Two Hours

THE UNIVERSITY OF MANCHESTER

FRACTAL GEOMETRY

3rd June 2019

09:45 - 11:45

Answer **ALL FOUR** questions in **Section A** (48 marks in total) and
TWO of the **THREE** questions in **Section B** (32 marks in total).
If more than two questions from Section B are attempted then credit will be given for the best two answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL FOUR** questions

A1.

(a) Let (X,d) be a metric space and $E \subset X$. Let $s \geq 0, \delta > 0$.
Define what is meant by a δ-*cover* of E. Define $\mathcal{H}^s_\delta(E)$.

[3 marks]

(b) Define $\mathcal{H}^s(E)$, and justify why any limits that you used in your definition exist.

[4 marks]

(c) Give the definition of an outer measure on a metric space (X,d). Prove that $\mathcal{H}^s$ satisfies the second condition of the definition of an outer measure (relating $\mathcal{H}^s(A)$ and $\mathcal{H}^s(B)$ where $A \subset B$).

[5 marks]

(d) Define $\dim_H(E)$ in terms of $\mathcal{H}^s(E)$. Prove that if $A \subset B$ then $\dim_H(A) \leq \dim_H(B)$.

[5 marks]

A2.

(a) What does it mean for an outer measure μ on a space (X,d) to be a *mass distribution*? (You do not need to define outer measure.)

[3 marks]

(b) State and prove the mass distribution principle.

[6 marks]

(c) Use the mass distribution principle to prove that $\dim_H([0,1]) \geq 1$ (using the Euclidean metric on $\mathbb{R}$). You do not need to prove that the mass distribution you use is a mass distribution.

[4 marks]

A3.

(a) Let (X,d) be a metric space. Define the space $\mathcal{S}(X)$ and the Hausdorff metric d_H on $\mathcal{S}(X)$. For $X = \mathbb{R}$ with the Euclidean metric, $A = [1,2]$, $B = \{0\}$, write down $d_H(A,B)$.

[5 marks]

(b) What does it mean for an iterated function system to satisfy the open set condition? Write down an iterated function system Φ which satisfies the open set condition and for which the attractor is the Sierpinski triangle.

[4 marks]

(c) Given an iterated function system $\Phi = \{\phi_1, \cdots, \phi_N\}$ where each ϕ_i is a similarity on $(\mathbb{R}^n, d)$ with contraction ratio $c_i < 1$, state the definition of the similarity dimension $\dim_{sim}(\Phi)$. (Note: not all of the c_i are the same.) In the special case that each contraction ratio $c_i = c$, prove that your above general definition of similarity dimension satisfies

$$\dim_{sim}(\Phi) = \frac{\log N}{-\log c}.$$

[4 marks]

A4. Consider $\mathbb{R}$ equipped with the Euclidean metric and let $E \subset \mathbb{R}$. Consider the set

$$E + 1 := \{x + 1 : x \in E\}$$

Prove that $\dim_H(E) = \dim_H(E + 1)$.

[5 marks]

SECTION B

Answer TWO of the THREE questions

B5.

(a) Let maps $\phi_1, \phi_2, \phi_3 : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\begin{aligned} \phi_1(x,y) &= \left(\frac{x}{3}, \frac{y}{3}\right) \\ \phi_2(x,y) &= \left(\frac{x+1}{3}, \frac{y+2}{3}\right) \\ \phi_2(x,y) &= \left(\frac{x+2}{3}, \frac{y+1}{3}\right) \end{aligned}$$

Let

$$E = \bigcup_{i=1}^{3} \phi_i(E) \subset \mathbb{R}^2.$$

Sketch the first two levels in the construction of E.

[3 marks]

(b) Prove that $\overline{\dim}_B(E) \leq 1$.

[6 marks]

(c) **Without** using the mass distribution principle, prove that $\dim_H(E) \geq 1$. Why can you now conclude that $\dim_B(E)$ exists?

[7 marks]

B6.

(a) Let $\alpha \in (0,1]$. What does it mean for a map $f : (X,d) \to (X',d')$ to be α-Hölder continuous with Hölder constant K? Let $f : (X,d) \to (X',d')$ be α-Hölder continuous and $E \subset X$. Prove that

$$\mathcal{H}^{\frac{s}{\alpha}}(f(E)) \leq K^{\frac{s}{\alpha}} \mathcal{H}^s(E)$$

[6 marks]

(b) State a corollary to the above statement relating $\dim_H(E)$ and $\dim_H(f(E))$.

[2 marks]

(c) Let F denote the middle-$\frac{1}{3}$ Cantor set. By defining a Hölder continuous map from F to another space, give a lower bound for the dimension of F. You should prove that the map is Hölder continuous with the necessary constants.

[8 marks]

B7.

(a) State and prove Hutchinson's Theorem. You may use the fact that the space $\mathcal{S}(X)$ is complete.

[8 marks]

(b) Let $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$ be given by $\phi_1(x) = \frac{x}{3}$, $\phi_2(x) = \frac{x+1}{2}$. Let $E = \bigcup_{i=1}^{2} \phi_i(E)$. Prove that $\dim_H(E) \geq \dim_{sim}(\Phi)$ where $\Phi = \{\phi_1, \phi_2\}$.

[8 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

TOPOLOGY

21 May 2019

9:45-11:45

Answer **ALL** SIX questions

University approved calculators may be used.

1.

(a) Define what is meant by a *topology* on a set X.

(b) Define what is meant by the *usual* topology on a subset $X \subset \mathbb{R}^n$.

(c) Define what is meant by saying that a function $f\colon X \to Y$ between topological spaces is *continuous*. Define what is meant by saying that f is a *homeomorphism*.

(d) Consider the set $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. Show that τ defines a topology on X. Find all continuous maps from $X \to \{1, 2\}$ with the usual topology on $\{1, 2\}$.

[15 marks]

2.

(a) Define what is meant by saying that a topological space X is *path-connected*.

(b) What is meant by saying that path-connectedness is a *topological property*?

(c) Define what is meant by the set $\pi_0(X)$ of *path-components* of a topological space X.

(d) Prove that a continuous map of topological spaces $f\colon X \to Y$ induces a map $f_*\colon \pi_0(X) \to \pi_0(Y)$ between the sets of path-components, taking care to prove that your function is well-defined. Prove that if f is a homeomorphism then f_* is a bijection.

(e) Use (d) to show that path-connectedness is a topological property.

(f) Consider the topological space (X, τ) from Question 1. (d), what are the path-components of this space? Justify your answer.

[15 marks]

3.

(a) Suppose that $q\colon X \to Y$ is a surjection from a topological space X to a set Y . Define the *quotient topology* on Y determined by q. State the *universal property* of the quotient topology.

(b) Suppose that $f\colon X \to Z$ is a continuous surjection from a compact topological space X to a Hausdorff topological space Z . Define an equivalence relation $\sim$ on X so that f induces a bijection $F\colon X/{\sim} \to Z$ from the identification space $X/{\sim}$ of this equivalence relation to Z. Prove that F is a homeomorphism. [*State clearly any general results which you use.*]

(c) Prove that the quotient space $(S^1 \times [0, 1])/(S^1 \times \{1\})$ is homeomorphic to the closed unit disc D^2 (where S^1 and D^2 have the usual topology).

[15 marks]

4.

(a) Define what is meant by saying that a topological space is *Hausdorff*.

(b) Suppose that X and Y are topological spaces. Define the *product topology* on the Cartesian product $X \times Y$. [*It is not necessary to prove that this is a topology.*]

(c) Consider the Sierpinski topology $\tau = \{\emptyset, \{a\}, \{a, b\}\}$ on $X = \{a, b\}$. Write down the product topology on $X \times X$.

(d) Prove that if X and Y are Hausdorff spaces, then the product space $X \times Y$ is Hausdorff, as well.

(e) Prove that, if a topological space X is Hausdorff, then all singleton subsets $\{x\}$ of X (where $x \in X$) are closed. Give an example to show that the converse of this statement is false; that is, if a topological space has all singleton subsets closed, it is not necessarily Hausdorff.

[15 marks]

5.

(a) Define what is meant by a *compact subset* of a topological space.

(b) Prove by using only the definition of compactness, that $\{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is not compact.

(c) Prove that every compact subset of a Hausdorff topological space is closed.

(d) Give an example to show that the Hausdorff condition in (c) cannot be omitted.

[15 marks]

6.

(a) Suppose that X_1 is a subspace of a topological space X. Define what is meant by saying that X_1 is a *retract* of X.

(b) Show that $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ is a retract of the punctured plane $\mathbb{R} \setminus \{0\}$.

(c) Explain how a continuous function of topological spaces $f \colon X \to Y$ induces a homomorphism of fundamental groups $f_* \colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ for $x_0 \in X$. You should indicate why f_* is a well-defined homomorphism.

(d) Use the functorial properties of the fundamental group to prove that, if X_1 is a retract of X, then, for any $x_0 \in X_1$, the homomorphism induced by the inclusion map

$$i_* \colon \pi_1(X_1, x_0) \to \pi_1(X, x_0)$$

is injective.

(e) Hence prove that S^1 is not a retract of the closed disc D^2 .

[*You may quote any calculation of fundamental groups that you need, without proof.*]

[15 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

TOPOLOGY

21 May 2019

9:45-11:45

Answer **ALL** SIX questions

University approved calculators may be used.

1.

(a) Define what is meant by a *topology* on a set X.

(b) Define what is meant by the *usual* topology on a subset $X \subset \mathbb{R}^n$.

(c) Define what is meant by saying that a function $f: X \to Y$ between topological spaces is *continuous*. Define what is meant by saying that f is a *homeomorphism*.

(d) Consider the set $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. Show that τ defines a topology on X. Find all continuous maps from $X \to \{1, 2\}$ with the usual topology on $\{1, 2\}$.

[15 marks]

2.

(a) Define what is meant by saying that a topological space X is *path-connected*.

(b) What is meant by saying that path-connectedness is a *topological property*?

(c) Define what is meant by the set $\pi_0(X)$ of *path-components* of a topological space X.

(d) Prove that a continuous map of topological spaces $f: X \to Y$ induces a map $f_*: \pi_0(X) \to \pi_0(Y)$ between the sets of path-components, taking care to prove that your function is well-defined. Prove that if f is a homeomorphism then f_* is a bijection.

(e) Use (d) to show that path-connectedness is a topological property.

(f) Consider the topological space (X, τ) from Question 1. (d), what are the path-components of this space? Justify your answer.

[15 marks]

3.

(a) Suppose that $q: X \to Y$ is a surjection from a topological space X to a set Y . Define the *quotient topology* on Y determined by q. State the *universal property* of the quotient topology.

(b) Suppose that $f: X \to Z$ is a continuous surjection from a compact topological space X to a Hausdorff topological space Z . Define an equivalence relation $\sim$ on X so that f induces a bijection $F: X/\!\sim\, \to Z$ from the identification space $X/\!\sim$ of this equivalence relation to Z. Prove that F is a homeomorphism. [*State clearly any general results which you use.*]

(c) Prove that the quotient space $(S^1 \times [0, 1])/(S^1 \times \{1\})$ is homeomorphic to the closed unit disc D^2 (where S^1 and D^2 have the usual topology).

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[15 marks]

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(b) Prove by using only the definition of compactness, that $\{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is not compact.

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[15 marks]

6.

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(b) Show that $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ is a retract of the punctured plane $\mathbb{R} \setminus \{0\}$.

(c) Explain how a continuous function of topological spaces $f \colon X \to Y$ induces a homomorphism of fundamental groups $f_* \colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ for $x_0 \in X$. You should indicate why f_* is a well-defined homomorphism.

(d) Use the functorial properties of the fundamental group to prove that, if X_1 is a retract of X, then, for any $x_0 \in X_1$, the homomorphism induced by the inclusion map

$$i_* \colon \pi_1(X_1, x_0) \to \pi_1(X, x_0)$$

is injective.

(e) Hence prove that S^1 is not a retract of the closed disc D^2 .

[*You may quote any calculation of fundamental groups that you need, without proof.*]

[15 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY

04 June 2019

09.45 – 11.45

Answer **ALL FOUR** questions in Section A (50 marks in total)
Answer **TWO** of the THREE questions in Section B (40 marks in total)
If more than TWO questions in Section B are attempted,
the credit will be given for the best TWO answers.

The use of electronic calculators is not permitted.

SECTION A

Answer **ALL** *FOUR* questions

A1.

(a) Explain what is meant by saying that G is a Riemannian metric on a manifold M.

(b) Let G be a Riemannian metric on a 2-dimensional manifold M such that in local coordinates (u, v)

$$G = ||g_{\alpha\beta}|| = \begin{pmatrix} A(u,v) & B(u,v) \\ C(u,v) & D(u,v) \end{pmatrix},$$

where $A(u,v), B(u,v), C(u,v), D(u,v)$ are smooth functions.
Show that $C(u,v) \equiv B(u,v)$ and $A(u,v) > 0$, $D(u,v) > 0$.

(c) Show that $\det ||g_{\alpha\beta}|| \neq 0$.

[10 marks]

A2.

(a) Explain what is meant by a Riemannian manifold being locally Euclidean.

(b) Consider the surface of the cylinder $x^2 + y^2 = 9$ in $\mathbf{E}^3$

$$\mathbf{r}(h,\varphi)\colon \qquad \begin{cases} x = 3\cos\varphi \\ y = 3\sin\varphi \\ z = h \end{cases}, -\infty < h < \infty\,, 0 \le \varphi < 2\pi$$

Calculate the Riemannian metric G on this surface induced by the Euclidean metric in $\mathbf{E}^3$ in coordinates (h, φ).
Show that the surface of the cylinder is locally Euclidean by giving an example of local coordinates (u, v), which are Euclidean coordinates.

(c) Give an example of other coordinates $(\tilde{u}, \tilde{v})$ such that these coordinates are also Euclidean coordinates.

[10 marks]

A3.

(a) Explain what is meant by an affine connection on a manifold.

(b) Let ∇ be an affine connection on a manifold M. Show that for arbitrary vector fields $\mathbf{X}$ and $\mathbf{Y}$ and for an arbitrary function f,

$$e^{-f}\nabla_{\mathbf{X}}\left(e^{f}\mathbf{Y}\right) = (\partial_{\mathbf{X}} f)\mathbf{Y} + \nabla_{\mathbf{X}}\mathbf{Y}\,.$$

(c) Explain what is meant by a canonical flat connection on Euclidean space $\mathbf{E}^n$.
Calculate the Christoffel symbols Γ^r_{rr} and $\Gamma^r_{\varphi\varphi}$ of the canonical flat connection in the Euclidean space $\mathbf{E}^2$, where r, φ are polar coordinates ($x = r\cos\varphi, y = r\sin\varphi$).

[15 marks]

A4.

(a) State the relation between the Riemannian curvature tensor of the Levi-Civita connection of a surface in $\mathbf{E}^3$ and its Gaussian curvature.

(b) Consider in $\mathbf{E}^3$ the surface of the saddle $z = kxy$ ($k \neq 0$):

$$\mathbf{r}(u,v)\colon \quad \begin{cases} x = u \\ y = v \\ z = kuv \end{cases}, \quad -\infty < u, v < \infty$$

Using the shape operator calculate the Gaussian curvature at the point $\mathbf{p}$ of the saddle with coordinates $x = y = z = 0$.
Calculate the scalar curvature R of the Riemannian curvature tensor of the induced Riemannian metric on the saddle at the point $\mathbf{p}$.

(c) Explain why the saddle is not a locally Euclidean surface.

[15 marks]

SECTION B

Answer **TWO** of the three questions

B5.

(a) Consider the stereographic projection of the sphere $x^2 + y^2 + z^2 = 1$ (without the north pole) on the plane $\mathbf{R}^2$: (u, v) are standard coordinates on $\mathbf{R}^2$ and for arbitrary point (x, y, z) on the sphere except the point $z = 1$,

$$\begin{cases} u = \frac{x}{1-z} \\ v = \frac{y}{1-z} \end{cases}, \quad \begin{cases} x = \frac{2u}{u^2+v^2+1} \\ y = \frac{2v}{u^2+v^2+1} \\ z = \frac{u^2+v^2-1}{u^2+v^2+1} \end{cases}.$$

The metric

$$G = \frac{4(du^2 + dv^2)}{(u^2 + v^2 + 1)^2}, \tag{1}$$

is induced on the plane $\mathbf{R}^2$ from the metric on the sphere by the stereographic projection. Calculate the total area of $\mathbf{R}^2$ with respect to the Riemannian metric (1).

(b) Let C be the intersection of the sphere with the plane $z = x$.
Let C' be the image of the curve C under the stereographic projection.
Show that C' is a circle.

(c) The curve C' divides the plane $\mathbf{R}^2$ into two domains. Find the areas of these domains of $\mathbf{R}^2$ with respect to the Riemannian metric (1).

[20 marks]

B6.

(a) Give a detailed formulation of the Levi-Civita Theorem. In particular write down the expression for the Christoffel symbols Γ^i_{km} of the Levi-Civita connection in terms of the Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

(b) Consider sphere of radius R in stereographic coordinates. It is equipped with metric

$$G = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2} \,. \tag{2}$$

Show that Christoffel symbols of Levi-Civita connection ∇ vanish at the point $u = v = 0$ in coordinates (u, v).

(c) Consider on the sphere a connection $\tilde{\nabla}$ such that the Riemannian metric (2) is invariant with respect to this connection, and the Christoffel symbols $\tilde{\Gamma}^u_{vv}$ and $\tilde{\Gamma}^v_{uu}$ of the connection $\tilde{\nabla}$ vanish at all points (u, v).
Show that the connection $\tilde{\nabla}$ does not coincide with the Levi-Civita connection ∇.
Does it contradict to Levi-Civita Theorem that the Riemannian metric (2) is invariant with respect to two different connections?
Justify the answer.

[20 marks]

B7.

(a) Let (M, G) be a Riemannian manifold, let C be a curve on M starting at the point $\mathbf{p}_1$ and ending at the point $\mathbf{p}_2$.
Explain what is meant by parallel transport $P_{C_{\mathbf{p}_1\mathbf{p}_2}}$ along the curve C.

(b) Show that in the case that $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$ (C is a closed curve), then $P_{C_{\mathbf{p}_1\mathbf{p}_2}} = P_{C_\mathbf{p}}$ is an orthogonal operator in the tangent space $T_\mathbf{p}M$.

(c) Let M be the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$ with induced Riemannian metric, and let the curve C on M be an intersection of the sphere with a plane $x + z = R$. Take the point $\mathbf{p} = (R, 0, 0)$ on the curve C and consider vectors $\mathbf{Y} = \frac{\partial}{\partial y}$ and $\mathbf{Z} = \frac{\partial}{\partial z}$ which are tangent to the sphere at the point $\mathbf{p}$. Find the result of the parallel transport of these vectors along the closed curve C.

[20 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

CODING THEORY

20 May 2019

14:00 – 16:00

Answer ALL THREE questions in Section A (40 marks in total). Answer TWO of the THREE questions in Section B (40 marks in total). If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

University approved calculators may be used

SECTION A

Answer **ALL** questions in this section (40 marks in total)

A1. (a) Define what is meant by:

— a *word* $\underline{x}$ of length n in a finite alphabet F;

— a *code* C of length n in a finite alphabet F;

— the *Hamming distance* $d(\underline{x}, \underline{y})$ between two words $\underline{x}$, $\underline{y}$ of length n;

— a *nearest neighbour* of a word $\underline{x}$ in a code C.

(b) Let C be the binary repetition code of length 6. List the codewords of C. Write down words $\underline{x}, \underline{y} \in \mathbb{F}_2^6$ such that $\underline{x}$ has exactly one nearest neighbour in C, and $\underline{y}$ has more than one nearest neighbour in C.

[10 marks]

A2.

(a) Explain what is meant by a *standard array* for a linear $[n, k, d]_q$-code C. State the number of rows and the number of columns in a standard array for C. Explain how to decode a received vector using the standard array.

You are now given that $D = \{0000,\ 0111,\ \underline{v},\ 1110\}$ is a binary linear code.

(b) Prove that the vector $\underline{v}$ is 1001. Assuming that a codevector of D is sent via $\mathrm{BSC}(p)$, show that, with probability $p^2 - p^4$, the received vector will contain an error which is not detected.

(c) Construct a standard array for D. Use the standard array you have constructed to give an example where one bit error occurs in a codevector $\underline{c}$ of D and is not corrected by the standard array decoder.

(d) Assume that $\underline{y}, \underline{z} \in \mathbb{F}_2^4$, $\underline{y} \neq \underline{z}$, and that H is a parity check matrix for the code D such that $\underline{y}H^T = \underline{z}H^T$. Can $\underline{y}$ and $\underline{z}$ occur in the same column of the standard array that you constructed in part (c)? Justify your answer briefly.

[20 marks]

A3.

