x^{id}.

a) Write down the expected present value (EPV) for a whole of life lump sum death benefit of £10,000 payable immediately on death for a life aged x in the able state.

[4 marks]

b) Another policy also pays out a lump sum on death but the amount payable varies depending on the circumstances. In the event of death having never been ill, the payment is £20,000. In all other circumstances the payment on death is £10,000.

Write down the EPV of these revised death benefits, again in integral form for a life aged x in the able state.

[6 marks]

[Total 10 marks]

4. Consider a 3 state multiple decrement model with states a, active, r, retired and d, dead.

a) Calculate the dependent probabilities of dying and of retiring when aged 60 and 61, where the independent probabilities of death are $q^d_{60} = 0.008022$ and $q^d_{61} = 0.009009$ and the force of retirement $\mu^{ar}_x = 0.1$ for all $x > 0$.

[10 marks]

b) Now consider the introduction of a fourth state, withdrawal, in our model so that the model remains a multiple decrement model. If withdrawal is assumed to occur only at each year end and the independent probability of withdrawal is 0.1 for all $x > 0$, calculate the dependent probability of a person aged 60 exact dying when aged 61.

[5 marks]

[Total 15 marks]

5. Use the pension scheme service table to calculate the following:

a) The probability that a life aged 40 exact withdraws from the pension scheme in the following year.

[3 marks]

b) The probability that a life aged 50 exact retires in normal health at any time between the ages of 60 and 65 inclusive.

[3 marks]

c) The estimated salary earned in the 12 month period through to retirement when aged 65 exact for a person now aged 30 exact with a salary in the previous year of £15,000.

[3 marks]

[Total 9 marks]

6. A pension scheme provides a lump sum benefit on retirement which is calculated as follows:

$$\text{Lump sum payable on retirement at age } x = 0.05 \times FPS \text{ at } x \times N.$$

where:

- FPS is an estimate of the average salary earned over the three years prior to retirement and
- N is the length of service in years and part years, and is calculated from the date of joining the pension scheme through to the estimated date of retirement.

Use the pension service table to:

a) Write down a vector showing an estimate of the amount of lump sum payable on retirement when aged 62 and all subsequent ages through to normal retirement at age 65 inclusive for a person now aged 50 exact and who joined the pension scheme 25 years ago. The person has a salary of £20,000 earned in the previous year.

[8 marks]

b) Write down the vector for the expected present value (EPV) of a lump sum payment of £1 on retirement when aged 62 and all subsequent ages through to normal retirement at age 65 inclusive. Then calculate the EPV of the benefits calculated in part a). Use an interest rate of 4% p.a.

[8 marks]

[Total 16 marks]

7. An insurer writes a 2 year unit linked endowment policy for a life aged 50. Level premiums are paid yearly in advance. Benefits are paid at the end of the year.

Construct a table setting out the development of the unit fund each year and then a table showing the cash flows each year for the non unit fund. Briefly show your workings so that the derivation of each item in the tables can be seen and state what the profit vector for the policy is.

Details are as follows:

- Premium = £ 2,000 p.a.
- Allocation is 50% in year 1 and 105% in year 2
- Bid offer spread = 5%
- Management charges 0.5% deducted from the unit fund at the end of the year, before any claim payments are made.
- Initial expenses are 15% of first premium and £ 500 on receipt of the first premium.
- Renewal expenses are 1% of the second premium and £ 50 on payment of this premium.

- Dependent probability of death = 0.01 in year 1 and 0.012 in year 2 and the dependent probability of withdrawal is 0.1 in each year.
- A minimum sum assured of £3,000 is payable on death or on maturity of the policy (both payable at the policy year end).
- A surrender value of 75% of premiums paid is made at the policy year end.
- Assumed rate of return of 5% p.a. on the unit fund and 3% p.a. on the non-unit fund.

[Total 15 marks]

8. Give two reasons why insurance companies zeroise their reserves.

[Total 4 marks]

9. Define what temporary initial selection and class selection are and give an example of each in the area of life assurance.