(a) Define what is meant by a *line* in the vector space $\mathbb{F}_q^r$. Explain how to construct a check matrix H for a Hamming $\mathrm{Ham}(r, q)$-code. State without proof the number of rows and the number of columns of H.

(b) Write down a check matrix of a ternary $\mathrm{Ham}(2, 3)$-code.

(c) Write down a check matrix of a $\mathrm{Ham}(2, 5)$-code $C \subseteq \mathbb{F}_5^6$ such that C contains the vector 111111.

[10 marks]

SECTION B

Answer **TWO** of the three questions in this section (40 marks in total).
If more than TWO questions from this section are attempted, then credit will be given for the best TWO answers.

B4. In this question, F is a finite alphabet of cardinality q, and $C \subseteq F^n$ is a code of cardinality $M > 1$.

(a) Define what is meant by the *minimum distance*, $d(C)$, of the code C.

(b) Prove the Singleton bound which says that $M \leqslant q^{n-d(C)+1}$.

(c) Let r be an integer between 0 and n. Define the *Hamming sphere* $S_r(\underline{\mathrm{u}})$ of radius r centred at $\underline{\mathrm{u}} \in F^n$. State and prove the formula for the cardinality of $S_r(\underline{\mathrm{u}})$ in terms of r, n and q.

(d) State without proof the Hamming bound on M in terms of n, q and $d(C)$. Define what is meant by saying that C is a *perfect code*.

(e) Show that if $n = q = 60$ and C is perfect, then $d(C) \neq 7$. Hint: you may use the Singleton bound.

[20 marks]

B5. In this question, $C \subseteq \mathbb{F}_q^n$ is a linear code of dimension k.

(a) Define what is meant by the *inner product* $\underline{\mathrm{x}} \cdot \underline{\mathrm{y}}$ of two vectors $\underline{\mathrm{x}}, \underline{\mathrm{y}} \in \mathbb{F}_q^n$, and what is meant by the *dual code* $C^\perp$. State without proof the length and the dimension of $C^\perp$.

(b) Recall that C is *self-orthogonal* if $C \subseteq C^\perp$. Prove that if C has a generator matrix G such that GG^T is a zero matrix, then C is self-orthogonal.

(c) A self-orthogonal code $D \subseteq \mathbb{F}_q^4$ has generator matrix $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 1 & 2 & -1 \end{bmatrix}$. Find a prime number $q > 2$ for which this is possible. Then write down a check matrix for D, explaining how it is obtained.

(d) Assume that $q = 2$ and $C \subseteq C^\perp$. Let $W_C(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i$ be the weight enumerator of C. Prove that $A_2 \leqslant k$ and that $A_2(A_2 - 1)/2 \leqslant A_4$.

[20 marks]

B6. In this question, we assume that vectors in $\mathbb{F}_q^n$ are identified with polynomials, in $\mathbb{F}_q[x]$, of degree less than n in the usual way.

(a) Let $C \subseteq \mathbb{F}_q^n$ be a cyclic code. Explain what is meant by a *generator polynomial* of C. Show that there exists at most one generator polynomial of C.

(b) Let $Z \subseteq \mathbb{F}_7^8$ be the cyclic code with generator polynomial $x - 1$. Write down a generator matrix for Z. Prove that $Z = \{(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8) \in \mathbb{F}_7^8 : \sum_{i=1}^{8} v_i = 0 \text{ in } \mathbb{F}_7\}$.

You may use the factorisation $(x-1)(x+1)(x^2+1)(x^2-3x+1)(x^2+3x+1)$ of $x^8 - 1$ into irreducible monic polynomials in $\mathbb{F}_7[x]$ to answer the following questions.

(c) Determine: (i) the total number of cyclic codes $C \subseteq \mathbb{F}_7^8$; (ii) the number of cyclic codes $C \subseteq \mathbb{F}_7^8$ such that $C \subseteq Z$ where Z is the code from part (b). Justify your answers briefly.

(d) Prove that none of the cyclic codes in $\mathbb{F}_7^8$ is a Hamming code. Any facts from the course can be freely used.

[20 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

HYPERBOLIC GEOMETRY

30 May 2019

14:00 – 16:00

Answer **THREE** of the **FOUR** questions.
If all four questions are attempted then credit will be given for the three best answers.

Electronic calculators may be used, provided that they cannot store text.

Notation: Throughout, $\mathbb{H}$ denotes the upper half-plane, $\partial\mathbb{H}$ denotes the boundary of $\mathbb{H}$, $\mathbb{D}$ denotes the Poincaré disc, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$.

1.

(i) What does it mean to say that γ is a Möbius transformation of $\mathbb{H}$?

[2 marks]

(ii) Let $\gamma \in \text{Möb}(\mathbb{H})$. How is $\tau(\gamma)$ defined? Briefly state (without proof) how to use $\tau(\gamma)$ to determine if γ is hyperbolic, parabolic or elliptic.

[5 marks]

Let

$$\gamma_1(z) = \frac{7z-8}{2z-1}, \quad \gamma_2(z) = \frac{2z-1}{-7z+8}.$$

Determine whether each of γ_1 and γ_2 is hyperbolic, parabolic or elliptic.

[4 marks]

(iii) Suppose that $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{H})$. Prove that the composition $\gamma_1\gamma_2 \in \text{Möb}(\mathbb{H})$.

[6 marks]

(iv) It was proved in the course that, in $\mathbb{C}$, vertical straight lines and circles with real centres have equations of the form

$$\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

Suppose that A is either a vertical straight line or a circle with a real centre and that $\gamma \in \text{Möb}(\mathbb{H})$. Show that $\gamma(A)$ is also either a vertical straight line or a circle with a real centre in $\mathbb{C}$.

[8 marks]

(v) One can also prove the following fact (you do not need to prove this yourself): Let A a circle in $\mathbb{C}$ (not necessarily with a real centre). Let $\gamma \in \text{Möb}(\mathbb{H})$. Then $\gamma(A)$ is either a straight line in $\mathbb{C}$ (not necessarily vertical) or a circle in $\mathbb{C}$ (not necessarily with a real centre).

Now suppose that A is a circle in $\mathbb{H}$ that touches the real axis at a single point, as illustrated in Figure 1. Using the above fact, describe (in words) precisely all the possibilities for $\gamma(A)$, justifying your answer.

[5 marks]

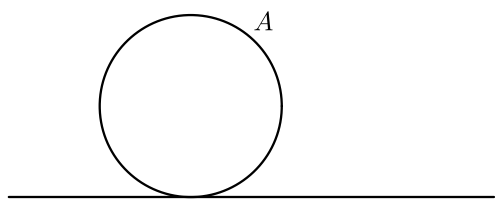


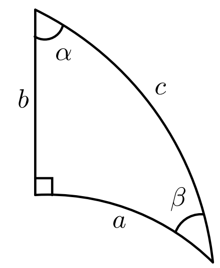
Figure 1: See Question 1(v).

2. Throughout this question you may use, without proof, the formula

$$\cosh d_{\mathbb{H}}(z, w) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}.$$

You may also assume that in a hyperbolic right-angled triangle with angles $\alpha, \beta, \pi/2$ and sides of hyperbolic length a, b, c (as illustrated below) we have

$$\sin\alpha = \frac{\sinh a}{\sinh c},$$
$$\cos\alpha = \frac{\tanh b}{\tanh c}.$$



You may also use the trig and hyperbolic trig identities: $\cosh^2 x - \sinh^2 x = 1$, $\sinh 2x = 2\sinh x\cosh x$, $\sin(x+y) = \sin x\cos y + \cos x\sin y$, $\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$.

(i) Let Δ be an arbitrary hyperbolic right-angled triangle with sides of hyperbolic lengths a, b, c. Suppose that the side of hyperbolic length c is opposite the right-angle. Prove the Hyperbolic Pythagoras' Theorem:

$$\cosh c = \cosh a \cosh b.$$

(If you reduce Δ to a special case by applying a Möbius transformation then you should briefly explain why you can do this.)

[8 marks]

(ii) Let Δ be an arbitrary acute hyperbolic triangle with internal angles α, β, γ and opposite sides of hyperbolic length a, b, c (see Figure 2 on the following page). Prove the hyperbolic sine rule:

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}.$$

Hint: drop a perpendicular from a vertex to the opposite side to create two right-angled triangles.

[12 marks]

(iii) Let Δ be a hyperbolic triangle with internal angles α, β and γ. State, without proof, the Gauss-Bonnet theorem for a hyperbolic triangle.

[2 marks]

(iv) Consider a tiling of the hyperbolic plane by regular hyperbolic triangles with k triangles meeting at each vertex. Use the Gauss-Bonnet Theorem to show that $k \geq 7$.

[4 marks]

(v) Sketch a tiling of the Poincaré disc by regular hyperbolic triangles when $k = \infty$.

[4 marks]

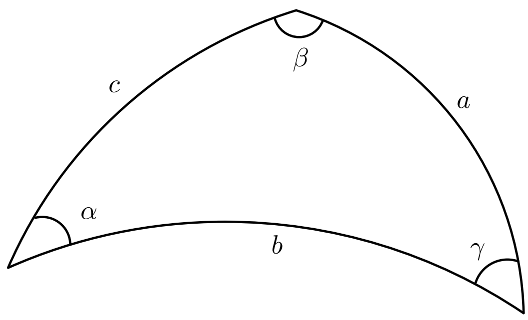


Figure 2: An acute hyperbolic triangle; see Question 2(ii).

3.

(a) (i) Let Γ be a Fuchsian group acting on $\mathbb{H}$. What does it mean to say that an open set $F \subset \mathbb{H}$ is a fundamental domain for Γ?

[2 marks]

(ii) Briefly explain how to construct a Dirichlet region for a Fuchsian group Γ. (Your answer should include a definition of the terms $[p, \gamma(p)]$, $L_p(\gamma)$, $H_p(\gamma)$, $D(p)$ that were defined in the lectures.)

[6 marks]

(iii) It was proved in the lectures that the perpendicular bisector of the arc of geodesic $[z_1, z_2]$ between two points $z_1, z_2 \in \mathbb{H}$ is given by

$$\{z \in \mathbb{H} \mid d_{\mathbb{H}}(z, z_1) = d_{\mathbb{H}}(z, z_2)\}. \tag{1}$$

Write $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Show that (1) can be written in the form

$$\{z \in \mathbb{H} \mid y_2|z - z_1|^2 = y_1|z - z_2|^2\}. \tag{2}$$

(You may use the fact that

$$\cosh d_{\mathbb{H}}(z, w) = 1 + \frac{|z - w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}.)$$

[4 marks]

(b) (i) Let $\gamma(z) = z+b$ where $b \in \mathbb{R}$. Let $p = i$. Use the formula (2) to calculate the perpendicular bisector of $[p, \gamma(p)]$.

[4 marks]

(ii) Let $\Gamma = \{\gamma_n \mid \gamma_n(z) = z + 4n,\ n \in \mathbb{Z}\}$. Use the procedure you outlined in (a)(ii) and the result of (b)(i) above to show that $D(i) = \{z \in \mathbb{H} \mid -2 < \mathrm{Re}(z) < 2\}$.

[6 marks]

(iii) Draw the resulting tessellation in $\mathbb{H}$. What does this tessellation look like in $\mathbb{D}$?

[4 marks]

(iv) Describe (either in words or by drawing a picture) what $D(p)$ will look like for an arbitrary choice of $p \in \mathbb{H}$.

Find a fundamental domain for Γ which is *not* of the form $D(p)$ for some $p \in \mathbb{H}$.

[4 marks]

4.

(i) Let $\mathcal{E} = v_1 \to v_2 \to \cdots \to v_n$ be an elliptic cycle. Let the internal angle at v_j be $\angle v_j$.

Define $\mathrm{sum}(\mathcal{E})$.

What does it mean to say that $\mathcal{E}$ satisfies the *elliptic cycle condition*?

What does it mean to say that $\mathcal{E}$ is an *accidental cycle*?

[4 marks]

(ii) Consider the hyperbolic hexagon illustrated in Figure 3 with the side-pairings indicated. Here $s, t, u \geq 2$ are real numbers and $\theta_1 + \theta_2 + \theta_3 \leq \pi$. (You may assume that such a hyperbolic hexagon exists.)

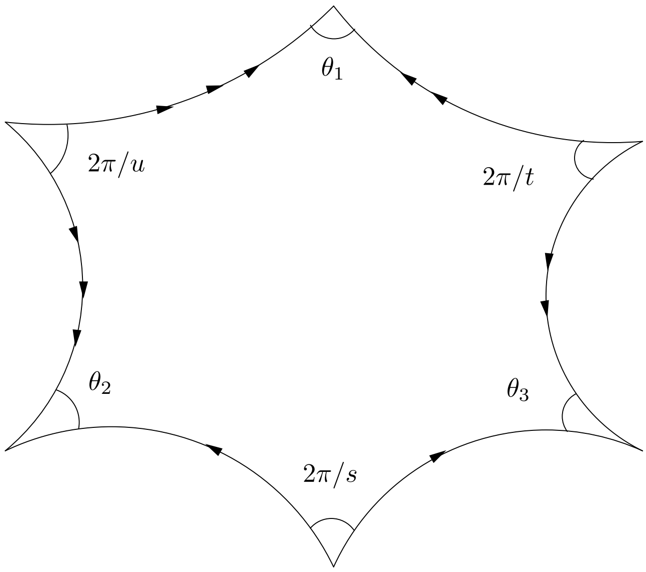


Figure 3: A hyperbolic hexagon; see Question 4(ii).

Show that there are 4 elliptic cycles. In each case, calculate the elliptic cycle transformations and the angle sum.

[12 marks]

(iii) Determine conditions on $\theta_1, \theta_2, \theta_3, s, t, u$ so that the side-pairing transformations generate a Fuchsian group Γ. Give a presentation for Γ in terms of generators and relations.

[6 marks]

(iv) How is the signature of a cocompact Fuchsian group defined?

Assume that $\theta_1, \theta_2, \theta_3, s, t, u$ are such that the side-pairing transformations in Figure 3 generate a Fuchsian group Γ. Use Euler's formula to calculate the genus g of $\mathbb{H}/\Gamma$. Hence write down the possibilities for the signature of Γ. Briefly describe $\mathbb{H}/\Gamma$ in each case.

[8 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

ALGEBRAIC GEOMETRY

17 May, 2019

14:00–16:00

Answer THREE of the FOUR questions. If more than THREE questions are attempted, then credit will be given for the best THREE answers.

Electronic calculators may be used in accordance with the University regulations

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1. In this question K denotes an arbitrary field and unless specified otherwise, varieties are defined over K.

(a) (i) Give the definition of the *affine algebraic variety* $\mathcal{V}(J) \subseteq K^n$ determined by an ideal $J \lhd K[x_1, x_2, \ldots, x_n]$.

(ii) Define the *ideal* $\mathcal{I}(V)$ and the *co-ordinate ring* $K[V]$ of an affine algebraic variety $V \subseteq K^n$.

(iii) Let $J = \langle (x - y + z)^2, x^2 - y^2 + z^2 \rangle \lhd K[x, y, z]$. Show that $\mathcal{I}(\mathcal{V}(J)) \neq J$.

(b) (i) Explain what it means to say that an affine algebraic variety W is *irreducible*.

(ii) Prove that if $\mathcal{I}(W)$ is not prime, then W is reducible. (You may use any other results from the course that you need.)

(iii) This part of the question is over $\mathbb{C}$. Find the irreducible components of the variety

$$W = \mathcal{V}(\langle xyz - xz, x^2 + 2y^2 - z^2 - 2 \rangle) \subset \mathbb{A}^3(\mathbb{C}).$$

(The irreducibility of certain varieties was proved in the course, you can use this if you identify correctly what kind of a variety each component is.)

(c) (i) Let $V \subseteq \mathbb{A}^m(K)$ and $W \subseteq \mathbb{A}^n(K)$ be affine algebraic varieties. Define the notion of a *morphism* $\varphi : V \to W$ and state how the ring homomorphism $\varphi^* : K[W] \to K[V]$ is defined. (You are *not* asked to prove that φ^* is a homomorphism.)

(ii) This part of the question is over $\mathbb{R}$. Let $C = \mathcal{V}(\langle x^2 + y^2 - 1 \rangle) \subset \mathbb{R}^2$ be the unit circle. Show that if $\varphi : \mathbb{A}^1(\mathbb{R}) \to C$ is a morphism, then it is a constant function, i. e., it maps the whole of $\mathbb{A}^1(\mathbb{R})$ to a single point. (*Hint:* Assume that there exists a non-constant morphism $\varphi : \mathbb{A}^1(\mathbb{R}) \to C$ and derive a contradiction.)

[20 marks]

2.

(a) Let K be a field. Let $V \subseteq \mathbb{A}^n(K)$ be an affine algebraic variety and let $P \in V$.

(i) Let $\mathbf{v} \in K^n$ be a non-zero vector. Let $\ell = \{P + t\mathbf{v} \mid t \in K\}$ be the line through P with direction vector $\mathbf{v}$. Explain what is meant by saying that ℓ is *tangent* to V at P and give the definition of the *tangent space* $T_P V$ to V at P.

(ii) Let $f_1, f_2, \ldots, f_r$ be a generating set for $\mathcal{I}(V)$ and let

$$J_P = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}|_P & \frac{\partial f_1}{\partial x_2}|_P & \cdots & \frac{\partial f_1}{\partial x_n}|_P \\ \frac{\partial f_2}{\partial x_1}|_P & \frac{\partial f_2}{\partial x_2}|_P & \cdots & \frac{\partial f_2}{\partial x_n}|_P \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1}|_P & \frac{\partial f_r}{\partial x_2}|_P & \cdots & \frac{\partial f_r}{\partial x_n}|_P \end{pmatrix}$$

be the Jacobian matrix evaluated at $P \in V$.

State and prove the criterion in terms of J_P which describes when the line $\ell = \{P + t\mathbf{v} \mid t \in K\}$ $(\mathbf{v} \in K^n \setminus \{\mathbf{0}\})$ is tangent to V at P.

(b) This part of the question is over $\mathbb{C}$. Let x, y be co-ordinates on $\mathbb{A}^2(\mathbb{C})$ and let t be the co-ordinate on $\mathbb{A}^1(\mathbb{C})$. Let

$$f = x^3 - 6xy^2 - y^4 - 8y^2$$

and let $C = \mathcal{V}(\langle f \rangle) \subset \mathbb{A}^2(\mathbb{C})$. (You may assume without proof that f is an irreducible polynomial, therefore C is irreducible and $\mathcal{I}(C) = \langle f \rangle$.)

(i) Show that $\varphi(t) = (t^4 + 2t^2, t^3)$ is a morphism $\varphi : \mathbb{A}^1(\mathbb{C}) \to C$.

(ii) Let $\psi(x, y) = \dfrac{xy + 4y}{2x + y^2}$. Show that ψ is a rational map $C \dashrightarrow \mathbb{A}^1(\mathbb{C})$ and that $\psi \circ \varphi : \mathbb{A}^1(\mathbb{C}) \dashrightarrow \mathbb{A}^1(\mathbb{C})$ is the identity map.

(iii) Find the three singular points of C.

[20 marks]

3. In this question K denotes an arbitrary field.

(a) (i) Let n be a positive integer and let $I \lhd K[X_0, X_1, \ldots, X_n]$ be a homogeneous ideal. State the definition of $\mathcal{V}(I)$, the *projective algebraic variety* defined by I.

(ii) Let $W \subseteq \mathbb{P}^n$ be a projective algebraic variety for some $n \geq 1$. Define $\mathcal{I}(W)$ and $K[W]$.

(b) Let m, n be positive integers and let A be a $(n+1) \times (m+1)$ matrix with entries from K other than the 0 matrix. Define Φ by $\Phi(X_0 : X_1 : \ldots : X_m) = (Y_0 : Y_1 : \ldots : Y_n)$, where $\begin{pmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_n \end{pmatrix} = A \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_m \end{pmatrix}$.

(i) Show that Φ is a rational map $\mathbb{P}^m \dashrightarrow \mathbb{P}^n$.

(ii) Prove that if $m = n$ and A is invertible, then Φ is a morphism.

(iii) Prove that if $m > n$ and the rows of A are not all scalar multiples of one of the rows, then Φ is not a morphism.

(c) We identify $\mathbb{P}^1(K)$ with $K \cup \{\infty\}$ by identifying $(X_0 : X_1)$ with $X_0/X_1 \in K$ if $X_1 \neq 0$ and $(X_0 : 0)$ with ∞ as usual. Recall that if z_1, z_2, z_3, $z_4 \in K \cup \{\infty\}$ with at least 3 of them pairwise distinct, their cross ratio is defined to be

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \in K \cup \{\infty\}$$

with appropriate rules for arithmetic involving ∞ and for division by 0.

Let z_1, z_2, z_3, $z_4 \in K \cup \{\infty\}$ with at least 3 of them pairwise distinct. Show that $(z_1, z_2; z_3, z_4) = (\varphi(z_1), \varphi(z_2); \varphi(z_3), \varphi(z_4))$ for any projective transformation $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$.

(d) Find $a, b, c, d \in \mathbb{C}$ such that $\varphi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, $\varphi(z) = \dfrac{az + b}{cz + d}$ satisfies $\varphi(-1) = -5$, $\varphi(3) = 3$ and $\varphi(5) = 4$.

[20 marks]

4.

(a) Let K be a field. Let $E \subset \mathbb{P}^2(K)$ be a non-singular cubic curve and let O be a point of E. (You should *not* assume that the equation of E is in a special form or that O is an inflection point of E.)

(i) Describe without proof how one can define the addition operation on the points of E to obtain a commutative group in which O is the zero element.

(ii) Let $A \in E$. Describe how $-A$ is calculated and prove that this point $-A$ indeed satisfies $A + (-A) = O$.

(b) Let F be the elliptic curve given by the affine equation

$$y^2 = x^3 - 4x^2 + 8x + 4$$

over $\mathbb{C}$ together with the point $O = (0 : 1 : 0)$ in homogeneous co-ordinates, which is taken to be the zero element of the group law as usual.

Let $A = (1, 3)$, $B = (4, -6)$. Calculate $A + B$ and $2A$. (You do *not* have to verify that A and B lie on F.)

(c) The elliptic curve G is defined by the affine equation

$$9y^2 - 18xy - 12y = 27x^3 - 9x^2$$

over $\mathbb{C}$. Find $p, q \in \mathbb{C}$ such that the curve defined by the equation $y^2 = x^3 + px + q$ is isomorphic to G.

[20 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

GREEN'S FUNCTIONS, INTEGRAL EQUATIONS AND APPLICATIONS

29 May 2019

14:00 – 16:00

Answer **ALL 6** questions.

The use of electronic calculators is not permitted.

Given equations:

- For functions f and g defined in either two or three dimensions, and a domain D with smooth boundary in either two or three dimensions

$$\int_D (f(\mathbf{x})\nabla^2 g(\mathbf{x}) - g(\mathbf{x})\nabla^2 f(\mathbf{x}))\ d\mathbf{x} = \int_{\partial D} (f(\mathbf{x})\nabla g(\mathbf{x}) - g(\mathbf{x})\nabla f(\mathbf{x})) \cdot \mathbf{n}(\mathbf{x})\ ds(\mathbf{x}) \qquad (1)$$

  where the boundary ∂D is oriented by the outward pointing unit normal $\mathbf{n}(\mathbf{x})$ at each point $\mathbf{x}$ in ∂D.

- The surface area of the unit sphere $\mathbb{S}^2$ is 4π:

$$\int_{\mathbb{S}^2} ds = 4\pi.$$

Answer **ALL** questions (there are a total of 100 marks possible). To receive full marks you must show your work.

1.

(a) Is the following a regular Sturm-Liouville eigenvalue problem? Justify your answer.

$$\begin{cases} \cos(x)\phi''(x) - \sin(x)\phi'(x) + \lambda\phi(x) = 0, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \phi(-\pi/2) = 0, \quad \phi(\pi/2) = 0. \end{cases}$$

[5 marks]

(b) Suppose that G is the Green's function for a boundary value problem

$$\begin{cases} \mathcal{L}u(x) = f(x), \quad x \in (a,b) \\ \mathcal{B}, \end{cases}$$

where $\mathcal{L}$ is a second order linear ordinary differential operator, and $\mathcal{B}$ is a set of two boundary conditions. Also suppose that G^* is the adjoint Green's function.

(i) What boundary value problem does G^* satisfy?

[2 marks]

(ii) What is the relationship between G and G^*?

[3 marks]

(c) Give the free space Green's function for the Laplacian in three dimensions. You do not need to justify the formula for the Green's function, just write it down.

[3 marks]

(d) Consider the integral equation

$$u(x) = \int_1^2 e^{-(x-y)^2} u(y)\ \mathrm{d}y + f(x).$$

(i) What type of integral equation is this?

[2 marks]

(ii) What is the kernel?

[2 marks]

[Total 17 marks]

2. Consider the following boundary value problem:

$$\begin{cases} u''(x) + 4xu'(x) + 2(1 + 2x^2)u(x) = f(x), \quad x \in (0, 1) \\ u(0) = 0, \ u(1) = 0 \end{cases}$$

A complementary solution for the ODE in this boundary value problem is

$$u_c(x) = c_1 e^{-x^2} + c_2 x e^{-x^2}.$$

Find the Green's function.

[Total 12 marks]

3. Suppose that p and p' are continuous, and $p(x) > 0$ for all x. Consider the following boundary value problem:

$$\text{(BVP)} \quad \begin{cases} p(x)u''(x) + p'(x)u'(x) = f(x), \quad x \in (0,1) \\ u(0) = 0, \ u(1) = 0. \end{cases}$$

(a) Show that (BVP) is fully self-adjoint.

[6 marks]

(b) Use the Fredholm Alternative to show that there exists a unique solution for (BVP).

[6 marks]

[Total 12 marks]

4. Consider the following integral equation:

$$\text{(INTEQ)} \quad u(x) = \lambda \int_0^1 x^3 y^2 u(y)\ \mathrm{d}y + f(x).$$

(a) For what values of $\lambda \in \mathbb{C}$ is there a unique solution $u(x)$ to (INTEQ) for any continuous function f?

[5 marks]

(b) For the values of λ found in part (a), find a formula for the solution $u(x)$ of (INTEQ).

[5 marks]

(c) When λ is not one of the values found in part (a) find a condition on f so that there is a solution for (INTEQ). Also write a general solution for (INTEQ) in this case.

[5 marks]

[Total 15 marks]

5. For time harmonic waves on an infinite string where the physical displacement from equilibrium is given by the real part of $e^{-i\omega t}u(x)$, the function u will satisfy the Helmholtz equation

$$\text{(HELMHOLTZ)} \quad u''(x) + k(x)^2\, u(x) = f(x),$$

provided the oscillations are small. Assume that both $k(x) = k_0$ (a constant) and $f(x) = 0$ when $|x|$ is sufficiently large.

(a) Describe the radiation conditions on u which model the situation in which all waves are created by the source f. You should give the formulas for the radiation conditions, and explain how they are derived.

[8 marks]

(b) Assume that $u(x) = e^{ik_0 x} + u_{sc}(x)$, where u_{sc} satisfies the radiation conditions from part (a), is a solution of (HELMHOLTZ) with $f(x) = 0$ for all x. Show that $u(x)$ satisfies a Fredholm integral equation of the second kind. You may use without justification the fact that the Green's function for (HELMHOLTZ) with $k(x) = k_0$ everywhere and the radiation conditions is

$$G(x, x_0) = \frac{e^{ik_0|x-x_0|}}{2ik_0}.$$

[8 marks]

(c) What is the Born approximation for the solution of the integral equation from part (b)? You do not need to evaluate the integral which appears in the approximation.

[6 marks]

[Total 22 marks]

6. In three dimensions we can also consider the Helmholtz equation

$$\nabla^2 u(\mathbf{x}) + k^2 u(\mathbf{x}) = Q(\mathbf{x})$$

where $\mathbf{x} \in \mathbb{R}^3$. In this question we will assume that k is constant and use the notation

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi} \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}$$

where $\mathbf{x}$ and $\mathbf{x}_0$ are in $\mathbb{R}^3$.

(a) Show that G satisfies

$$\nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}_0) + k^2 G(\mathbf{x}, \mathbf{x}_0) = 0$$

when $\mathbf{x} \neq \mathbf{x}_0$.

[9 marks]

(b) Let f be a function on $\mathbb{R}^3$ which is twice continuously differentiable, and such that $f(\mathbf{x}) = 0$ when $|\mathbf{x}|$ is sufficiently large. Also, suppose that for $\epsilon > 0$, $B_\epsilon(\mathbf{x}_0)$ is the ball of radius ϵ around $\mathbf{x}_0$. Show that

$$\int_{\mathbb{R}^3 \setminus B_\epsilon(\mathbf{x}_0)} G(\mathbf{x}, \mathbf{x}_0) \left(\nabla^2 f(\mathbf{x}) + k^2 f(\mathbf{x})\right) \, \mathrm{d}\mathbf{x} = \frac{e^{ik\epsilon}}{4\pi\epsilon^2} \int_{\partial B_\epsilon(\mathbf{x}_0)} (1 - ik\epsilon) f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0) \, ds(\mathbf{x}).$$

[8 marks]

(c) Take the limit $\epsilon \to 0^+$ in the formula from part (b) to show that

$$\int_{\mathbb{R}^3} G(\mathbf{x}, \mathbf{x}_0) \left(\nabla^2 f(\mathbf{x}) + k^2 f(\mathbf{x})\right) \, \mathrm{d}\mathbf{x} = f(\mathbf{x}_0).$$

[HINT: You should change variables $\mathbf{r} = (\mathbf{x} - \mathbf{x_0})/\epsilon$.]

[5 marks]

[Total 22 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

WAVE MOTION

5th June 2019

09:45-11:45

Answer ALL **SIX** questions.

Electronic calculators are permitted, provided they cannot store text.

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1. You are given that a harmonic signal (in the usual notation) $y(x,t) = A\exp[i(kx-\omega t)]$, satisfies the equation

$$\frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} - b\frac{\partial^3 y}{\partial x^3} = 0\,.$$

Here $a, b > 0$, x is a spatial coordinate and t is time.

(i) Obtain the dispersion relation for this problem.

[2 marks]

(ii) Define both the phase speed c and group velocity c_g, and hence determine each in terms of a, b and k.

[2 marks]

(iii) Sketch the behaviour of ω, c and c_g vs $k > 0$. Is this wave system dispersive?

[3 marks]

(iv) Which propagates faster, phase or energy?

[1 marks]

[Total: 8 marks]

2. Consider a physical signal $f(x,y,t) = A\exp[i(kx+ly-\omega t)]$, in the usual notation, which satisfies the governing equation

$$\frac{\partial f}{\partial t} - \beta R^2\frac{\partial f}{\partial x} = R^2\frac{\partial}{\partial t}\left(\nabla^2 f\right)\,,$$

where R and β are positive constants and ∇^2 is the two-dimensional Laplacian.

(i) Obtain the dispersion relation, and show that it can be written in the form

$$(k-k_0)^2 + l^2 = A\,,$$

where k_0 and A are to be determined in terms of β, R, ω.

[3 marks]

(ii) For fixed values of $0 < \omega < \omega_c$ (where ω_c is to be determined in terms of β and R) sketch solutions of the dispersion relation in the $k-l$ wavenumber plane. Indicate a wavenumber vector and a group velocity vector on your sketch.

[5 marks]

(iii) Using the sketch, explain how shorter wavelength waves behave differently to those with a longer wavelength.

[4 marks]

[Total: 12 marks]

3. Small-amplitude water waves exist on an inviscid, incompressible fluid layer of depth h. The waves (of amplitude A) have a velocity potential

$$\phi(x,z,t) = \frac{gA}{\omega}\frac{\cosh(\hat{K}[z-h])}{\cosh \hat{K}h}\cos(\hat{K}x-\omega t),$$

where $\hat{K}\tanh(\hat{K}h) = \omega^2/g$, with g being the acceleration due to gravity. The coordinates (x, z) have z measured downwards from the mean location of the free surface.

In the motion resulting from the potential ϕ above, write down the equations of motion for a particle with position $x(t), z(t)$. Find approximations for $x(t)$ and $z(t)$ near a point (x_0, z_0) and hence give an equation for the particle paths at leading order. Sketch the shape of the paths for varying z_0.

[Total: 10 marks]

4. Consider an impermeable sphere centred at the origin, surrounded by an inviscid, compressible fluid of sound speed c. The radius $a(t)$ of the sphere changes, and this generates sound waves in the fluid; these waves are to be described in terms of a potential $\phi(r, t)$, where r is the distance from the origin.

(i) State the boundary conditions to be satisfied by $\phi(r, t)$.

[2 marks]

(ii) Show that the wave equation (given below) can be satisfied by a velocity potential of

$$\phi(r,t) = -\frac{1}{4\pi r}m(t-r/c)\,,$$

where m is a (twice differentiable) function.

[4 marks]

(iii) For small deflections of the sphere radius about a mean value a_0, obtain an ordinary-differential equation for $m(t)$ in terms of the volume of the sphere.

[4 marks]

You may use: The wave equation for a spherically symmetric velocity potential is

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)\,.$$

[Total: 10 marks]

5. A wave motion exists in an incompressible, inviscid, fluid (occupying $z > 0$), covered by an elastic sheet, such that the usual linearised surface conditions are altered to be

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{on} \quad z = 0\,,$$

and

$$-\rho\frac{\partial \phi}{\partial t} = -\rho g \eta + T\frac{\partial^2 \eta}{\partial x^2} - R\frac{\partial^4 \eta}{\partial x^4} \quad \text{on} \quad z = 0\,.$$

Here ρ, g, T and R are positive constants associated with the density of the fluid, the local gravitational acceleration, tension in the sheet and rigidity of the sheet respectively and x is measured along the interface. In addition, ϕ is the velocity potential of the wave and η is the free surface deflection caused by the wave.

(i) State the equation to be satisfied by ϕ in $z > 0$ and the condition on ϕ as $z \to \infty$. Show that the surface conditions above can be reduced to a single boundary condition for ϕ.

[4 marks]

(ii) Use separation of variables to obtain a solution $\phi(x, z, t)$ that is periodic in x and t. Hence obtain the dispersion relation for these waves in terms of a frequency ω and a wavenumber K in the x direction.

[6 marks]

(iii) Show that (for $K > 0$) the phase speed has a minimum value, and find the value $K = K_{min}$ associated with this minimum phase speed.

[5 marks]

(iv) Obtain an expression for the ratio c_g/c in terms of T, R, g, ρ, K. Evaluate this expression for $K = K_{min}$, $K \to 0$, $K \to \infty$, and interpret the results.

[5 marks]

[Total: 20 marks]

6. A compressible fluid of sound speed c and mean density ρ_0 fills a pipe with rigid impermeable walls. A Cartesian coordinate system is assumed, such that x is measured along the axis of the pipe.

(i) For linear acoustic waves in the pipe, state the equation to be satisfied in the fluid and the boundary condition to be imposed at the walls for a velocity potential ϕ. Hence verify that, for a pipe with a constant cross section, a solution for the velocity potential exists in the form

$$\phi(x,t) = B\exp\Big\{i\omega(t - x/c)\Big\},$$

for any constant B, where ω is a frequency, t is time.

[4 marks]

(ii) If the pipe has cross-sectional area A then give the mass flux Q down the pipe in terms of ρ_0, A and ϕ.

[2 marks]

(iii) You are given that the pressure perturbation of the acoustic field is

$$p = -\rho_0\frac{\partial\phi}{\partial t}.$$

Now consider a pipe that has a piecewise constant cross sectional area, being A_1 for $x \le 0$, A_2 for $x > 0$, with a velocity potential of the form

$$\phi(x,t) = \phi_I(x,t) + \phi_R(x,t),$$

for $x \le 0$ and

$$\phi(x,t) = \phi_T(x,t),$$

for $x > 0$.

For an incoming wave $\phi_I(x,t) = B_I\exp\{i\omega(t - x/c)\}$, give appropriate expressions for ϕ_T and ϕ_R. Hence determine the amplitudes of ϕ_T and ϕ_R relative to ϕ_I, given that p and Q must be continuous at $x = 0$.

Evaluate $|\phi_T|/|\phi_I|$ for the particular case of $A_1 = 2A_2$.

[10 marks]

(iv) Interpret your results in the special cases $A_2 \ll A_1$ and $A_2 \approx A_1$.

[4 marks]

[Total: 20 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL BIOLOGY

21 May 2019

14.00 – 16.00

Answer **ALL** three questions in Section A (30 marks in total).
Answer **TWO** of the THREE questions in Section B (50 marks in total).
If more than TWO questions from Section B are attempted,
then credit will be given for the best TWO answers.

Electronic calculators may be used in accordance with the University regulations

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SECTION A

Answer **ALL** three questions

A1. A fish population of size $N(t)$ that is subject to harvesting at rate $c > 0$ is described by the ordinary differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right) - c. \tag{1}$$

(a) Give a physical interpretation of the parameters r and K. [2 marks]

(b) Writing $N = Kx(\tau)$ and $t = \tau/r$, re-write (1) in the form $\mathrm{d}x/\mathrm{d}\tau = f(x)$, where f includes the parameter $\alpha = c/rK$. [3 marks]

(c) By considering the form of the function $f(x)$, find the largest value of α for which a stable fish population can be maintained. [5 marks]

A2. A gene with expression level $X(t)$ that represses its own transcription is described by the delay-differential equation

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{1}{1 + X(t-T)} - \frac{1}{1 + X_0},$$

where X_0 and T are positive constants.

(a) By setting $X(t) = X_0 + z(t)$, where $|z| \ll X_0$, derive the linear delay-differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{z(t-T)}{(1 + X_0)^2}.$$

[3 marks]

(b) Assuming $z(t) = ae^{\lambda t}$ for some constants a and λ (which may be complex), derive a condition that must be satisfied by λ. [2 marks]

(c) Find the smallest value of T for which z exhibits neutrally stable oscillations (for which λ is purely imaginary). [3 marks]

(d) Show that the resulting oscillations have period $4T$. [2 marks]

A3. A model for an animal population migrating steadily across a spatial domain is given by

$$-cu_z = Du_{zz} + ru(1-u). \tag{2}$$

Here $c > 0$ is a migration speed, D is a diffusion coefficient, and $u(z) \geq 0$ measures population size (relative to a reference value) as a function of distance z.

(a) To describe the solution where $u \ll 1$, we may replace $(1-u)$ in (2) with 1. Seek a solution of the resulting equation of the form $u = Ae^{\lambda z}$ for some constant A, subject to the condition that $u \to 0$ as $z \to \infty$. You may assume that λ is real. Show that c cannot be less than $2\sqrt{Dr}$.

[4 marks]

(b) Suppose now that $c = 2\sqrt{Dr}$ and find λ. Give a physical interpretation of the quantity $\sqrt{D/r}$.

[3 marks]

(c) For this value of c, and writing $u = 1 + \hat{u}$, linearise (2) assuming $|\hat{u}| \ll 1$. Setting $\hat{u} = Be^{\mu z}$ for some constant B, and assuming $\hat{u} \to 0$ as $z \to -\infty$, show that $\mu = \alpha\sqrt{r/D}$ where α is a constant to be determined.

[3 marks]

SECTION B

Answer **TWO** of the three questions

B4. An infectious disease spreads through a population of individuals. The population may be divided into those who have not yet caught the disease (i.e. are susceptible, $S(t)$), those who are infected ($I(T)$) and those who have recovered ($R(t)$). These three sub-populations evolve according to

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta IS, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta IS - \gamma I, \quad \frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I.$$

(a) Let $S + I + R = N$. Show that N is constant during the course of the disease outbreak. [2 marks]

(b) Writing $S(t) = Nu(\tau)$, $I(t) = Nv(\tau)$, where $t = \tau/\gamma$, derive the simplified system

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = -R_0 uv, \quad \frac{\mathrm{d}v}{\mathrm{d}\tau} = v(R_0 u - 1),$$

where R_0 is a dimensionless constant (to be determined) that represents the virulence of the disease. [5 marks]

(c) You may assume that $u(0) = 1$, $v(0) = 0$ and $R_0 > 1$. By constructing $\mathrm{d}v/\mathrm{d}u$, or otherwise, show that the disease evolves under the constraint

$$v = 1 - u + \frac{\log u}{R_0}. \tag{3}$$

Plot graphs of $u - 1$ and $(\log u)/R_0$ to demonstrate that (3) has a solution $u = u_\star$, $v = 0$ satisfying $0 < u_\star < 1/R_0$. [7 marks]

(d) Use (3) to describe the evolution of the outbreak, identifying in particular the largest proportion $v_{\max}$ of the population that is infected at any time. You may find it helpful to sketch a trajectory in the (u, v)-plane. What does the quantity $N(1 - u_\star)$ represent? [6 marks]

(e) Show that $v_{\max}$ is an increasing function of R_0. [2 marks]

(f) Returning to the original formulation of the problem, imagine that recovered individuals slowly lose their immunity to the disease, and so return to the susceptible population at rate ν (where ν has dimensions of 1/time). Write down the modified model for the evolution of $S(t)$, $I(t)$ and $R(t)$. [3 marks]

B5. An enzyme E catalyses the transformation of a substrate S to a product P through the reaction

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C_1 \xrightarrow{k_2} P + E.$$

A drug D is added to the system which binds competitively to the enzyme according to

$$D + E \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} C_2.$$

Here k_1, k_{-1}, k_2, k_3 and k_{-3} are reaction rates and C_1 and C_2 are different complexes. Initially, concentrations of S, E, C_1, C_2, D, P are $S_0, E_0, 0, 0, D_0, 0$ respectively.