[Total 6 marks]

Formula sheet Contingencies 2

1. Compound interest

i is the interest rate per annum, $v = (1+i)^{-1}$ and $\delta = \log(1+i)$

$a_{\overline{n}|} = v + v^2 + \ldots + v^n$ is the present value of an annuity certain in arrears with term n years

2. Mortality tables and laws

l_x is the number of lives at age x

$d_x = l_x - l_{x+1}$ is the number of lives who die between ages x and $x+1$

${}_tp_x$ is the probability that a life aged x will survive at least t years and ${}_tq_x$ is the probability that a life aged x will die within t years

${}_{m|n}q_x$ is the probability that a life aged x will die between ages $x+m$ and $x+m+n$

K_x is the curtate future lifetime for a life aged x; $e_x = \mathbb{E}[K_x]$

T_x is the complete (exact) future lifetime for a life aged x; $\mathring{e}_x = \mathbb{E}[T_x]$

μ_x is the force of mortality at age x:

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s} \text{ş}\right)$$

3. Assurances

A_x is the EPV of a whole life assurance that pays at the end of the year of death of a life aged x

$\bar{A}_x$ is the EPV of a whole life assurance that pays at the moment of death of a life aged x

$$\mathbb{E}\left[v^{K_x+1}\right] = A_x;\ \mathrm{Var}\left(v^{K_x+1}\right) = A_x^{i^*} - (A_x)^2$$

$$\mathbb{E}\left[v^{T_x}\right] = \bar{A}_x;\ \mathrm{Var}\left(v^{T_x}\right) = \bar{A}_x^{i^*} - (\bar{A}_x)^2$$

(where $i^* = (1+i)^2 - 1$. Similar relationships hold for other assurances.

A_{xy} is the EPV of a whole life assurance that pays at the end of the year in which the first death of two lives aged x and y occurs.

$A_{\overline{xy}}$ is the EPV of a whole life assurance that pays at the end of the year in which the second death of two lives aged x and y occurs.

A^1_{xy} is the EPV of a whole life assurance that pays at the end of the year in which a life aged x dies assuming that a life aged y remains alive at that time.

A^2_{xy} is the EPV of a whole life assurance that pays at the end of the year in which a life aged x dies assuming that a life aged y is already dead at that time.

4. Annuities

a_x is the EPV of an annuity for life in arrears for a life aged x

$\ddot{a}_x = 1 + a_x$ is the EPV of an annuity for life in advance for a life aged x

$\bar{a}_x$ is the EPV of a continuously paying annuity for life for a life aged x

$$\mathbb{E}[\ddot{a}_{\overline{K_x+1}|}] = \ddot{a}_x;\ \mathrm{Var}(\ddot{a}_{\overline{K_x+1}|}) = \frac{A_x^{i^*} - (A_x)^2}{(1-v)^2}$$

$$\mathbb{E}[\bar{a}_{\overline{T_x}|}] = \bar{a}_x;\ \mathrm{Var}(\bar{a}_{\overline{T_x}|}) = \frac{\bar{A}_x^{i^*} - (\bar{A}_x)^2}{\log(v)^2}$$

(where $i^* = (1+i)^2 - 1$. Similar relationships hold for temporary annuities with status ${}_{x:\overline{n}|}$)

a_{xy} is the EPV of an annuity for the joint lifetime of two lives aged x and y, paid in arrears.

$a_{\overline{xy}}$ is the EPV of an annuity payable in arrears so long as at least one of two lives aged x and y is alive.

$a_{x|y}$ is the EPV of a reversionary annuity payable to a life aged y after the death of a life aged x, paid in arrears.

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m};$$

$$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - v^n\, {}_np_x\right).$$

$$A_x = 1 - (1-v)\ddot{a}_x; \quad \bar{A}_x = 1 - \delta\bar{a}_x$$

END OF EXAMINATION PAPER

Two hours

UNIVERSITY OF MANCHESTER

RISK THEORY

24 May 2017

9:45 – 11:45

Answer ALL FIVE questions
The total number of marks in the paper is 90.

University approved calculators may be used

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1. An insurance company is modelling the evolution of the surplus for a portfolio of products by means of a Cramér-Lundberg capital process denoted by $(U_t)_{t\geq 0}$. That is, we have

$$U_t = u + ct - \sum_{k=1}^{N_t} X_k \quad \text{for } t \geq 0$$

where t is measured in years, $u \geq 0$ is the initial capital, $c > 0$ the premium rate, $(N_t)_{t\geq 0}$ a Poisson process with intensity $\lambda > 0$ and $\{X_k\}_{k=1,2,\ldots}$ is a sequence of iid random variables representing the claim sizes.

Suppose in this question that all incoming claims have size 1 i.e. that $X_k = 1$ for all $k = 1, 2, \ldots$.

(a) Determine the probability that at most one claim occurs during the first year.