(a) Using the principles of mass action kinetics, write down the six ordinary differential equations (ODEs) describing the evolution of the concentrations of six chemical species (S, E, C_1, C_2, D and P) in terms of five different reactions. [6 marks]

(b) Write down the stoichiometric matrix for this system. Hence, or otherwise, identify three conserved quantities, and reduce the system to

$$\begin{aligned}
\dot{C_1} &= k_1(S_0 - C_1 - P)(E_0 - C_1 - C_2) - k_2C_1 - k_{-1}C_1, \\
\dot{C_2} &= k_3(D_0 - C_2)(E_0 - C_1 - C_2) - k_{-3}C_2, \\
\dot{P} &= k_2C_1.
\end{aligned}$$

[8 marks]

(c) Assuming that C_1 and C_2 are close to equilibrium, and that S and D are sufficiently abundant for their concentrations to be assumed constant (i.e. $S_0 \gg C_1 + P$ and $D_0 \gg C_2$), show that

$$S_0(E_0 - C_1 - C_2) = LC_1,$$
$$D_0(E_0 - C_1 - C_2) = KC_2,$$

where $K = k_{-3}/k_3$ and $L = (k_{-1} + k_2)/k_1$. Hence derive the equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{k_2S_0E_0}{S_0 + L + (D_0L/K)}.$$

[8 marks]

(d) Briefly discuss the impact of the drug on the yield of product, making comparison with the case $D_0 = 0$. [3 marks]

B6. In a simple formulation of Turing's reaction-diffusion mechanism for morphogenesis, $X(\theta, t)$ and $Y(\theta, t)$ are the concentrations of two morphogens that interact via

$$\frac{\partial X}{\partial t} = f(X, Y) + \mu \frac{\partial^2 X}{\partial \theta^2}, \qquad \frac{\partial Y}{\partial t} = g(X, Y) + \nu \frac{\partial^2 Y}{\partial \theta^2}, \tag{4}$$

where μ and ν are positive real numbers. X and Y are defined over the spatial domain $0 \leq \theta \leq 2\pi$ and satisfy periodic boundary conditions, so that

$$X(0, t) = X(2\pi, t), \quad Y(0, t) = Y(2\pi, t).$$

You may assume there are numbers $X^\star$ and $Y^\star$ such that $f(X^\star, Y^\star) = g(X^\star, Y^\star) = 0$.

(a) Show that the spatially uniform solution $X(\theta, t) = X^\star$, $Y(\theta, t) = Y^\star$ satisfies (4). [2 marks]

(b) Consider solutions of the form $X(\theta, t) = X_\star + x(\theta, t)$ and $Y(\theta, t) = Y_\star + y(\theta, t)$, where x and y are assumed to be small in magnitude. Linearise (4a,b) to obtain equations of the form

$$\frac{\partial x}{\partial t} = ax + by + \mu \frac{\partial^2 x}{\partial \theta^2}, \qquad \frac{\partial y}{\partial t} = cx + dy + \nu \frac{\partial^2 y}{\partial \theta^2}. \tag{5}$$

Explain how the parameters a, b, c and d appearing in (5) relate to the functions f and g. [8 marks]

(c) Consider solutions to (5) of the form $x(\theta, t) = u(t)\cos(n\theta)$ and $y(\theta, t) = v(t)\cos(n\theta)$ with n a positive integer. Show that the functions $v(t)$ and $u(t)$ evolve according to

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (a - \mu n^2)u + bv, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = cu + (d - \nu n^2)v.$$

[4 marks]

(d) Give two conditions on a, b, c and d that guarantee that the spatially uniform solution is stable when $n = 0$. [6 marks]

(e) Assuming the conditions in (d) hold, show that stability is guaranteed for all $n > 0$ provided

$$ad - bc > \frac{\mu\nu}{4}\left(\frac{a}{\mu} + \frac{d}{\nu}\right)^2.$$

[5 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

SYMMETRY IN GEOMETRY AND NATURE

16 May 2019

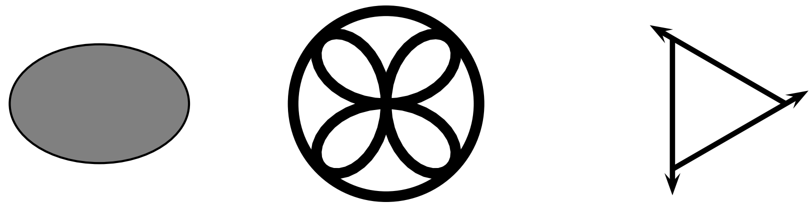
14.00 – 16.00

Answer **ALL FIVE** questions

Electronic calculators may be used, provided that they cannot store text.

1.

(a) Name the symmetry groups of the following three planar shapes, and state in each case the order of the group and whether the group is Abelian.

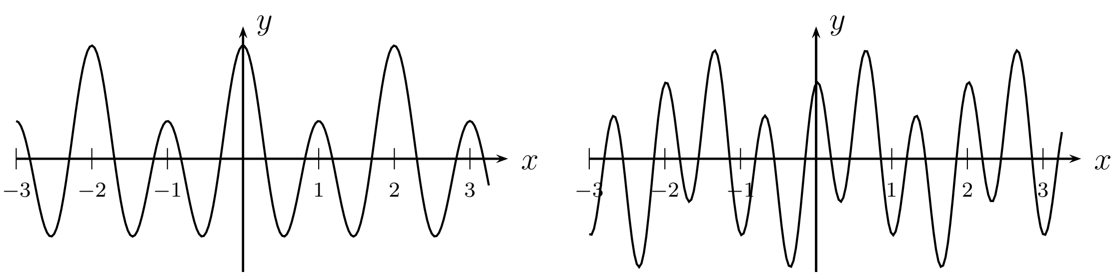


(i) (ii) (iii)

[6 marks]

(b) Given a function $f : \mathbb{R} \to \mathbb{R}$, its *extended symmetry group* consists of all pairs (T, σ) for which T is a transformation of $\mathbb{R}$ and $\sigma \in \mathbb{Z}_2 = \{\pm 1\}$ satisfying $f(Tx) = \sigma f(x)$ (in particular, $f(Tx) = \pm f(x)$).

Describe the extended symmetry group of each of the functions f and g of one variable whose graphs are shown below. (It is sufficient to give generators of each of the groups.)



$y = f(x)$ $y = g(x)$

[6 marks]

2. Suppose a finite group G acts on a set X.

(a) Define the orbit $G \cdot x$ and stabilizer G_x of an element $x \in X$. Show that the stabilizer is always a subgroup of G.

[5 marks]

(b) State the orbit-stabilizer theorem.

[3 marks]

(c) Let $h \in G$ and $y = h \cdot x$. Show that the subset

$$G_{x,y} = \{g \in G \mid g \cdot x = y\}$$

is a left coset of G_x, and explain briefly how this can be used to prove the orbit-stabilizer theorem (details of the proof are not required).

[6 marks]

(d) The 'octahedral group' $\mathbb{O}$ of order 24 acts by rotations on the cube. On each face of the cube the 2 diagonals are drawn. Use the orbit-stabilizer theorem to find the order of the stabilizer of one of these diagonals.

[4 marks]

3.

(a) Define the action on $\mathbb{R}^2$ of an element $(A \mid \mathbf{v}) \in \mathsf{E}(2)$, where $A \in \mathsf{O}(2)$ and $\mathbf{v} \in \mathbb{R}^2$. Show that multiplication in the group is given by

$$(A \mid \mathbf{v})(B \mid \mathbf{u}) = (AB \mid \mathbf{v} + A\mathbf{u}),$$

and deduce the Seitz symbol of the inverse transformation $(A \mid \mathbf{v})^{-1}$.

[6 marks]

(b) Consider the following lattice in the plane:

$$L = \left\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z},\ y \in \mathbb{Z},\ x + y \in 3\mathbb{Z}\right\}.$$

(i) Show that the two vectors $\mathbf{u}_1 = (3, 0)$ and $\mathbf{u}_2 = (0, 3)$ both belong to L. Sketch a diagram showing all the points of L that lie in the rectangle $-3 \le x \le 3$ and $-3 \le y \le 3$ and find a point in L which is not contained in $\mathbb{Z}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(ii) Find two vectors $\mathbf{a}$ and $\mathbf{b}$, as required for the classification of symmetry types of lattices; that is, such that $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$, and $|\mathbf{a}| \le |\mathbf{b}| \le |\mathbf{b} - \mathbf{a}| \le |\mathbf{b} + \mathbf{a}|$, proving carefully that your choice satisfies this.

(iii) Identify which of the 5 types of lattice L is (Oblique, Rectangular, Centred rectangular, Square or Hexagonal).

[16 marks]

4. Consider a system of 3 coupled cells governed by the system of differential equations

$$\begin{cases} \dot{x} &= x - yz \\ \dot{y} &= y - xz \\ \dot{z} &= z - xy. \end{cases} \qquad (*)$$

(a) Show that the system $(*)$ has S_3 symmetry, and draw a cell diagram illustrating the system.

[4 marks]

(b) Deduce that the diagonal, where $x = y = z$, is invariant under the evolution of the system, stating carefully any results used.

[4 marks]

(c) Find all the equilibrium points of the system. Divide them into group orbits and find the Burnside type of the set of equilibrium points.

[6 marks]

(d) Let $\mathbf{x}(t)$ be the solution to this system with initial value $(x, y, z) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Find which of the equilibria is equal to $\lim_{t\to\infty} \mathbf{x}(t)$. [You are not expected to solve the differential equation.]

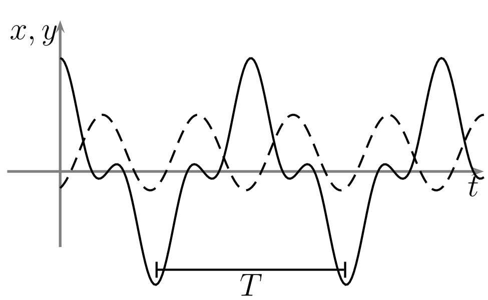
[4 marks]

5.

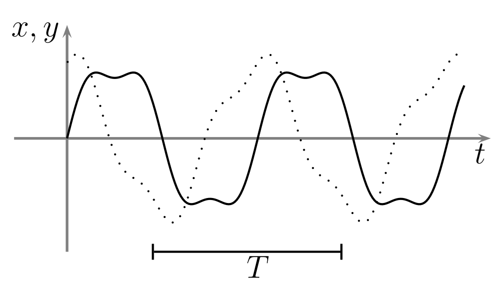
(a) Define the symmetry group of a periodic solution $\gamma(t)$ with fundamental period T of a differential equation with symmetry group G.

[3 marks]

(b) Suppose a mechanical system in the plane has potential energy, with D_2 symmetry; that is $V(x,y) = V(-x,y) = V(x,-y)$. Let γ be a periodic orbit with period T and symmetry group $\widetilde{\mathsf{C}_2}$, and let $\delta(t)$ be a periodic solution of the same period and with symmetry $\widetilde{D_1}$. Write down in each case what the symmetry group tells us about $x(t)$ and $y(t)$ for each periodic orbit. Below is shown numerical data taken from two experiments with this symmetry showing both $x(t)$ and $y(t)$. Determine which solution is $\gamma(t)$ and which is $\delta(t)$, giving a brief justification of your answer.



Experiment 1



Experiment 2

[7 marks]

END OF EXAMINATION PAPER

MATH36022

TWO Hours

THE UNIVERSITY OF MANCHESTER

NUMERICAL ANALYSIS 2

3rd June 2019

09:45–11:45

Answer **ALL** questions in **SECTION A** (40 marks in total)
and
Answer **TWO** of the three questions in **SECTION B** (20 marks each).
If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

Electronic calculators may be used in accordance with the University regulations

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SECTION A

Answer **ALL** five questions

A1.

(a) Let $w(x)$ be a given weight function. Define the $L_{2,w}$ norm of a function $f \in C[a,b]$, denoted by $\| f \|_{2,w}$. State any properties that $w(x)$ must satisfy. In addition, explain what is meant by the phrase g is a "best weighted L_2 approximation" to f from a subspace $G \subset C[a,b]$.

[4 marks]

(b) Suppose we want to find an approximation to $f \in C[a,b]$ of the form

$$g(x) = \sum_{i=0}^{n} c_i \phi_i(x),$$

where $\{\phi_i(x)\}_{i=0}^{n}$ are linearly independent. Define the normal equations that can be solved to find the best weighted L_2 approximation. Explain why it is advantageous to use orthogonal polynomials ϕ_i (where ϕ_i has degree i) and in that case, state the solution c_i, $i = 0, 1, \ldots, n$.

[5 marks]

A2. Recall that the Chebyshev polynomial of degree n is defined by $T_n(x) = \cos(n \cos^{-1}(x))$ for $x \in [-1,1]$ and $n = 0, 1, 2, \ldots$.

Define the set of $n+1$ Chebyshev points x_i, $i = 0, 1, \ldots, n$ associated with the Chebyshev polynomial $T_{n+1}(x)$ and briefly explain why it is beneficial to use Chebyshev points as interpolation points when constructing polynomial approximations p_n of degree $\leq n$ to a function $f \in C[-1,1]$.

[7 marks]

A3. Divide $[0,1]$ into n equal intervals $[x_i, x_{i+1}]$ of width h, where $x_i = ih$, $i = 0, 1, \ldots, n$ and consider the repeated trapezium rule

$$I(f) = \int_0^1 f(x)\,dx \approx \sum_{i=0}^{n-1} \frac{h}{2}\big(f(x_i) + f(x_{i+1})\big) = T(h).$$

Assuming that f is sufficiently differentiable, the Euler-Maclaurin summation formula gives an asymptotic expansion for the error of the form

$$I(f) - T(h) = a_2 h^2 + a_4 h^4 + a_6 h^6 + \ldots$$

for some real constants $a_2, a_4, a_6, \ldots$.

(a) Use the Euler-Maclaurin formula to determine what happens to the error when we double the number of intervals $[x_i, x_{i+1}]$.

[3 marks]

(b) By taking an appropriate linear combination of the rules $T(h)$ and $T(\frac{h}{2})$, derive a new quadrature scheme that gives an error that is $\mathcal{O}(h^4)$.

[4 marks]

A4. Suppose that we approximate $I(f) = \int_0^1 f(x)dx$ using the following three-point quadrature rules

$$\frac{1}{6}\left(f(0) + 4f(1/2) + f(1)\right), \qquad \frac{1}{3}\left(2f(1/4) - 4f(1/2) + 2f(3/4)\right),$$

and suppose that small errors are made in evaluating f at the chosen quadrature points x_i. Let ϵ_i be the error in $f(x_i)$ and suppose $|\epsilon_i| \leq 10^{-4}$. Briefly analyse the effect of this data error for both quadrature schemes. Why are negative weights not desirable?

[7 marks]

A5. The ℓ-step Adams linear multistep methods for the initial value problem $y' = f(x, y)$, $x \geq x_0$, $y(x_0) = y_0$ have the form

$$y_{n+1} = \sum_{i=1}^{\ell} a_i y_{n+1-i} + h \sum_{i=0}^{\ell} b_i f_{n+1-i},$$

where a_i, b_i are given constants, h is the step size, and $f_n = f(x_n, y_n)$.

(a) Explain how we can derive the ℓ-step Adams–Bashforth method and the ℓ-step Adams–Moulton method by replacing the integrand in

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))\, dx$$

with a suitable polynomial. Briefly comment on the accuracy of the two ℓ-step methods and their computational costs.

[6 marks]

(b) By introducing the appropriate polynomial, derive the 1-step Adams–Bashforth method. By what name is this scheme better known?

[4 marks]

SECTION B

Answer **TWO** of the three questions

B6.

(a) State **two** reasons why Padé approximants offer more flexibility than single polynomials for approximating functions. In addition, explain **briefly** why it is beneficial to consider continued fraction representations of Padé approximants.

[4 marks]

(b) Consider

$$r_n(x) = \cfrac{c_1 x}{1 + \cfrac{c_2 x}{1 + \cfrac{c_3 x}{1 + \cdots + \cfrac{c_{n-2} x}{1 + \cfrac{c_{n-1} x}{1 + c_n x}}}}}$$

with coefficients

$$c_1 = 1, \qquad c_{2j} = \frac{j}{2(2j-1)}, \quad c_{2j+1} = \frac{j}{2(2j+1)}, \qquad j = 1, 2, \ldots$$

Show that

$$r_3(x) = \frac{x + x^2/6}{1 + 2x/3}$$

and hence that $r_3(x)$ is the $[2/1]$ Padé approximant of $f(x) = \log(1 + x)$.

[7 marks]

(c) The Top-Down algorithm for evaluating the continued fraction

$$r_n(x) = b_0 + \cfrac{a_1 x}{b_1 + \cfrac{a_2 x}{b_2 + \cfrac{a_3 x}{b_3 + \cdots + \cfrac{a_{n-1} x}{b_{n-1} + \cfrac{a_n x}{b_n}}}}} \tag{1}$$

is given by

$p_{-1} = 1$, $q_{-1} = 0$, $p_0 = b_0$, $q_0 = 1$
for $j = 1\colon n$
    $p_j = b_j p_{j-1} + a_j x p_{j-2}$
    $q_j = b_j q_{j-1} + a_j x q_{j-2}$
end
$r_n = p_n q_n^{-1}$

Prove by induction that p_n/q_n, where p_n, q_n are computed via the Top-Down algorithm, is equal to $r_n(x)$ defined in (1).

[9 marks]

B7. Consider the integral

$$I(f) = \int_a^b w(x) f(x)\, dx,$$

where $w(x)$ is a valid weight function and let $\{\phi_i(x)\}$ be the system of polynomials that are orthogonal with respect to $w(x)$ on $[a, b]$.

(a) Using the idea of interpolatory quadrature, derive an expression for the weights w_i in the n-point rule

$$I(f) \approx G_n(f) = \sum_{i=1}^{n} w_i f(x_i),$$

that ensures the rule is exact for all polynomials of degree $\leq n-1$.

[4 marks]

(b) If the weights w_i are chosen as in part (a) and the nodes $x_1, \ldots, x_n$ are chosen as the zeros of the polynomial $\phi_n(x)$ of degree n, prove that the resulting Gauss rule $G_n(f)$ is in fact exact for polynomials of degree $\leq 2n-1$.

[8 marks]

(c) Derive the two point Gauss–Hermite rule

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx \approx w_1 f(x_1) + w_2 f(x_2),$$

using the method of undetermined coefficients. You may use the results

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}, \qquad \int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}$$

without proof.

HINT: You may find it useful to define a quadratic polynomial $\Phi(x) = x^2 + ax + b$ whose roots coincide with x_1 and x_2, and find a and b.

[8 marks]

B8. Consider the numerical method

$$y_{n+1} = y_n + \frac{h}{2}\big(f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\big), \qquad (\star)$$

for the initial value problem $y'(x) = f(x, y(x)), x \geq x_0$ with $y(x_0) = y_0$, where h is the chosen stepsize and y_n is an approximation to $y(x_n)$ at the grid point x_n for $n = 0, 1, \ldots$.

(a) Explain how the method in ($\star$) can be interpreted as:

(1) an 'improved' Euler scheme,

(2) a 2-stage Runge-Kutta scheme.

[6 marks]

(b) Define the terms *local truncation error* (denoted $\tau(h)$) and *order* of a numerical method for initial value problems.

[3 marks]

(c) Show that for the numerical method ($\star$),

$$\tau(h) = y(x_{n+1}) - y(x_n) - \frac{h}{2}y'(x_n) - \frac{h}{2}f(x_n + h, y(x_n) + hy'(x_n)).$$

[3 marks]

(d) Using part (c), show that the numerical method in ($\star$) has order two.

HINT: you may find the following identity useful

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f.$$

[8 marks]

END OF EXAMINATION PAPER

MATH37012

Two hours

THE UNIVERSITY OF MANCHESTER

MARKOV PROCESSES

03 June 2019

9.45 - 11.45

Answer **ALL** four questions.

The use of electronic calculators is prohibited.

---

1. a)

(i) Define the terms "recurrent state" and "transient state" in the context of discrete time Markov Chains.

[2 marks]

(ii) Define the stationary distribution of a discrete time Markov Chain.

[2 marks]

(iii) Using a transition diagram, give an example of a Markov Chain with two transient states, six recurrent states and $i, j \in S$ such that

$$P\left(X_n = j \text{ for some } n \geq 1 \mid X_0 = i\right) = \frac{2}{3}.$$

[2 marks]

(iv) Give an example of a four state Markov Chain such that, $\forall i, j \in S$,

$$lim_{n \to \infty} p_{ij}(n) \text{ does not exist.}$$

[2 marks]

(v) Give an example of a Markov Chain which has more than one stationary distribution.

[2 marks]

b) Consider the discrete time Markov Chain $\{X_n : n \geq 0\}$ with state space {1,2,3,4,5} and transition probability matrix

$$\begin{bmatrix} 0.4 & 0 & 0 & 0.6 & 0 \\ 0.3 & 0 & 0 & 0 & 0.7 \\ 0.5 & 0.1 & 0.4 & 0 & 0 \\ 0.4 & 0 & 0 & 0.6 & 0 \\ 0 & 0.7 & 0 & 0 & 0.3 \end{bmatrix}$$

(i) Show that state 5 is transient.

[2 marks]

(ii) Determine a stationary distribution for the Markov chain.

[3 marks]

(iii) Carefully stating any results to which you appeal, determine $\lim_{n\to\infty} p_{11}(n)$ and $\lim_{n\to\infty} p_{52}(n)$.

[4 marks]

(iv) Write down the mean recurrence time of state 1.

[1 marks]

2. Let $\{X_n, n \geq 0\}$ be a Markov chain, with state space the set of non-negative integers and transition probabilities

$$\begin{aligned} p_{0,0} &= p \\ p_{i,i+1} &= q, \quad 0 \leq i \\ p_{i,i-1} &= p, \quad 1 \leq i. \end{aligned}$$

where $0 < p < 1$ and $q = 1 - p$.

(i) Determine a stationary distribution for this Markov chain, stating the range of p for which a stationary distribution exists.

[6 marks]

(ii) Giving a reason, state the range of p for which the Markov chain is positive recurrent. For p in this range, write down the mean recurrence time of state 0.

[3 marks]

(iii) Prove that, if a Markov chain with transition matrix P has stationary distribution $\underline{\pi}$, then so does a Markov chain with transition matrix $\frac{1}{2}(I + P)$.

[3 marks]

(iv) Let $\{Y_n, n \geq 0\}$ be a Markov chain, with state space the set of non-negative integers and transition probabilities

$$\begin{aligned} p_{0,0} &= \frac{5}{6} \\ p_{i,i+1} &= \frac{1}{6}, \quad 0 \leq i \\ p_{i,i-1} &= \frac{1}{3}, \quad 1 \leq i \\ p_{i,i} &= \frac{1}{2}, \quad 1 \leq i. \end{aligned}$$

Using the result of part (iii) or otherwise, determine a stationary distribution for this Markov chain.

[4 marks]

(v) For the Markov chain of part (iv), determine the mean recurrence time of state zero and hence, or otherwise, the expected time of first visit to state zero starting at state 1.

[4 marks]

3. a) At time 0 a population comprises 1 individual. Lifetimes of individuals are independent and each is exponentially distributed with parameter λ. At death each individual splits into 2 new individuals. Let $X(t)$ denote the number of individuals alive in the population at time t.

(i) With careful reference to the lack of memory property of the exponential distribution, and to the distribution of the minimum of a number of independent exponential random variables, derive the generator matrix Q for this process.

[5 marks]

(ii) For $X(t) : t \geq 0$, write down the forward equations in matrix form and show they include

$$p'_{1,k}(t) = (k-1)\lambda p_{1,k-1}(t) - k\lambda p_{1,k}(t) \qquad 1 \leq k.$$

[5 marks]

b) A telephone network connects 4 subscribers. When not engaged in a conversation, each makes calls in a Poisson process with rate λ independently of the others. The recipient of a call is equally likely to be any of the other 3 subscribers. If that recipient is not engaged, a conversation begins; otherwise the call is lost. Call durations are independently exponentially distributed with parameter μ.

Let the Markov chain $X(t) : t \geq 0$, with state space $S = \{0, 1, 2\}$, denote the number of conversations in progress at time t.

(i) Stating any results to which you appeal, derive the generator matrix Q for $X(t)$ and calculate its stationary distribution.

[6 marks]

(ii) Determine the expected time of first visit to state 2, starting from state 0.

[4 marks]

4. a) Consider a Markov queue where customers arrive in a Poisson process, rate λ, each customer is served immediately on arrival (effectively, there is an unlimited number of servers), and service times are independent, exponentially distributed with parameter μ, and independent of the arrival process.

(i) Carefully stating any results to which you appeal, determine the generator matrix Q.

[3 marks]

(ii) Determine the stationary distribution of the number of customers in the system and write down its expected value.

[5 marks]

b) Let the Markov chain $X(t) : t \geq 0$, with state space $S = \{0, 1, 2 \ldots m\}$, denote the number of items in a storage facility. When the facility contains at least one item, the intervals between removals are exponentially distributed with parameter λ. Following the removal of the final item, the facility is replenished with m items after an exponential, parameter μ, distributed interval. All the exponential random variables are independent.

(i) Determine the stationary distribution of the Markov chain.

[5 marks]

(ii) Determine the expected time taken for the facility to first empty following replenishment.

[3 marks]

(iii) The Markov chain is observed when stationary. Determine the expected time elapsed before the facility is first empty.

[4 marks]

END OF EXAMINATION PAPER

Statistical Tables to be provided

Two hours

THE UNIVERSITY OF MANCHESTER

Time Series Analysis

30 May 2019
09.45 – 11.45

Electronic calculators are permitted, provided they cannot store text.

Answer **ALL** Four questions in Section A (32 marks in all)

and

ALL Two questions in Section B (24 marks each).

Where verbal answers are required, they should be concise and clear—typically, a single phrase or sentence for each worthy thought.

SECTION A
Answer ALL four questions

A1. Let $\{X_t\}$ be a stationary time series, such that

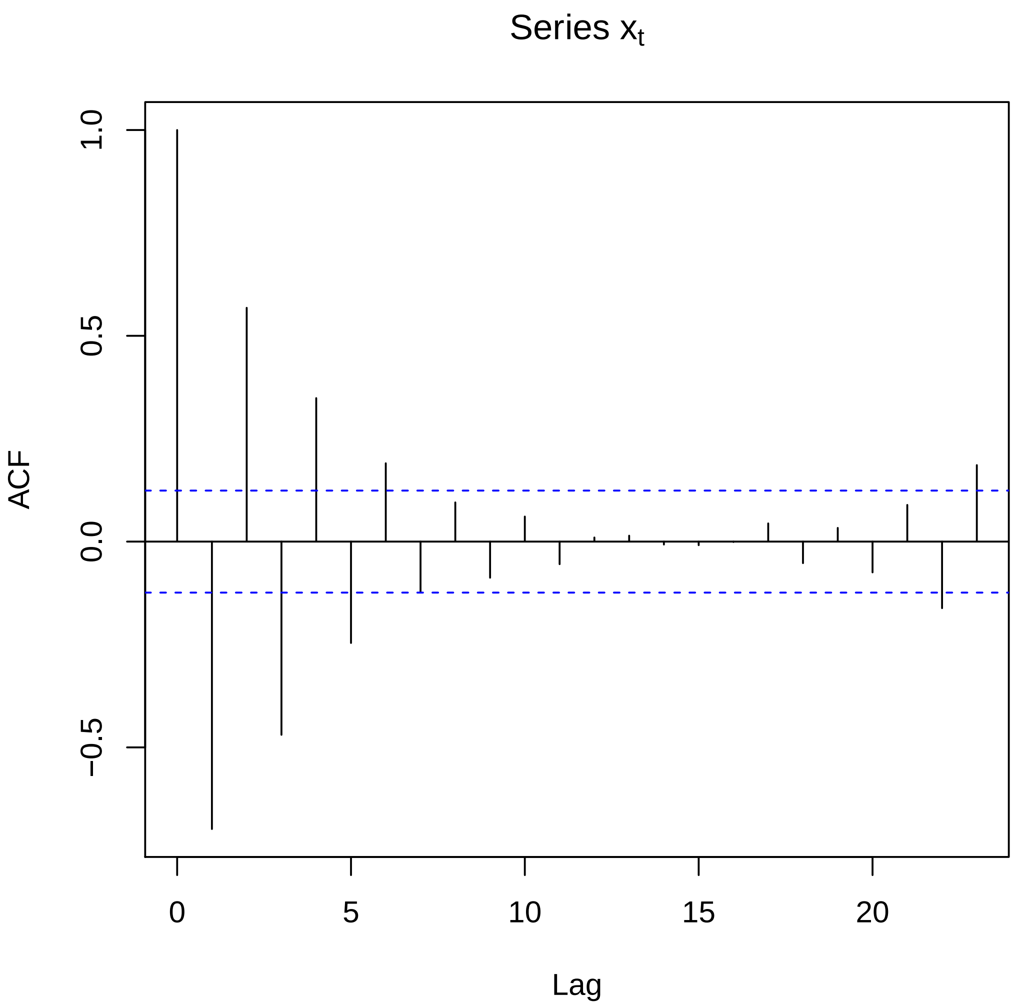
$$X_t = \phi_1 X_{t-1} + \varepsilon_t,$$

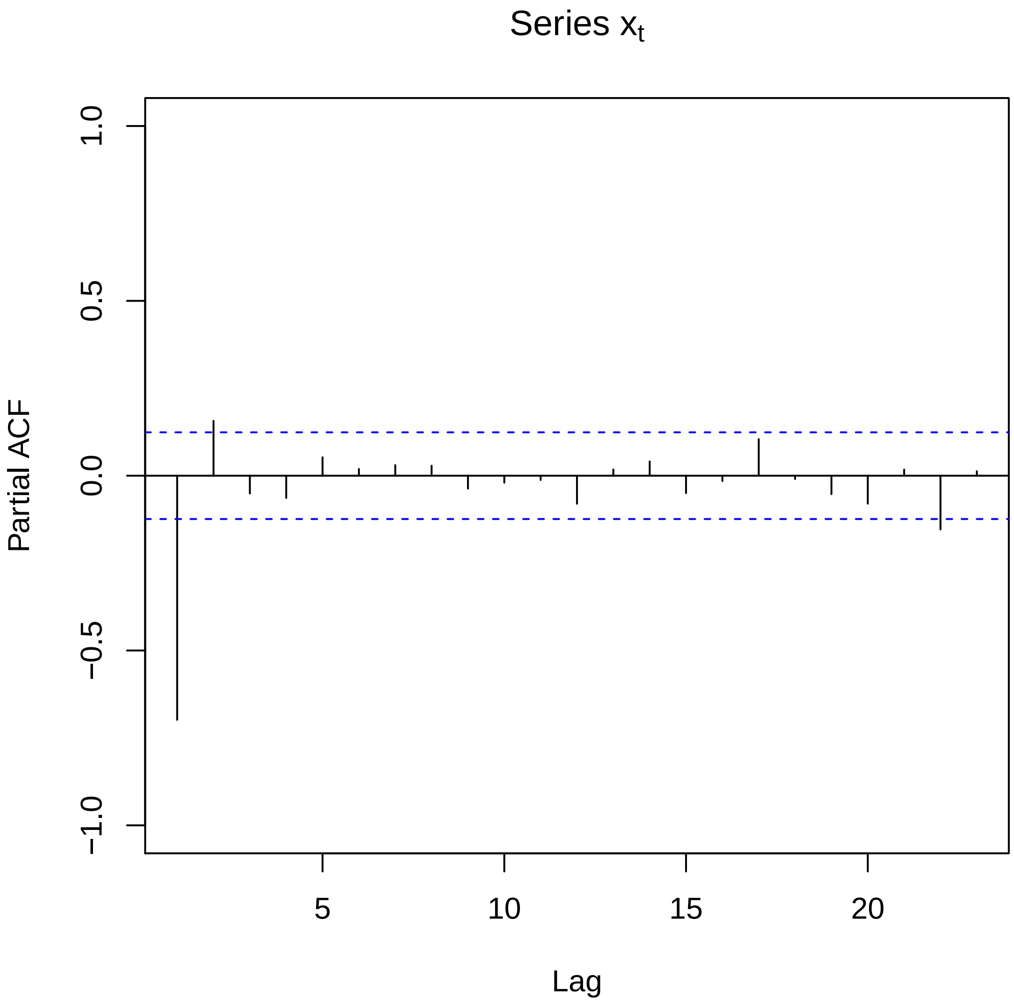
where $|\phi_1| < 1$, $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$, and $\mathrm{E}\,\varepsilon_t X_{t-k} = 0$ for $k > 0$. Let $\{u_t\}$ be white noise, $\mathrm{WN}(0, \sigma_u^2)$, independent of $\{X_t\}$ and $\{\varepsilon_t\}$. Consider the time series $\{Y_t\}$, such that

$$Y_t = X_t + u_t.$$

Find the autocovariance function of $\{Y_t\}$ for all lags (give the expressions in terms of ϕ_1 and the variances of the white noise sequences). [8 marks]

A2. The graphs below show the sample autocorrelations and sample partial autocorrelations of an observed time series, $\{x_t\}$. Based on these graphs, what models would you consider for $\{x_t\}$? Which model would be your first choice? (Justify your answers concisely.)





[8 marks]

A3.

a) (2 marks) The sample autocorrelations, $\hat{\rho}_k$, computed from a sequence of independent identically distributed random variables of length n, are approximately independent and have normal (Gaussian) distributions with mean zero and variance $1/n$. Consider the statistic

$$\mathrm{BP} = n \sum_{k=1}^{h} \hat{\rho}_k^2.$$

Give an informal argument that the distribution of the statistic BP is approximately χ_h^2.

b) (6 marks) An ARMA(2,1) model was fitted to an observed time series, $\{x_t\}$, $t = 1, \ldots, 200$. As part of the diagnostic checking of the fitted model, the values of the Ljung-Box statistic,

$$Q_{LB}(h) = n(n+2) \sum_{k=1}^{h} \frac{\hat{\rho}_k^2}{n-k}, \tag{1}$$

were computed for $h = 1, \ldots, 8$. The table below gives the values of the statistics $Q_{LB}(h)$ and the corresponding p-values.

	1	2	3	4	5	6	7	8
X-squared	0.0131	0.0736	0.0736	0.1334	1.0152	1.1412	4.1835	4.2584
p.value	0.9090	0.9639	0.9948	0.7150		0.7671	0.3817	0.5128

Complete the table by computing the omitted p-value for $h = 5$. What do $\hat{\rho}_k$, $k = 1, \ldots, h$ stand for in Equation (1)? What can you infer about the quality of the fitted model?

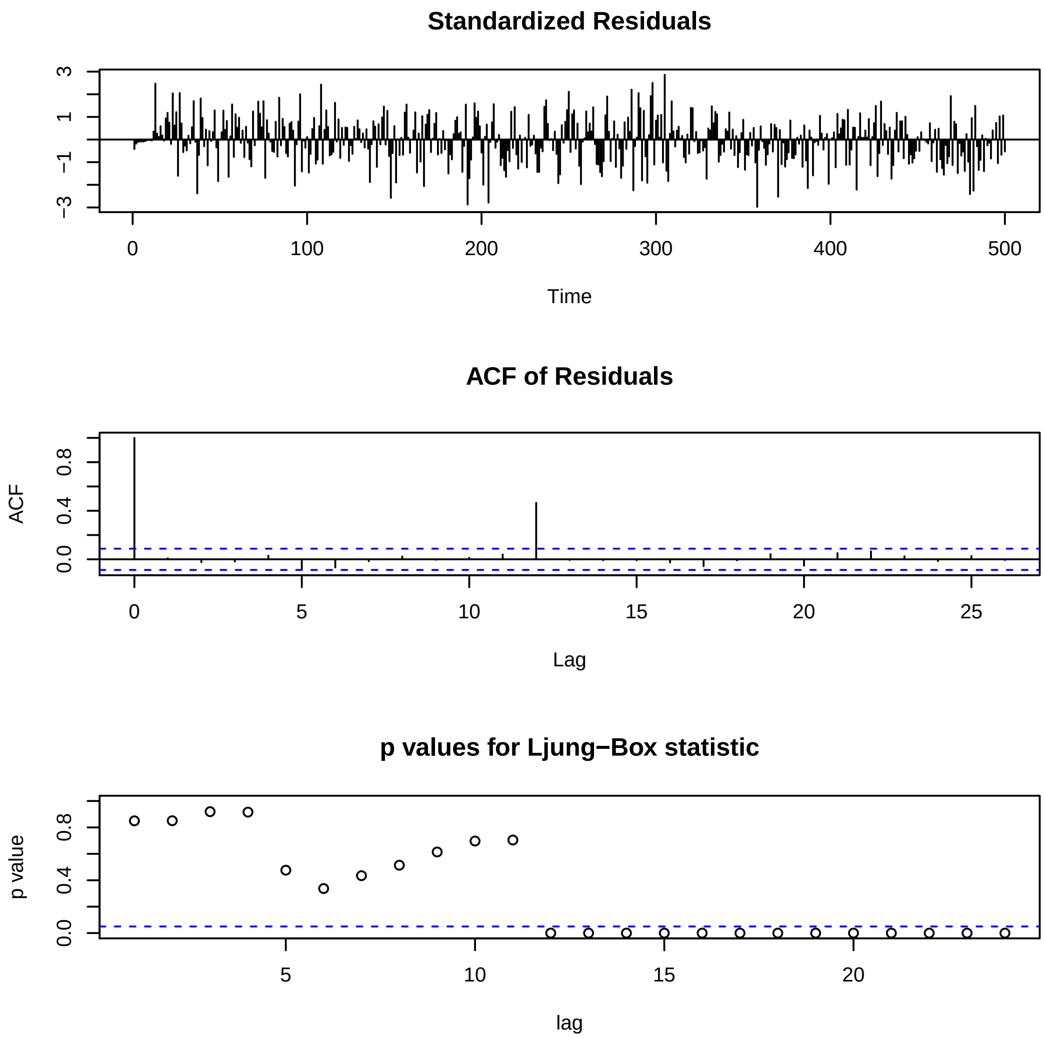
[8 marks]

A4. A seasonal ARIMA model was fitted to a monthly time series. A summary of the fitted model and some residual diagnostics are given below.
Write down the mathematical equation of the fitted model in operator form. Is there evidence that the model is not adequate? If so, suggest a modification of the model that may lead to a better fit.

```
Call:
arima(x = y, order = c(2, 1, 1), seasonal = list(order = c(0, 1, 0), period = 12))

Coefficients:
         ar1      ar2     ma1
      0.8498  -0.4475  0.5438
s.e.  0.0508   0.0482  0.0466

sigma^2 estimated as 1.582:  log likelihood = -803.8,  aic = 1615.61
```



[8 marks]

SECTION B
Answer ALL two questions

B5.

a) (6 marks) Consider the time series $\{X_t\}$, $t \geq 0$, defined by the equations

$$X_0 = 0$$
$$X_t = 0.5X_{t-1} + \varepsilon_t, \qquad t \geq 1,$$

where $\{\varepsilon_t\} \sim \mathrm{WN}(0, 1)$.
Check if the time series $\{X_t\}$, $t \geq 0$, is stationary and report your conclusion. (*No marks will be awarded for unsubstantiated yes/no guesses.*)

b) (18 marks) Let Y be a random variable with mean $\mathrm{E}\,Y = a$ and variance $\mathrm{Var}(Y) = \sigma_y^2$. Consider the time series $\{X_t\}$, $t \geq 0$, defined by the equations

$$X_0 = Y$$
$$X_t = c + \phi X_{t-1} + \varepsilon_t, \qquad t \geq 1,$$

where c is a constant, $|\phi| < 1$, $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$, and $\mathrm{E}\,\varepsilon_t Y = 0$ for all $t \geq 1$.

i) Obtain a condition on the mean, a, of Y, that ensures that $\mathrm{E}\,X_t$ is constant.

ii) Assuming that $\mathrm{E}\,X_t$ is constant (i.e., a is chosen according to part (i) above), show that $\mathrm{Cov}(X_t, X_{t-1}) = \phi\,\mathrm{Var}(X_{t-1})$.

iii) Find necessary and sufficient conditions for stationarity of $\{X_t\}$, $t \geq 0$.

[24 marks]

B6. Consider the following seasonal ARIMA model for a time series $\{X_t\}$:

$$(1-\mathbf{B})(1-\mathbf{B}^7)(1-0.5B)X_t = \varepsilon_t,$$

where $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$.

a) (2 marks) What other property of $\{\varepsilon_t\}$ is assumed implicitly in the specification of ARIMA models, such as the above? (It may be referred to as orthogonality or causality condition.)

b) (2 marks) Identify the model as an ARIMA$(p, d, q)(p_s, d_s, q_s)_s$ model, i.e., give the values of s, p, d, q, p_s, d_s, and q_s.

c) (4 marks) Let $f(h) = \hat{X}_{60+h|60,59,\ldots}$ be the h-step ahead best linear predictor at time $t = 60$. Give recursive equations for computation of $f(h)$ for $h = 1, 2, \ldots$.

d) (4 marks) Give the variance of the prediction errors for horizons $h = 1$ and $h = 2$.

e) (6 marks) Give a state-space form of the model for $\{X_t\}$.

f) (6 marks) Consider processes $\{Y_t\}$ and $\{Z_t\}$ defined as follows

$$\begin{aligned} Y_t &= Y_{t-1} + \varepsilon_t \\ Z_t &= \phi Y_{t-1} + u_t, \end{aligned}$$

where $\phi \neq 0$, $\{\varepsilon_t\}$ and $\{u_t\}$ are white noise sequences, and $\{u_t\}$ is independent of $\{\varepsilon_t\}$ (and therefore of $\{X_t\}$). Show that $\{Y_t\}$ and $\{Z_t\}$ are cointegrated and give a cointegrated relation and the corresponding cointegration vector.