[4 marks]

(b) Show that the probability that the first claim does not lead to ruin equals

$$\begin{cases} e^{-\lambda(1-u)/c} & \text{if } u \in [0,1) \\ 1 & \text{if } u \geq 1. \end{cases}$$

[6 marks]

(c) For $k = 1, 2, \ldots$, let $p_k(u)$ denote the probability that none of the first k claims lead to ruin as a function of the initial capital u. We also allow for negative initial capitals by setting $p_k(u) = 0$ for $u < 0$. Prove the following recursive relationship:

$$p_k(u) = \int_0^\infty p_{k-1}(u + cs - 1)\lambda e^{-\lambda s}\,\mathrm{d}s \quad \text{for all } u \geq 0 \text{ and } k = 1, 2, \ldots.$$

Hint: do some clever conditioning.

[9 marks]

(d) Assume now that $c > \lambda$. The company is considering to purchase excess of loss reinsurance, where the retention level is denoted M and the possible choices are $M \in [0,1]$. The reinsurance company requires premium at a rate $\beta c(1-M)$ for their services, for some $\beta > 1$. We denote the capital process with excess of loss reinsurance at retention level M in force by $U^{(M)}$.

(i) For which values of $M \in [0,1]$ does $U^{(M)}$ satisfy the Net Profit Condition?

[4 marks]

(ii) Suppose that the initial capital is $u = 0$. Determine the optimal choice $M \in [0,1]$ for the insurance company i.e. the value for M for which the ruin probability is minimal.

[6 marks]

[Total 29 marks]

2. Consider again a Cramér-Lundberg capital process as in question 1 above, but now with claim intensity rate $\lambda = 1.5$, premium rate $c = 1$ and claim size distribution given by the pdf

$$f(x) = \begin{cases} e^{-x}/(1-e^{-1}) & \text{if } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the Laplace exponent ξ of U, and state its domain.

[5 marks]

(b) Denote by $\psi(u)$ the ruin probability when the initial capital is u.

(i) Prove that

$$\psi(u) \leq 1.5\frac{e-2}{e-1} \quad \text{for all } u \geq 0.$$

[4 marks]

(ii) Prove that we also have

$$\psi(u) \leq e^{-1.3u} \quad \text{for all } u \geq 0.$$

[6 marks]

[Total 15 marks]

3. Let X denote a risk i.e. a non-negative random variable. We define the premium principle π_1 as $\pi_1(X) := \mathbb{E}[X] + a$ for all risks X, where $a > 0$ is a constant. Furthermore the premium principle π_2 is defined as

$$\pi_2(X) := \int_0^\infty (1 - F_X(x))^{1/b} \, \mathrm{d}x \quad \text{for all risks } X,$$

where $b > 1$ is a constant and F_X denotes the cdf of X.

(a) (i) State the no rip-off property.

[3 marks]

(ii) Does π_1 satisfy the no rip-off property? Prove your answer.

[4 marks]

(b) In this part, let $X \sim \mathrm{Exp}(\alpha)$ for some $\alpha > 0$.

(i) Let $Y \sim \mathrm{Exp}(\beta)$. Find β such that $\pi_2(X) = \mathbb{E}[Y]$.

[2 marks]

(ii) Briefly explain why Y (with β as computed in part (b)(i) above) could be considered more risky than X.

[2 marks]

(c) (i) State the scale invariance property.

[3 marks]

(ii) Does π_2 satisfy the scale invariance property? Prove your answer.

[4 marks]

[Total 18 marks]

4. In a Bayesian estimation problem where as usual the distribution of a random variable X depends on a sample θ from a random variable Θ, suppose that a sample x from X is observed. Consider the loss function l given by

$$l(\theta, a) = g(\theta)(\theta - a)^2,$$

where g is some strictly positive function. Show that the Bayes estimate $\hat{\theta}_B(x)$ for θ under l is given by

$$\hat{\theta}_B(x) = \frac{\mathbb{E}[g(\Theta)\Theta \mid X = x]}{\mathbb{E}[g(\Theta) \mid X = x]}.$$

[Total 8 marks]

5. An insurance company has established the following model: the number of bite incidents a pet dog is involved in during a year follows a Poisson(Θ) distribution, where the risk parameter Θ itself follows an Exponential distribution with parameter 2. It is assumed that for any given pet dog, the number of bite incidents in one year is independent from the number of bite incidents in another year.

(a) Compute the expected number of bite incidents a randomly selected dog will be involved in next year.

[5 marks]

Suppose for the sequel that Brutus is a pet dog who was involved in 4 bite incidents last year.

(b) Compute the Bühlmann credibility estimate for the number of bite incidents Brutus will be involved in next year.

[9 marks]

(c) Using that Brutus was involved in 4 bite incidents last year, determine the Bayes estimate for Brutus' risk parameter value under squared error loss. (Note: don't get confused if you find the same answer as in part (b)).

[6 marks]

[Total 20 marks]

END OF EXAMINATION PAPER