[24 marks]

END OF EXAMINATION PAPER

Two Hours

Statistical tables are provided at end of paper

THE UNIVERSITY OF MANCHESTER

GENERALISED LINEAR MODELS

24 May 2019

9:45–11:45

Answer ALL THREE questions in Section A.
Answer TWO of the THREE questions in Section B.
If more than TWO questions from section B are attempted
then credit will be given to the best TWO answers.

Electronic calculators may be used in accordance with the University regulations

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SECTION A

Answer **ALL** of the three questions

A1.

(a) Give the general form of a distribution from the exponential family and indicate which is the natural parameter and which is the dispersion parameter. [4 marks]

(b) Show that the distribution with probability mass function

$$P(Y = y) = e^{-\lambda y}(1 - e^{-\lambda}),\ y = 0, 1, 2, \dots$$

belongs to the exponential family, with $\lambda > 0$ being the natural parameter. [4 marks]

(c) Use Property 1 of distributions from the exponential family to find the mean and variance of the distribution in (b) and give your answers in terms of λ. [4 marks]

(d) Find the canonical link for a generalised linear model (GLM) with response following the distribution in (b). [3 marks]

(e) Write the mean response μ for the GLM in (d) with canonical link in terms of the linear predictor η and state what the formula can be used for after fitting the model. [3 marks]

(f) The Newton-Raphson algorithm is identical to which two other algorithms for the fitting of the model in (d) with canonical link? Give the names of the algorithms in full. [2 marks]

[20 marks in total for this question.]

A2.

(a) Write down a generalised linear model for $y_i \sim Pois(\mu_i)$ on a single explanatory variable x taking values x_i, $i = 1, \ldots, n$. Use the square root link $g(\mu) = \sqrt{\mu}$ and linear predictor $\eta = \beta_0 + \beta_1 x$ in this model. [3 marks]

(b) Derive the log-likelihood function $\ell(\underset{\sim}{\beta})$ for the model in (a), where $\underset{\sim}{\beta} = (\beta_0, \beta_1)^\mathsf{T}$. Make sure you use the link function specified above. [3 marks]

(c) Following on from (a), let

$$\xi_i = \frac{1}{2}\left(\beta_0 + \beta_1 x_i + \frac{y_i}{\beta_0 + \beta_1 x_i}\right), \ i = 1, \ldots, n.$$

Find the mean vector and variance-covariance matrix of $\underset{\sim}{\xi} = (\xi_1, \ldots, \xi_n)^\mathsf{T}$. [3 marks]

(d) By differentiating the log-likelihood function in (b), show that the gradient vector or score is

$$\frac{\partial \ell}{\partial \underset{\sim}{\beta}} = 4\sum_{i=1}^{n}(\xi_i - \beta_0 - \beta_1 x_i)\begin{pmatrix} 1 \\ x_i \end{pmatrix},$$

where ξ_i is as defined in (c). [4 marks]

(e) Use the result in (d) to show that the MLE of $\underset{\sim}{\beta}$ satisfy

$$X^\mathsf{T} X \underset{\sim}{\beta} = X^\mathsf{T} \underset{\sim}{\xi},$$

where X is the data matrix and $\underset{\sim}{\xi}$ is as defined in (c). [3 marks]

(f) Explain in detail how the equation in (e) is solved iteratively and interpret what happens at each iteration. [4 marks]

[20 marks in total for this question.]

SECTION B

Answer **TWO** of the three questions

B1. The Gamma distribution with probability density function (pdf) written as

$$f(y;\ \alpha,\lambda) = \frac{1}{\Gamma(\alpha)}\lambda^{\alpha}y^{\alpha-1}e^{-\lambda y},\ y>0$$

is known to belong to the exponential family with dispersion parameter $\phi = \alpha$ and $a(\phi) = 1/\phi$. Both parameters α and λ are positive, and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\mathrm{d}x$ is the Gamma function.

(a) Find the natural parameter θ and the function $b(\theta)$. [4 marks]

(b) Use Property 1 of distributions from the exponential family or otherwise to show that the mean is $\mu = \alpha/\lambda$ and find the variance function $V(\mu)$. [4 marks]

(c) Show that the scaled deviance of a generalised linear model fitted to Gamma responses $y_1,\ldots,y_n$ is

$$D^* = 2\alpha\sum_{i=1}^{n}\left(\frac{y_i}{\hat{y}_i} - \log\frac{y_i}{\hat{y}_i} - 1\right),$$

where $\hat{y}_i$, $i = 1,\ldots,n$ are fitted values. [4 marks]

(d) Suppose the model in (c) with linear predictor $\beta_0 + \beta_1 x$ is fitted to $n = 22$ observations with

$$\sum_{i=1}^{n}\frac{y_i}{\hat{y}_i} = 31.84 \text{ and } \sum_{i=1}^{n}\log\frac{y_i}{\hat{y}_i} = 5.37.$$

Use these results to estimate α. [4 marks]

(e) If the true value of α is 3, does the model above appear to provide an adequate fit to the data? Use the scaled deviance to answer this question. [4 marks]

[20 marks in total for this question.]

B2. The Insurance dataset from the MASS library has information on the following variables.

- Claims = number of claims on a policy during a set period of time.
- Holders = number of policy holders on a policy (who can drive the car insured and make a claim).
- District = geographic location (4 categories, 4=large cities).
- Group = Car group by engine size (from 1=small to 4=large)
- Age = Age group of policy holder (from 1=young to 4=old)

A Poisson model with an offset was fitted to the data with the following results.

```
> fit <- glm(Claims ~ District + Group + Age + offset(log(Holders)),
+      data = Insurance, family = poisson)
> summary(fit)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.82174    0.07679 -23.724  < 2e-16 ***
District2    0.02587    0.04302   0.601 0.547597
District3    0.03852    0.05051   0.763 0.445657
District4    0.23421    0.06167   3.798 0.000146 ***
Group2       0.16134    0.05053   3.193 0.001409 **
Group3       0.39281    0.05500   7.142 9.18e-13 ***
Group4       0.56341    0.07232   7.791 6.65e-15 ***
Age2        -0.19101    0.08286  -2.305 0.021149 *
Age3        -0.34495    0.08137  -4.239 2.24e-05 ***
Age4        -0.53667    0.06996  -7.672 1.70e-14 ***

(Dispersion parameter for poisson family taken to be 1)
Null deviance: 236.26  on 63  degrees of freedom
Residual deviance:  51.42  on 54  degrees of freedom
```

(a) Write the model that was fitted in mathematical/statistical terms, stating clearly any assumptions associated with it. You may use claims_i and holders_i to denote the claims and policy holder numbers respectively on policy i with factor levels district_i, group_i and age_i, $i = 1, \ldots, 64$, and α_j, β_j, γ_j $(j = 1, 2, 3, 4)$ for the main effects of the factors. [4 marks]

(b) Explain the use of the offset term. [4 marks]

(c) Does the model provide an adequate fit to the data? Give a reason for your answer. [4 marks]

(d) What can be said of the different levels of District? [4 marks]

(e) Find the combination of factor levels that gives the highest probability of a claim per policy holder and estimate this probability. [4 marks]

[20 marks in total for this question.]

B3.

(a) A random sample of n subjects are classified according to characteristic A with I levels and characteristic B with J levels. Let y_{ij} be the number of subjects in the combined category $(i,\ j)$, $i = 1, \ldots, I$, $j = 1, \ldots, J$. What is the name of the joint distribution of the random variables behind these observations? Write down the joint probability. [3 marks]

(b) Show that the joint distribution in (a) can be obtained from independent Poisson random variables $\{Y_{ij},\ i = 1, \ldots, I,\ j = 1, \ldots, J\}$ by conditioning on their sum. [5 marks]

(c) A random sample of 433 men were observed for obesity and snoring and the data are as follows.

```
Obese      Snoring       Total (y)
-----------------------------------------
 no          no              77
yes          no              10
 no         yes             272
yes         yes              74
```

(i) Use the following results to test independence between obesity and snoring at the 5% level. [4 marks]

```
> fit <- glm(y~obese*snoring, family=poisson)
> anova(fit, test="Chisq")

Analysis of Deviance Table
Terms added sequentially (first to last)

              Df Deviance Resid. Df Resid. Dev Pr(>Chi)
NULL                              3     344.83
obese          1  174.224         2     170.60  < 2e-16 ***
snoring        1  165.810         1       4.79  < 2e-16 ***
obese:snoring  1    4.793         0       0.00  0.02857 *
```

(ii) Interpret the obeseyes:snoringyes term below as a log odds ratio and estimate the odds ratio. [4 marks]

```
> summary(fit)
Coefficients:
                     Estimate Std. Error z value Pr(>|z|)
(Intercept)            4.3438     0.1140  38.117  < 2e-16 ***
obeseyes              -2.0412     0.3361  -6.073 1.26e-09 ***
snoringyes             1.2620     0.1291   9.776  < 2e-16 ***
obeseyes:snoringyes    0.7395     0.3608   2.050   0.0404 *
```

(iii) What are the fitted probabilities under this model? [4 marks]

[20 marks in total for this question.]

df	*q − value*										
ν	0.005	0.01	0.025	0.05	0.10	0.50	0.90	0.95	0.975	0.99	0.995
1	$.0^{4}39$	$.0^{3}16$	$.0^{3}98$	.0039	.0158	.455	2.706	3.841	5.024	6.635	7.879
2	.0100	.0201	.0506	.103	.211	1.386	4.605	5.991	7.378	9.210	10.597
3	.0717	.115	.216	.352	.584	2.366	6.251	7.815	9.348	11.345	12.838
4	.207	.297	.484	.711	1.064	3.357	7.779	9.488	11.143	13.277	14.860
5	.412	.554	.831	1.145	1.610	4.351	9.236	11.070	12.832	15.086	16.750
6	.676	.872	1.237	1.635	2.204	5.348	10.645	12.592	14.449	16.812	18.548
7	.989	1.239	1.690	2.167	2.833	6.346	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	7.344	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	8.343	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	9.342	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	10.341	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	11.340	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	12.340	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	13.339	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	14.339	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	15.338	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	16.338	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	17.338	25.989	28.869	31.526	34.805	37.156
19	6.844	7.638	8.907	10.117	11.651	18.338	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	19.337	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	20.337	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	21.337	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.688	13.091	14.848	22.337	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	23.337	33.196	36.415	39.364	42.980	45.558
25	10.520	11.524	13.120	14.611	16.473	24.337	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	25.336	35.563	38.885	41.943	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	26.336	36.741	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	27.336	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	28.336	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	29.336	40.256	43.733	46.979	50.892	53.672
31	14.458	15.656	17.539	19.281	21.434	30.436	41.422	44.985	48.232	52.191	55.003
32	15.134	16.362	18.291	20.072	22.271	31.336	42.585	46.194	49.480	53.486	56.328
33	15.815	17.074	19.047	20.867	23.110	32.336	43.745	47.400	50.725	54.776	57.649
34	16.501	17.879	19.806	21.664	23.952	33.336	44.903	48.602	51.966	58.061	58.964
35	17.192	18.509	20.569	22.465	24.797	34.336	46.059	49.802	53.203	57.342	60.275
36	17.887	19.233	21.336	23.269	25.643	35.336	47.212	50.999	54.437	58.619	61.581
37	18.586	19.960	22.106	24.075	26.492	36.336	48.363	52.192	55.668	59.893	62.883
38	19.289	20.691	22.879	24.884	27.343	37.335	49.513	53.384	56.896	61.162	64.181
39	19.996	21.462	23.654	25.695	28.196	38.335	50.660	54.572	58.120	62.428	65.476
40	20.707	22.164	24.433	26.509	29.051	39.335	51.805	55.759	59.342	63.691	66.766
42	22.139	23.650	25.999	28.144	30.765	41.335	54.090	58.124	61.777	66.206	69.336
44	23.584	25.148	27.575	29.788	32.487	43.335	56.369	60.641	64.201	68.710	71.893
46	25.041	26.657	29.160	31.439	34.215	45.335	58.641	62.830	66.617	71.201	74.437
48	26.511	28.177	30.755	33.098	35.949	47.335	60.907	65.171	69.023	73.683	76.969
50	27.991	29.707	32.357	34.764	37.689	49.335	63.167	67.505	71.420	76.154	79.490
55	31.735	33.571	36.398	38.958	42.060	54.335	68.796	72.312	77.381	82.292	85.749
60	35.535	37.485	40.482	43.188	46.459	59.335	74.397	79.082	82.298	88.379	91.952
65	39.383	41.444	44.603	47.450	50.883	64.334	79.973	84.821	89.177	94.422	98.105
70	43.275	45.442	48.758	51.739	55.329	69.334	85.527	90.531	95.023	100.43	104.22

Table 1: **Percentage points $\chi^2_{\nu,q}$ of the chi-squared distribution. The percentage point $\chi^2_{\nu,q}$ of the chi-squared distributed random variable χ^2_ν is defined as** $\Pr(\chi^2_\nu \leq \chi^2_{\nu,q}) = q$

Percentage points of the F-distribution $q = 0.95$

ν_2 \ Numerator degrees of freedom ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.54	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.50	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.86	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Table 2: **The percentage point $F_{\nu_1,\nu_2;q}$ of the F distributed random variable F_{ν_1,ν_2} with ν_1 and ν_2 degrees of freedom is defined as $\Pr(F_{\nu_1,\nu_2} \leq F_{\nu_1,\nu_2;q}) = q$.**

END OF EXAMINATION PAPER

Two Hours

Statistical tables are attached

UNIVERSITY OF MANCHESTER

MEDICAL STATISTICS

31st May 2019

09:45-11:45

Please use SEPARATE booklets for your answers to each Section.

Answer **ALL** five questions in **SECTION A** (40 marks in total).

Answer **TWO** of the three questions in **SECTION B** (40 marks in total). If more than two questions from Section B are attempted, then credit will be given for the two best answers.

The total number of marks for the paper is 80.

University-approved calculators may be used.

A1.

A randomised controlled trial of an **electronic** glucose monitor (**E**) against standard **finger**-prick glucose monitoring (**F**) in patients with Type 1 diabetes has been designed to use deterministic minimisation based on two baseline factors. The minimisation factors are trial centre (3 levels: Manchester; Birmingham, Cambridge) and HbA1c level (2 levels: 7.5%-9.0%; >9.0% to 11.0).

Note: HbA1c is glycated haemoglobin, a measure of average glucose level over the past 2-3 months.

The number of participants with each baseline factor level after 20 participants have been allocated to the intervention groups are given in the table below.

Baseline factor	Trial Centre						HbA1c level			
	Manchester		Birmingham		Cambridge		7.5% - 9.0%		>9.0% - 11.0%	
Treatment	E	F	E	F	E	F	E	F	E	F
Number of participants	4	2	3	3	3	5	5	6	5	4

(i) How many participants have been allocated to each treatment group?

[1 mark]

(ii) The characteristics of the next participant (participant 21) entered into the trial is:

Trial Centre: Manchester

HbA1c level: 7.5%-9.0%.

Determine the intervention group to which this participant has to be allocated.

[2 marks]

(iii) What type of bias has an increased risk when deterministic minimisation is used rather than stratified randomisation? How would the use of stochastic minimisation reduce this risk of bias?

[2 marks]

(iv) If stochastic randomisation is used with allocation probabilities 0.7 and 0.3, use the random number 0.60131 (from a uniform distribution on [0,1]) and a suitably-defined rule to allocate participant 21 to a treatment group.

[2 marks]

[Total 7 marks]

A2.

Van de Port (2012) reported testing for differences between treatment groups for 35 baseline variables, 4 of which had p-values less than 0.05. In the Discussion section of their paper they report some limitations of their trial, which included the text:

“Firstly, although the trial was well powered, including 250 patients with few drop-outs, we found significant baseline differences in favour of the circuit training group for a few secondary outcomes. All analyses, however, were adjusted for these covariates at baseline.”

(i) Explain why performing significance tests for baseline differences between treatment groups is not recommended in randomised controlled trials.

[2 marks]

(ii) When should the between-groups comparison of continuous outcome measures in randomized controlled trials be adjusted for baseline variables? Is it appropriate to use significance testing to select the baseline covariates to include in an adjusted analysis with the aim of reducing potential bias? (Justify your answer.)

[3 marks]

[Total 5 marks]

A3.

A randomised controlled trial is carried out to compare two doses of a new vaccine for the prevention of H1N1 influenza. The effectiveness of the vaccine is tested by measuring the immune response in the blood measured 9 days following vaccination (as this measures successful vaccination and protection against influenza). The results are summarised in the table below: the numbers in the table are the respective counts for the different combinations of Dose (15mg; 30mg) and Immune response (Yes; No), with the totals randomised to each of the dose levels.

Dose		15mg	30mg
Immune response	Yes	200	220
	No	50	30
	Total	250	250

(i) Estimate the odds ratio for the effectiveness of a 30mg dose as compared to a 15mg dose.

[1 mark]

(ii) Calculate the 95% confidence interval of this odds ratio.

[5 marks]

(iii) Interpret the results of this trial.

[2 marks]

[Total 8 marks]

A4.

In a meta-analysis of *k* trials, suppose that $\hat{\theta}_i$ is an estimate of the treatment effect for the i^{th} trial and let $Var[\hat{\theta}_i]$ be its sampling variance. The minimum variance estimate is defined by:

$$\hat{\theta}_{MV} = \frac{\sum_{i=1}^{k} w_i \hat{\theta}_i}{\sum_{i=1}^{k} w_i}$$

where $w_i = 1/Var[\hat{\theta}_i]$, and $Var[\hat{\theta}_{MV}] = \frac{1}{\sum_{i=1}^{k} \frac{1}{Var[\hat{\theta}_i]}}$.

The table below summarises the outcome of three trials of Neuromuscular Electrical Stimulation (NMES) versus no treatment for the condition of patellofemoral pain (PFP). The outcome measures was a Visual Analogue Scale (VAS) Pain score (range 0-10, with 0 representing no pain and 10 representing worst imaginable pain) during normal activities, assessed at the end of treatment. The treatment effect for each study (θ_i, i=1, 2, 3) is the difference in mean VAS pain for the two treatments (NMES group - No treatment group).

Study (Date of publication)	Difference $\hat{\theta}_i$	$Var[\hat{\theta}_i]$
Akarcali(2002)	1.12	0.3969
Bily (2008)	0.55	1.2769
Tunay (2003)	1.90	0.1296

(i) Compute the minimum variance estimate of the overall treatment effect, $\hat{\theta}_{MV}$, and determine its 95% confidence interval, stating any assumptions that you make.

[6 marks]

(ii) What can you conclude from the meta-analysis regarding the performance of NMES?

[2 marks]

(iii) It is known that the minimal clinically important difference for VAS pain is in the range 1.5 to 2.0. By comparing these values with the 95% confidence interval that you computed in (i), provide interpretation of the confidence interval in the context of clinical importance.

[2 marks]

[Total 10 marks]

A5.

Consider a randomized controlled trial. Suppose the patient population can be divided into three latent sub-groups as follows:

- *Compliers*: patients who will comply with the allocated treatment;
- *Always control treatment*: patients who will receive control treatment regardless of allocation;
- *Always new treatment*: patients who will receive the new treatment regardless of allocation.

This assumes that there are no defiers, namely patients who will always receive the opposite of the treatment to which they are randomized. Assuming that the proportion and characteristics of compliers, *always control treatment*, *always new treatment* is the same in both arms and that randomization can only affect the outcome through the receipt of treatment, show that:

(i) An intention-to-treat estimate of the treatment effect is biased towards the null hypothesis of no treatment effect.

[5 marks]

(ii) A per-protocol estimate of the treatment effect may be biased either towards or away from the null hypothesis of no treatment effect.

[5 marks]

[Total 10 marks]

B1.

For an *AB*/*BA* crossover trial, a standard model for a continuous outcome y_{ij} for the i^{th} participant in the j^{th} period is:

$y_{i1} = \mu + \xi_i + \varepsilon_{i1}$ for a participant in sequence *AB* in period 1

$y_{i2} = \mu + \tau + \phi + \xi_i + \varepsilon_{i2}$ for a participant in sequence *AB* in period 2

$y_{i1} = \mu + \tau + \xi_i + \varepsilon_{i1}$ for a participant in sequence *BA* in period 1

$y_{i2} = \mu + \phi + \xi_i + \varepsilon_{i2}$ for a participant in sequence *BA* in period 2

where μ is the mean for the sequence *BA* in period 1, τ is the treatment effect of A compared to B, and ϕ is the period effect, with ξ_i and ε_{ij} being independent random variables with $\xi_i \sim N[0, \sigma_B^2]$ and $\varepsilon_{ij} \sim N[0, \sigma_\varepsilon^2]$. Defining $d_i = y_{i2} - y_{i1}$, let $\bar{d}_{AB}, \bar{d}_{BA}, \mu_{AB}^d$ and μ_{BA}^d be the sample and population means of d_i for sequences *AB* and *BA* respectively.

(i) Show that $\hat{\tau} = \frac{\bar{d}_{AB} - \bar{d}_{BA}}{2}$ is an unbiased estimator of τ, that is

$$E[\hat{\tau}] = E\left[\frac{\bar{d}_{AB} - \bar{d}_{BA}}{2}\right] = \tau .$$

Hence also show that a test of the null hypothesis H_0: $\mu_{BA}^d = \mu_{AB}^d$ is the same as a test of the null hypothesis of no treatment effect H_0: $\tau = 0$.

[4 marks]

(ii) Show that the variance of $\hat{\tau}$ is given by

$$Var[\hat{\tau}] = Var\left[\frac{\bar{d}_{AB} - \bar{d}_{BA}}{2}\right] = \frac{\sigma_\varepsilon^2}{2}\left(\frac{1}{n_{AB}} + \frac{1}{n_{BA}}\right) .$$

(You may assume the general result $Var(\bar{x}) = \frac{\sigma^2}{n}$ where $Var(x) = \sigma^2$, and n is the sample size.)

[5 marks]

(iii) Given that the variance of the estimator based on the first period observations from the two groups, which is equivalent to a parallel-design trial treatment effect estimator, is given by:

$$(\sigma_B^2 + \sigma_\varepsilon^2)\left(\frac{1}{n_{AB}} + \frac{1}{n_{BA}}\right),$$

what does the result given in (iii) suggest is an advantage of using an *AB*/*BA* crossover design rather than a parallel-design trial? State the condition when this advantage is at its greatest.

[3 marks]

(iv) An AB/BA crossover trial was designed to compare two treatments for hypertension (high blood pressure): Drug A and Drug B. Thirty-eight participants, all suffering from hypertension, were randomised: 18 to the sequence Drug A then Drug B, and 20 to the sequence Drug B then Drug A. A measurement of the primary outcome measure, diastolic blood pressure (in mm Hg), was taken at the end of each treatment period and are summarised in the table below. Test the hypothesis H_0: $\tau = 0$ versus H_1: $\tau \neq 0$ using a 5% significance level. Also compute and briefly interpret the 95% confidence interval for τ.

Sequence	*Period 1*		*Period 2*		*Period 2 – Period 1*		
	Mean	*S.D.*	*Mean*	*S.D.*	*Mean*	*S.D.*	*N*
AB	84.50	7.01	83.39	6.57	-1.11	2.47	18
BA	84.10	8.94	86.65	8.12	2.55	2.26	20

[8 marks]

[Total 20 marks]

B2.

A randomised controlled trial is planned to compare a treatment (A) with the current standard therapy (B). For an outcome measure *Y*, let $\overline{y_A}$, $\overline{y_B}$, μ_A, μ_B be the sample and population means of *Y* for each treatment. Let s and σ be the common within-group sample and population standard deviations of *Y*. Assume that the null hypothesis of no treatment effect will be tested by the statistic $T = \frac{\overline{y_A} - \overline{y_B}}{s\lambda}$, with $\lambda = \sqrt{1/n_A + 1/n_B}$, where n_A and n_B are the number of participants allocated to each treatment.

(i) Write down an expression for the power (1-β) of the test for a two-tailed alternative hypothesis to detect a treatment effect τ, stating the assumptions you make.

[4 marks]

(ii) Assuming that participants are allocated in the ratio of 1: k such that $n_B = k.n_A$, show that the total sample size required to give a power of (1-β) for a two-tailed α-sized test is:

$$n = \frac{(k+1)^2}{k}\frac{\sigma^2}{\tau^2}\left(z_{\alpha/2} + z_\beta\right).$$

[5 marks]

(iii) Show that the minimum sample size is obtained by using an equal allocation ratio.

[5 marks]

(iv) In a randomised controlled trial it is planned to allocate 240 participants to two treatments. The pooled within-group standard deviation is thought to be 12 units. Estimate the power of a study to detect a treatment effect of 5 units for two-tailed 0.05-sized test, if an allocation ratio of 1:2 (treatment: control) is used.

[4 marks]

(v) In a randomised controlled trial, it is decided to allocate twice as many participants to the control group as to the treatment group, using blocked randomisation with a block size of 3. Briefly describe how to create a randomisation list to implement this randomisation scheme by using random numbers with a simple allocation rule.

[2 marks]

[Total 20 marks]

B3.

In a parallel-group *non-inferiority* trial, a new treatment *T* is being compared with a control treatment *C* using a normally distributed outcome measure *Y*. Assume that large values of *Y* represent a worse outcome for the participant. Let $\overline{y_T}$, μ_T and n_T and $\overline{y_C}$, μ_C and n_C be the sample mean, population mean and the sample size for the new treatment and for the control treatment, respectively. Suppose σ is the population standard deviation of both treatments. The treatment effect is defined as $\tau = \mu_T - \mu_C$.

(i) Provide an example, with justification, of a situation where a non-inferiority trial rather than a superiority trial should be used.

[2 marks]

(ii) Explain why a significance test of the hypothesis $H_0 \colon \tau = 0$ versus $H_1 \colon \tau > 0$ would not be appropriate in a *non-inferiority* trial.

[3 marks]

(iii) For a non-inferiority trial, the null hypothesis $H_0 \colon \tau \geq \tau_N$ is rejected if the upper limit for a $100(1-\alpha)\%$ one-sided confidence interval for τ is less than the limit of non-inferiority τ_N, for which the probability, under H_0, is given by:

$$\Pr[\text{Reject } H_0 | \, \tau \,] = \Phi\left(\left(\frac{\tau_N}{\sigma\lambda} - z_\alpha\right) - \frac{\tau}{\sigma\lambda}\right),$$

where Φ is the cumulative distribution function of the standard normal distribution and $\lambda = \sqrt{1/n_T + 1/n_C}$.

Given this, show that $\Pr[\text{Reject } H_0 | \tau]$ has a maximum under H_0 when $\tau = \tau_N$. Hence, show that this procedure has a Type I error probability less than or equal to α .

[7 marks]

(iv) A randomised controlled non-inferiority trial was carried out to test whether *Nurse-led Telephone follow-up* (Telephone Group) was non-inferior to the current standard care of *Consultant-led Hospital follow-up* (Hospital Group) for women following treatment for cancer of the endometrium (the inner lining of the uterus). The primary outcome was measured on a continuous scale using a questionnaire-based measure of anxiety, with higher scores representing greater anxiety. Follow-up data were collected on 111 participants from the Telephone Group and 106 from the Hospital Group. The computer output from the statistical analysis is given below.

```
Two-sample t test with equal variances
------------------------------------------------------------------------------
   Group |     Obs        Mean    Std. Err.   Std. Dev.   [90% Conf. Interval]
---------+--------------------------------------------------------------------
telephon |     111    33.00901    1.045759    11.01775    31.27428    34.74374
hospital |     106     35.5283    1.265849    13.03271    33.42763    37.62897
---------+--------------------------------------------------------------------
combined |     217    34.23963    .8201926    12.08219    32.88472    35.59454
---------+--------------------------------------------------------------------
    diff |           -2.519293    1.635633               -5.221312    .1827265
------------------------------------------------------------------------------
    diff = mean(telephon) - mean(hospital)                        t =  -1.5403
Ho: diff = 0                                     degrees of freedom =      215

    Ha: diff < 0                 Ha: diff != 0                 Ha: diff > 0
 Pr(T < t) = 0.0625         Pr(|T| > |t|) = 0.1250          Pr(T > t) = 0.9375
```

A difference of less than 3.5 points on the anxiety scale was considered by researchers to be an unimportant difference between treatments. Using the computer output given above, test whether the Nurse-Led Telephone follow-up was non-inferior to the Consultant-Led Clinic Hospital-based follow-up for the primary outcome measure. Interpret your findings in the context of the trial. Can you also conclude superiority of the Nurse-led Telephone follow-up on the primary outcome measure?

[8 marks]

[Total 20 marks]

END OF EXAMINATION PAPER

Two hours

Tables of the cumulative distribution function are provided.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL MODELLING IN FINANCE

4 June 2019

14:00 - 16:00

Answer **ALL SIX** questions.

Electronic calculators may be used in accordance with the University regulations

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1. By using no arbitrage argument, derive *put-call parity* for the European put option, P, and the European call option, C, with the same exercise price X and expiry date T. Assume the constant risk-free interest rate r.

[6 marks]

2. Draw the expiry payoff diagrams for each of the following portfolios:

(i) Short two shares, long three calls with an exercise price X.

(ii) Short two shares, long three puts with exercise price X_1, long four calls with exercise price X_2. Consider only the case $X_1 < X_2$.

[12 marks]

3. The change in price of an asset S satisfies the stochastic differential equation

$$dS = 0.5Sdt + (1+t)\sqrt{S}dW,$$

where $W(t)$ is a Wiener process. Using Ito's Lemma along with the delta-hedge argument, derive the partial differential equation satisfied by the price $V(S,t)$ of an option on the underlying asset.

[12 marks]

4. Consider the Black-Scholes equation for the price for an option, V of an option, on a stock S that pays no dividends, namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where r and σ are both constants.

(i) Show by substitution that

$$V(S,t) = Ae^{rt+\beta \ln S},$$

where A and β are constants, is a solution of the Black-Scholes equation.

(ii) Determine the values of the constant β.

(iii) Find the partial differential equation satisfied by $V(Z,t)$, where $Z = -\ln S$.

[20 marks]

5. You are given that the solution for a vanilla (non-dividend paying) European call option is

$$C(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln(S/X) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S/X) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and X is the exercise price and $N(x)$ is the cumulative distribution function, namely

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy.$$

(i) Find the present value of a three-month European call option on a stock with the exercise price \$100, when the current stock price is \$100 and $\sigma = 1$. The risk-free interest rate is 0% p.a.

(ii) Find the limits

$$\lim_{X\to 0} C(S,t), \qquad \lim_{\sigma\to\infty} C(S,t), \qquad \lim_{\sigma\to 0} C(S,t).$$

(iii) By using the formula

$$\Delta = \frac{\partial C}{\partial S} = N(d_1),$$

find the explicit expression for

$$\Gamma = \frac{\partial^2 C}{\partial S^2}$$

and sketch the graph of Δ as a function of S in the limit $\sigma \to 0$.

(iv) Show that

$$\frac{\partial C}{\partial X} = f(t)N(d_2)$$

and find the function $f(t)$.

[25 marks]

6. Suppose that an asset of value S pays out a dividend $DSdt$ over the time period dt (i.e. a constant dividend yield D). A put option $P(S,t)$ on this asset satisfies the modified Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r-D)S\frac{\partial P}{\partial S} - rP = 0,$$

where r and σ are constants.

(i) Let $P(S,t)$ be a solution of the modified Black-Scholes equation for $D = r$. We set $S = e^z$, $\tau = T - t$ (T is the expiry date), $P = v(z,\tau)$. Find the partial differential equation satisfied by $v(z,\tau)$.

(ii) Find the solution of the modified Black-Scholes equation for $D = r$ in the form

$$P(S) = AS^\beta,$$

where A and β are constants. Find the two possible values of β.

(iii) Consider a *perpetual* American put option on an asset paying a continuous constant dividend yield, which is given by $D = r$. If the exercise boundary is denoted by S_f, write down the two conditions to be imposed on S_f. State why only one of the solutions in (ii) is retained, and which one it is.

(iv) Use the conditions at $S = S_f$ to determine both the value of the constant A and S_f itself. Sketch the graph of S_f as a function of β. What happens to S_f as $D \to 0$?

[25 marks]

END OF EXAMINATION PAPER

Two hours

Statistical Tables provided

THE UNIVERSITY OF MANCHESTER

ACTUARIAL MODELS 2

23 May 2019
9:45 - 11:45

Answer ALL FIVE questions.
The total number of marks in the paper is 80.

University approved calculators may be used.

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1. Consider the following observed values of the possibly left-truncated and right-censored survival times of 12 independent homogeneous individuals, where + denotes a censored value:

(1, 3] (5, 9+] (5,10] (5,10+] (4,11] (3,12] (5,12] (1,16] (4,17+] (4,20+] (0,23+] (4,25]

(a) Use the above survival data to calculate the Kaplan-Meier estimate of the survival function of the survival time distribution of the individuals and plot your estimate (you should show the details of your calculations).

[8 marks]

(b) Estimate, by using the Kaplan-Meier estimate, the probability that an individual fails before time 20.

[3 marks]

(c) Suppose the survival data corresponding to the first individual changes from $(1, 3]$ to $(0, 3]$. Explain if with this change in the data the estimate of the probability in part (b) goes either (i) up, (ii) down or (iii) remains the same. (Note that you can answer this question without needing to recompute the estimate.)

[3 marks]

(d) For a positive random variable X and a scalar $b \geq 0$, we have

$$\mathbb{E}\left[X\mathbf{1}_{\{X\leq b\}}\right] = \int_0^b \mathbb{P}(X > x)\mathrm{d}x - b\mathbb{P}(X > b).$$

Estimate, by using the Kaplan-Meier estimate and the above identity, the conditional expectation of the failure time of an individual given that the individual has failed before time 15.

[5 marks]

[Total 19 marks]

2. A clinical trial took place testing a particular treatment for Alzheimer's disease. For this trial a number of patients were randomly divided into a treatment group and a control group. The original plan was to observe patients for a duration of 24 months but due to a suspiciously high number of patients dying while getting this particular treatment, the study was halted after only 8 months. The researchers involved still want to use the data obtained during these 8 months for estimating and testing the effect of the treatment on the survival time distribution of Alzheimer's patients. To this end the survival times of patients who were still alive when the study was abruptly stopped are considered to be censored.

Explain briefly what independent censoring is and explain whether or not the particular type of censoring described above is likely to be independent or not.

[Total 5 marks]

3. Consider the following observed values of the survival times of five independent homogeneous individuals, where + denotes a censored value:

$$2 \quad 5 \quad 5+ \quad 6 \quad 7+$$

Assume the following parametric form of the common hazard function of the individuals:

$$\mu(t) = \mu(t; \alpha, \lambda) = \alpha\lambda^{\alpha} t^{\alpha-1}, \quad t \geq 0,$$

with parameters $\alpha, \lambda > 0$. Assume further that the times at which individuals are censored are all deterministic quantities and do not depend on the parameters α and λ.

(a) Derive, from first principles, an explicit expression (in term of α and λ) for the log-likelihood function of the parameters α and λ given the observations. (Note that directly applying the (log-)likelihood formula from the notes is not sufficient for getting full marks.) Here any constants that do not effect the maximum likelihood estimation can be ignored.

[8 marks]

(b) We want to test whether $\alpha = 1$ and $\lambda = 0.20$, i.e. whether the hazard rate is a constant given by $\lambda = 0.20$. Therefore we want to test the null hypothesis

$$H_0 : \mu(t) = \lambda \text{ for all } t \geq 0 \quad \text{versus} \quad H_A : \mu(t) \neq \lambda \text{ for some } t \geq 0.$$

where $\lambda = 0.20$. The following 1-sample test says to reject H_0 at significance level α if

$$\left| \frac{N_\infty - \lambda \int_0^\infty R_s \mathrm{d}s}{\sqrt{\lambda \int_0^\infty R_s \mathrm{d}s}} \right| > z_{\alpha/2},$$

where N_t denotes the number of individuals that are observed to have failed before or at time t, R_t denotes the number of individuals at risk just before time t and $z_\alpha > 0$ is such that $\Phi(z_\alpha) = 1 - \alpha$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

(i) Given the survival data at the beginning of the question, carry out this 1-sample test (with $\lambda = 0.20$) at significance level 0.05 and report your conclusions.

[5 marks]

(ii) Show that under H_0 the process $Z = \{Z_t : t \geq 0\}$ given by $Z_t = N_t - \lambda \int_0^t R_s \mathrm{d}s$ is a martingale. You can use here without proof that the process $\{M_t : t \geq 0\}$ given by $M_t = N_t - \int_0^t \mu(s) R_s \mathrm{d}s$ is a martingale.

[2 marks]

(iii) Show that under H_0, $\mathbb{E}[Z_t] = 0$ and $\mathrm{Var}(Z_t) = \lambda \mathbb{E}[\int_0^t R_s \mathrm{d}s]$ for any $t \geq 0$. (Hint: use part (ii).)

[5 marks]

[Total 20 marks]

4. For a study into recidivism a number of convicted persons who spent time in jail were observed for some period of time. The data below displays for each individual how long it took to commit another offence carrying a prison sentence after he/she had been released from jail. Here a + next to the survival time indicates that the corresponding person did not carry out another major offence while being observed. Also recorded and displayed in the table are the amount of time each person spent in prison and whether he/she has a low or high income.

time to reoffence (in months)	1	4+	6	12+	15
amount of time jailed (in months)	5	2	9	10	10
income (h=high, l=low)	l	l	h	h	h

For modelling the rate of recidivism (i.e. the hazard rate of the time to recommit an offence after being released from prison) consider a Cox proportional hazards model with the amount of time jailed and income acting as covariates.

(a) Give the form of the hazard rate of an (arbitrary) individual in this Cox proportional hazards model. Here you should clearly define each piece of notation.

[3 marks]

(b) Derive an explicit expression in terms of the two regression coefficients for the partial likelihood in this Cox proportional hazards model given the above data.

[7 marks]

(c) Although none of the survival times in the above table are left-truncated, explain within the context of this study, what it means if a survival time of an individual would be left-truncated.

[3 marks]

(d) Suppose, within this Cox proportional hazards model, that you want to use the likelihood ratio test to check whether or not the amount of income has a significant effect (for a given significance level) on the rate of recidivism. To this end, (i) give the null and alternative hypotheses, (ii) state the form of the test statistic and describe when one rejects the null hypothesis and (iii) explain for both possible outcomes of this test what you can conclude from it within the context of this particular study.

Note that you do not have to carry out this test, i.e. you do not have to compute a numerical value for the test statistic.

[6 marks]

[Total 19 marks]

P.T.O.

5. Assume a proportional frailty model for the mortality of a certain population where (i) the frailty random variable Z has an atom at 1 with mass $p \in [0,1]$ (i.e. $\mathbb{P}(Z=1)=p$) and has a density on $(0,\infty)$ given by $f_Z(z)=(1-p)\mathrm{e}^{-z}$ and (ii) the individual hazard function is given by

$$\mu(t|Z) = 0.01 \cdot t\mathrm{e}^{-bt} Z, \quad t \geq 0,$$

where $b \geq 0$.

(a) Find an expression for the population hazard function in terms of p and b. (Hint: treat the cases $b > 0$ and $b = 0$ separately.)

[8 marks]

The remaining questions can be answered without knowing the answer to part (a).

(b) A research team is investigating the age at which a particular disease amongst humans occurs. When looking at the population hazard function of the age at which the disease occurs they observe that this hazard rate has a peak at around age 20 and decays afterwards. The research team thinks the explanation for this is that the physical changes the body undergoes during early adulthood makes humans more susceptible to develop the disease, whereas at later ages when the human body has reached full maturity, the disease is less likely to strike.

Consider the proportional frailty model stated in the beginning of the question and consider the following four sets of parameter choices for p and b.

- $p = 1$ and $b = 0$;
- $p = 1$ and $b = 0.05$;
- $p = 0.75$ and $b = 0$;
- $p = 0.75$ and $b = 0.05$.

Which of the four sets of parameter choices would make the proportional frailty model align most closely with the explanation of the research team about the observed shape of the population hazard function corresponding to the age at which the disease occurs? Explain your answer.

[3 marks]

(c) (i) Explain how the frailty effect can provide a completely different explanation for the observed shape of the population hazard function corresponding to the age at which the disease occurs.

(ii) Which of the four sets of parameter choices in (b) would make the resulting proportional frailty model be in contradiction with the explanation of the research team but instead match with the frailty effect being the explanation for the observed shape of the aforementioned population hazard function? Explain your answer.

[6 marks]

END OF EXAMINATION PAPER [Total 17 marks]

2 Hours

THE UNIVERSITY OF MANCHESTER

CONTINGENCIES 2

17 May 2019
14:00 - 16:00

Answer ALL EIGHT questions
The total number of marks in the paper is 100.

Electronic calculators may be used, provided that they cannot store text.

Please note the following before you get started.

- You can find a formula sheet on the last page of this exam paper.
- The marking is based on your answer booklet only, so make sure to write down your working there.

1. Using the following extracts from two simplified mortality tables, one for life x and one for life y, calculate the probabilities below (in all cases assume the first life is life x and the second is life y):

age x	l_x	age y	l_y
30	10,000	20	10,000
31	9,900	21	9,950
32	9,800	22	9,900
33	9,700	23	9,850

a) $p_{32:22}$

[2 marks]

b) ${}_2p_{30:20}$

[2 marks]

c) ${}_{2|}q_{30:20}$

[4 marks]

d) ${}_{2|}q_{\overline{30:20}}$

[4 marks]

[Total 12 marks]

2. For two lives both aged 40 exact and assuming that both lives are statistically independent, calculate the expected present value of the following assurances, assuming a rate of interest of 3% p.a. and mortality of PFA90C20 for both lives:

a) $A_{40:40}$

[3 marks]

b) $A^{1}_{40:40}$

[2 marks]

c) $A_{\overline{40:40}}$

[4 marks]

[Total 9 marks]

3. An insurance company issues a policy to two lives x and y which pays a sum assured of £10,000 immediately on the first death. Premiums are payable continuously until the first death occurs.

Assume a mortality model for life x which has $\mu_x = 0.01$ for all $x \geq 0$ and for life y has $\mu_y = 0.012$ for all $y \geq 0$.

a) Show that the premium for this policy is independent of the force of interest δ.

[3 marks]

b) Calculate the premium.

[1 marks]

c) If the policy was a whole life last survivor policy paying the same sum assured immediately on death, with premiums also payable continuously until the second death, would the amount of premium be dependent on the value of δ? Give an explanation for your answer.

[4 marks]

[Total 8 marks]

4. Consider a 3-state sickness model with states H (healthy), S (sick) and D (dead) and the following forces of transition for a life aged x

- μ_x^{HS} on transition from healthy to sick,
- μ_x^{HD} on transition from healthy to dead,
- μ_x^{SH} on transition from sick to healthy,
- μ_x^{SD} on transition from sick to dead.

a) Write down an integral expression for the expected present value (EPV) of a lump sum sickness benefit of £20,000 payable immediately on becoming ill for the first time for a healthy life aged 40 exact.

[3 marks]

b) Write down an integral expression for the expected present value (EPV) of a lump sum sickness benefit of £10,000 payable immediately on becoming ill for the second or any subsequent time for a healthy life aged 40 exact.

[5 marks]

c) Assume the above transition intensities take the following constant values, for all $x \geq 0$:

- $\mu_x^{HS} = 0.10$
- $\mu_x^{HD} = 0.01$
- $\mu_x^{SH} = 0.08$
- $\mu_x^{SD} = 0.02$

Calculate the EPV for a sickness benefit of £2,000 p.a. for a life aged 30 exact *who is currently in the sick state* where the benefit is paid until the life either recovers or dies. Assume a force of interest of 0.03.

[6 marks]

[Total 14 marks]

5. Consider a 3-state multiple decrement model with states a (active), r (retired) and d (dead).

a) Given an independent probability of retiring when aged 60 of 0.1 and a constant force of mortality of 0.01 for all ages between 60 exact and 65 exact, calculate $(aq)_{60}^d$ and $(aq)_{60}^r$

[8 marks]

b) If retirement can only occur at the end of a year what is the probability of a life aged 60 exact being active at age 61 exact?

[4 marks]

[Total 12 marks]

6. A pension scheme provides a pension on normal health retirement, calculated as follows:

$$\text{Annual amount of pension payable for life from normal health retirement at age } x = \frac{N}{80} \times FPS_x$$

where

- FPS_x = the average salary earned over the three years ending at age x and
- N is the total period of time the life has been a member of the pension scheme.

Now, consider a person aged 50 exact who joined the pension scheme 20 years ago. You are told that the person has earned a salary of £25,000 in the previous year.

a) Using the data and functions provided in R, assuming an interest rate of 4% p.a. and assuming the member retires between the age of 63 exact and age 65 exact, calculate the expected present value (EPV) of the pension benefits the life will receive.

[14 marks]

b) The pension scheme is now changed so that there is a maximum value placed on FPS_x of £30,000. Again, assuming the member retires between the age of 63 exact and age 65 exact, calculate the EPV of the pension benefits the life will receive as a result of this change.

[4 marks]

c) Instead of FPS_x having a maximum value of £30,000, S_x, the salary earned between ages x and $x+1$, has this same maximum amount applied. Does this change:

i) increase the EPV calculated in part b), or

ii) decrease the EPV calculated in part b), or

iii) leave the EPV calculated in part b) unchanged?

[4 marks]

[Total 22 marks]

P.T.O.

7. An insurer writes a 5 year unit linked endowment policy for a life aged 35. Death benefits are payable at the end of the year of death and the maturity value on completion of the 5 year term. Level premiums are paid yearly in advance. Benefits are paid at the end of the year.

You are given the following tables of data (*note numbers shown in brackets are negative numbers*):

Unit Fund

year	Premium paid	Premium allocated	bid offer spread	net premium allocated to policyholder	fund before management charge	management charge	Unit Fund at year end
1	10,000	8,000	320	7,680	8,064	81	7,983
2	10,000	10,500	420	10,080	18,966	190	18,776
3	10,000	10,500	420	10,080	30,299	303	29,996
4	10,000	10,500	420	10,080	42,080	421	41,659
5	10,000	10,500	420	10,080	54,326	543	53,783

Non-Unit Fund

year	Non allocated premium	bid/offer spread	Expenses	interest on fund at 4%	management charge	Profit
1	2,000	320	3070	(30)	81	(699)
2	(500)	420	120	(8)	190	(18)
3	(500)	420	120	(8)	303	95
4	(500)	420	120	(8)	421	213
5	(500)	420	120	(8)	543	335

In addition assume :

- $(ap)_x = 0.99$ for all x
- Interest on reserves = 3%p.a.

a) State the allocation percentage and the bid offer spread (as a percentage) for this policy for each year.

[2 marks]

b) What is the assumed investment return on the Unit Fund each year implied by this data?

[3 marks]

c) What is the amount payable on death implied by this data?

[2 marks]

d) Zeroise the profit vector for this policy.

[6 marks]

e) If the insurance company decides to introduce a penalty to the policyholder on surrender equal to 10% of the value of the Unit Fund, what is the new Profit Vector. Assume the dependent probability of surrender, $(aq)_x^s = 0.05$ for all ages x.

[4 marks]

[Total 17 marks]

8.

a) List four factors that may affect the levels of mortality within a population.

[2 marks]

b) Mortality for a population has, for 20 years, consistently shown improvements. Recently these improvements have stopped. Referring to the factors affecting mortality, give four reasons why this might have happened.

[4 marks]

[Total 6 marks]

Formula sheet Contingencies 2

1. Compound interest

i is the interest rate per annum, $v = (1+i)^{-1}$ and $\delta = \log(1+i)$

$a_{\overline{n}|} = v + v^2 + \ldots + v^n$ is the present value of an annuity certain in arrears with term n years

2. Mortality tables and laws

l_x is the number of lives at age x

$d_x = l_x - l_{x+1}$ is the number of lives who die between ages x and $x+1$

${}_tp_x$ is the probability that a life aged x will survive at least t years and ${}_tq_x$ is the probability that a life aged x will die within t years

${}_{m|n}q_x$ is the probability that a life aged x will die between ages $x+m$ and $x+m+n$

K_x is the curtate future lifetime for a life aged x; $e_x = \mathbb{E}[K_x]$

T_x is the complete (exact) future lifetime for a life aged x; $\mathring{e}_x = \mathbb{E}[T_x]$

μ_x is the force of mortality at age x:

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s} \cdot ds\right)$$

3. Assurances

A_x is the EPV of a whole life assurance that pays at the end of the year of death of a life aged x

$\bar{A}_x$ is the EPV of a whole life assurance that pays at the moment of death of a life aged x

$$\mathbb{E}\left[v^{K_x+1}\right] = A_x;\ \operatorname{Var}\left(v^{K_x+1}\right) = A_x^{i^*} - (A_x)^2$$

$$\mathbb{E}\left[v^{T_x}\right] = \bar{A}_x;\ \operatorname{Var}\left(v^{T_x}\right) = \bar{A}_x^{i^*} - (\bar{A}_x)^2$$

(where $i^* = (1+i)^2 - 1$. Similar relationships hold for other assurances.

A_{xy} is the EPV of a whole life assurance that pays at the end of the year in which the first death of two lives aged x and y occurs.

$A_{\overline{xy}}$ is the EPV of a whole life assurance that pays at the end of the year in which the second death of two lives aged x and y occurs.

A^1_{xy} is the EPV of a whole life assurance that pays at the end of the year in which a life aged x dies assuming that a life aged y remains alive at that time.

A^2_{xy} is the EPV of a whole life assurance that pays at the end of the year in which a life aged x dies assuming that a life aged y is already dead at that time.

4. Annuities

a_x is the EPV of an annuity for life in arrears for a life aged x

$\ddot{a}_x = 1 + a_x$ is the EPV of an annuity for life in advance for a life aged x

$\bar{a}_x$ is the EPV of a continuously paying annuity for life for a life aged x

$$\mathbb{E}[\ddot{a}_{\overline{K_x+1}|}] = \ddot{a}_x;\ \operatorname{Var}(\ddot{a}_{\overline{K_x+1}|}) = \frac{A_x^{i^*} - (A_x)^2}{(1-v)^2}$$

$$\mathbb{E}[\bar{a}_{\overline{T_x}|}] = \bar{a}_x;\ \operatorname{Var}(\bar{a}_{\overline{T_x}|}) = \frac{\bar{A}_x^{i^*} - (\bar{A}_x)^2}{\log(v)^2}$$

(where $i^* = (1+i)^2 - 1$. Similar relationships hold for temporary annuities with status ${}_{x:\overline{n}|}$)

a_{xy} is the EPV of an annuity for the joint lifetime of two lives aged x and y, paid in arrears.

$a_{\overline{xy}}$ is the EPV of an annuity payable in arrears so long as at least one of two lives aged x and y is alive.

$a_{x|y}$ is the EPV of a reversionary annuity payable to a life aged y after the death of a life aged x, paid in arrears.

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m};$$

$$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - v^n\ {}_np_x\right).$$

$$A_x = 1 - (1-v)\ddot{a}_x; \quad \bar{A}_x = 1 - \delta\bar{a}_x$$

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

RISK THEORY

31 May 2019

14:00 – 16:00

Answer ALL FIVE questions
The total number of marks in the paper is 90.

University approved calculators may be used

Note: attached to this exam paper you find the list of distributions that you have previously used during the course.

1. For a portfolio of products, an insurance company models the evolution of capital levels by means of a Cramér-Lundberg capital process denoted $(U_t)_{t\geq 0}$. That is, the amount of capital at time t (measured in years) is given by

$$U_t = u + ct - S_t = u + ct - \sum_{k=1}^{N_t} X_k \quad \text{for } t \geq 0,$$

where $u \geq 0$ is the initial capital, $c > 0$ the premium rate, $(N_t)_{t\geq 0}$ a Poisson process with intensity $\lambda > 0$, and finally $\{X_k\}_{k=1,2,\ldots}$ is a sequence of independent and identically distributed (iid) positive random variables representing the claim sizes. Assume that the claim sizes have finite mean.

(a) Suppose, in this part (a) only, that the company has an initial capital of 3 and that premiums are received at a rate of 2 per year. Further, assume that during the first year two claims are submitted, one of size 2 after 3 months and one of size 3 after 9 months (for simplicity, a month is 1/12-th of a year). Draw the corresponding trajectories of $(N_t)_{t\geq 0}$, $(S_t)_{t\geq 0}$ and $(U_t)_{t\geq 0}$ for this year. Also indicate whether or not ruin happens during this first year, and if so at which point in time.

[6 marks]

(b) In the Cramér-Lundberg model, premiums are assumed to be paid continuously by the policy-holders. Consider (in this part (b) only) changing the model so that all premiums to be paid during the first year are paid at the *beginning* of the first year instead. Would the probability of ruin during the first year increase or decrease due to this change? Briefly motivate your answer.

[3 marks]

(c) Suppose that the Net Profit Condition holds. Show that the *expected* amount of capital at any time $t > 0$ is larger than the initial capital u.

[4 marks]

(d) Prove that the probability that the first incoming claim leads to ruin can be expressed as

$$\int_0^\infty \big(1 - F_{X_1}(u + cs)\big)\lambda e^{-\lambda s}\,\mathrm{d}s,$$

where F_{X_1} is the common cdf of the claim sizes.

Hint: consider conditioning on the arrival time of the first claim.

[5 marks]

[Total 18 marks]

2. The insurance company from question 1 owns a second portfolio of products, for which the evolution of capital levels is also modelled as a Cramér-Lundberg capital process $(U_t)_{t\geq 0}$ as introduced in question 1. For this portfolio, the premium rate is $c = 1/2$, the claim intensity is $\lambda = 1$ and the claim sizes have a common pdf (probability density function) f given by

$$f(x) = \begin{cases} \frac{1}{8}e^{-2x} + \frac{15}{4}e^{-4x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

The initial capital is $u \geq 0$.

(a) Verify whether or not the Net Profit Condition holds.

[4 marks]

(b) Determine the Laplace exponent ξ of $(U_t)_{t\geq 0}$ and state its domain.

[5 marks]

(c) Find the Lundberg coefficient R of $(U_t)_{t\geq 0}$.

[5 marks]

Note: if you could not do part (c), then use $R = 1$ in the rest of this question where you need the value of R (this is not the correct value).

(d) Consider the following four functions, each defined for all $u \geq 0$:

$$\psi_1(u) = \frac{17}{32}e^{-\frac{1}{2}u}, \quad \psi_2(u) = \frac{15}{64}e^{-\frac{3}{2}u} + \frac{9}{64}e^{-2u},$$
$$\psi_3(u) = \frac{18}{8}e^{-\frac{3}{2}u} - \frac{55}{32}e^{-2u}, \quad \psi_4(u) = \frac{25}{64}e^{-\frac{3}{2}u} + \frac{9}{64}e^{-\frac{5}{2}u}.$$

One of these four is the correct expression for the ruin probability of $(U_t)_{t\geq 0}$ (as a function of the initial capital $u \geq 0$). Determine which one that is. Motivate your answer.

Hint: consider using elimination i.e. check which functions cannot be the ruin probability.

[5 marks]

(e) Suppose now that $u = 0$. The company is contemplating whether a change in its current premium level of $c = 1/2$ could result in a lower ruin probability. However it knows from experience that the more premium it charges, the more its policyholders are keen to get value for their money, resulting in more claims being submitted and larger claim sizes.

More precisely, it is assumed that these effects can be approximated as follows for $c \in (0, 3)$. If the current value $c = 1/2$ is changed by an amount h (positive or negative), then

- the common mean of the claim sizes changes by $0.2h\mathbb{E}[X_1]$ (where $\mathbb{E}[X_1]$ is the current common mean of the claim sizes),
- and the common mean of the claim interarrival times changes by $-0.3h$.

Which premium level in the interval $(0, 3)$ yields the smallest ruin probability?

[7 marks]

[Total 26 marks]

3. This question concerns premium principles.

(a) State the Esscher premium principle, and state the Monotonicity property for premium principles.

[6 marks]

(b) Show that the standard deviation premium principle with loading factor $\beta = 1.1$ satisfies the Scale Invariance property, but not the No Rip-off property.

Hint: for the latter, consider constructing a counterexample.

[8 marks]

[Total 14 marks]

4. Every week you buy a bag containing 10 treats for your pet dog Kevin. You know that Kevin's attitude towards any given type of treats is pretty straightforward (yet unobservable), namely one of the following two states:

$$\theta_1 : \text{ he really likes them,} \quad \theta_2 : \text{ he likes them less.}$$

Your model uses the 'uninformative' prior that assigns probability 1/2 to both possible states.

Further, you model the amount of treats that Kevin eats per week by a Binomial$(10, 0.7)$ distribution for treats that he really likes and a Binomial$(10, 0.3)$ distribution for treats that he likes less.

Now, your supermarket has started selling a new type, Squirrel Treats, and you decide to buy them. By the end of the week Kevin has eaten 7 of the Squirrel Treats.

(a) Using the 0-1 loss function, show that Bayes estimate for Kevin's state is θ_1.

[6 marks]

(b) Suppose now that you would like to estimate Kevin's state using the loss function l given by

$$l(\theta, a) = \begin{cases} c & \text{if } a \neq \theta \\ 1 & \text{if } a = \theta \end{cases} \qquad \text{for some constant } c > 1.$$

Show that Bayes estimate for Kevin's state under this loss function is (also) θ_1.

[5 marks]

(c) The observation that we find the same result in parts (a) and (b) is not a coincidence. Indeed, consider a more general model in which Kevin's state can be one of $\theta_1, \theta_2, \ldots, \theta_n$ with *any* prior distribution and *any* likelihood distribution. Suppose that Kevin eats $x \in \{0, 1, \ldots, 10\}$ treats. Show that Bayes estimate for Kevin's state under the 0-1 loss function is the same as Bayes estimate for Kevin's state under the loss function given in part (b).

[5 marks]

[Total 16 marks]

5. Suppose that the total amount a policyholder claims per year can be modelled by an $\text{Exp}(\theta)$ distribution, where $\theta > 0$ is a parameter. It is assumed that these claim amounts are independent over the years. Further it is initially assumed that the parameters are distributed over all policyholders according to a $\text{Gamma}(\alpha, \beta)$ distribution, for some $\alpha, \beta > 0$.

Now, consider a policyholder who generated total claim amounts $x_1, x_2, \ldots, x_n$ during the past n years.

(a) Show that the Bayes credibility estimate for the total claim amount for next year for this policyholder is given by

$$\frac{\beta + \sum_{i=1}^{n} x_i}{\alpha + n - 1}. \tag{1}$$

Hint: you may find the following identity for the Gamma function helpful: $\Gamma(z+1) = z\Gamma(z)$ for any $z > 0$.

[10 marks]

(b) Without doing any computations, briefly explain why the Bühlmann credibility estimate for the total claim amount for next year for this policyholder is also given by (1).

[3 marks]

(c) Give an advantage that the Bühlmann credibility estimate in general (so not restricted to the problem in this question) has compared to the Bayes credibility estimate.

[3 marks]

[Total 16 marks]

List of distributions

Discrete distributions

Name	Notation	Parameters	Pmf	Mean	Variance
Bernoulli	Bernoulli(p)	$p \in (0,1)$	$f(0) = 1-p,\ f(1) = p$	p	$p(1-p)$
Binomial	Binomial(n,p)	$n \in \{1,2,\ldots\}$ $p \in (0,1)$	$f(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0,1,\ldots,n$	np	$np(1-p)$
Geometric	Geometric(p)	$p \in (0,1)$	$f(k) = (1-p)^{k-1}p$ for $k = 1,2,\ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Poisson(λ)	$\lambda > 0$	$f(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ for $k = 0,1,\ldots$	λ	λ

Continuous distributions

Name	Notation	Parameters	Pdf	Mean	Variance
Beta	Beta(α,β)	$\alpha,\beta > 0$	$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$ for $x \in (0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Exponential	Exp(λ)	$\lambda > 0$	$f(x) = \lambda e^{-\lambda x}$ for $x > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma	Gamma(α,β)	$\alpha,\beta > 0$	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ for $x > 0$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
Normal	$\mathcal{N}(\mu,\sigma^2)$	$\mu \in \mathbb{R}$ $\sigma^2 > 0$	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-(x-\mu)^2/(2\sigma^2)\right)$ for $x \in \mathbb{R}$	μ	σ^2
Uniform	Unif(a,b)	$a,b \in \mathbb{R}$ $a < b$	$f(x) = \frac{1}{b-a}$ for $x \in (a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$

Notes

- For the pdf's above we understand $f(x) = 0$ for those $x \in \mathbb{R}$ for which no value is specified. For example, in the case of a Beta(α,β) distribution we have that $f(x) = 0$ for $x \leq 0$ and for $x \geq 1$.
- Γ denotes the Gamma function which can be defined as
$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, \mathrm{d}t \quad \text{for } x > 0.$$
Note that for integers $x \in \{1,2,\ldots\}$ we have that $\Gamma(x) = (x-1)!$.
- B denotes the Beta function which can be defined as
$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{d}t \quad \text{for } x,y > 0.$$
Note that for integers $x, y \in \{1,2,\ldots\}$ we have that
$$B(x,y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}.$$

END OF EXAMINATION PAPER