Two hours

Information

THE UNIVERSITY OF MANCHESTER

LINEAR ANALYSIS

26 May 2016

09:45 - 11:45

Answer ALL 7 questions in Section A (25 marks in total). Answer 3 of the 4 questions in Section B (75 marks in total). If more than 3 questions from Section B are attempted, then credit will be given for the best 3 answers.

The use of electronic calculators is permitted provided they cannot store text.

SECTION A

Answer **ALL** seven questions

A1. State what is meant by a norm on a vector space.

[3 marks]

A2. Let X be a Banach space and let M be a closed subset of X. Show that M is a complete space.

[4 marks]

A3. Let

$$x = \left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots\right).$$

For which $p \in [1, \infty]$ does x belong to $\ell^p(\mathbb{R})$? Justify your answer.

[4 marks]

A4. State what is meant by the norm of a linear functional on a normed vector space.

[2 marks]

A5. For any continuous function $\varphi : [0,1] \to \mathbb{R}$ set

$$f(\varphi) = \int_0^1 \varphi(x^3)\ dx.$$

Show that f is a bounded linear functional and compute its norm.

[4 marks]

A6. Let $c_0 \subset \ell^\infty(\mathbb{R})$ be the set of all sequences $(x_1, x_2, \dots)$ such that $x_i \to 0$ as $i \to \infty$. Show that c_0 is an infinite-dimensional space. (You may assume that it is a vector space.)

[4 marks]

A7. Let L be a linear subspace of an inner product space H. State what is meant by $L^\perp$ and show that $L^\perp$ is a linear subspace of H.

[4 marks]

SECTION B

Answer **THREE** of the four questions

B8.

(a) State what is meant by the nth Bernstein polynomial $B_n(f;x)$ for a function $f:[0,1]\to\mathbb{R}$.

(b) Show that $B_n(2^x+x;x)=(x2^{1/n}+1-x)^n+x$.

(c) State, without proof, the Weierstrass Approximation Theorem.

(d) Let $\mathcal{A}$ be a subset of $C(X)$. State what it means to say that $\mathcal{A}$ is an algebra.

(e) State, without proof, the Stone-Weierstrass Theorem.

(f) Let $f:[0,1]^n\to\mathbb{R}$ be a continuous function on the n-dimensional unit cube $[0,1]^n$. Use the Stone-Weierstrass Theorem to show that f can be uniformly approximated by a polynomial in n variables.

[25 marks]

B9.

(a) State what it means for two norms on a vector space to be equivalent.

(b) Prove that any norm on $\mathbb{R}^n$ is equivalent to $\|\cdot\|_1$.

(c) Let $C^1[0,1]$ denote the space of differentiable functions from $[0,1]$ to $\mathbb{R}$ such that f' is continuous. For $f\in C^1[0,1]$, put

$$\|f\|_{C^1}=\|f\|_\infty+\|f'\|_\infty.$$

Prove that $\|\cdot\|_{C^1}$ is a norm. *(You may assume that $\|\cdot\|_\infty$ is a norm on $C[0,1]$.)*

(d) Prove that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{C^1}$ are not equivalent on $C^1[0,1]$.

[25 marks]

B10.

(a) State what is meant by an inner product on a vector space over $\mathbb{R}$ or $\mathbb{C}$.

(b) State what is meant by a Hilbert space.

(c) Let H be a Hilbert space and let $\{f_n\}_{n=1}^{\infty}$ be a linearly independent subset of H. Describe the Gram-Schmidt algorithm for constructing an orthonormal subset of H.

(d) State and prove Bessel's inequality.

[25 marks]

B11. Let H be a Hilbert space over $\mathbb{C}$ and let $T : H \to H$ be a bounded linear operator.

(a) State what is meant by the spectrum, $\mathit{spec}(T)$, of T.

(b) Prove that any eigenvalue of T is an element of the spectrum of T.

(c) Let P be a polynomial on $\mathbb{C}$. Prove that

$$\mathit{spec}(P(T)) = \{P(\lambda) : \lambda \in \mathit{spec}(T)\}.$$

Let $H = \ell^2(\mathbb{C})$ and

$$T(x_1, x_2, x_3, \dots) = (x_1, 0, x_3, 0, x_5, 0, \dots).$$

(d) Compute the norm of T.

(e) Using parts (b) and (c) or otherwise, prove that $\mathit{spec}(T) = \{0, 1\}$.

[25 marks]

END OF EXAMINATION PAPER

Two hours

UNIVERSITY OF MANCHESTER

ANALYTIC NUMBER THEORY

2 Jun 2016

14:00 – 16:00

Answer **THREE** of the FOUR questions. (90 marks in total.) If more than THREE questions are answered, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

Throughout the paper you may assume that the Dirichlet Convolution of two multiplicative functions is multiplicative.

1.

i) Prove, by Partial Summation, that

$$\sum_{n\leq x} \log n = x \log x - x + O(\log x).$$

ii) Define the arithmetic function Λ, von Mangoldt's function.

Prove that

$$\sum_{d|n} \Lambda(d) = \log n,$$

for all $n \geq 1$.

iii) Use parts i and ii to prove

$$\sum_{n\leq x} \Lambda(n) \left[\frac{x}{n}\right] = x \log x - x + O(\log x),$$

and thus deduce Merten's result

$$\sum_{n\leq x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

(You may assume Chebyshev's result in the form $\psi(x) = O(x)$ for all $x \geq 2$.)

[30 marks]

2.

i) a) Prove by partial summation that

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t} dt,$$

where $\pi(x) = \sum_{p\leq x} 1$ and $\theta(x) = \sum_{p\leq x} \log p$.

b) Deduce that

$$\pi(x) = \frac{\theta(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

(You may assume Chebyshev's result in the form $\theta(x) = O(x)$ for all $x > 2$.)

c) Prove that $\pi(x) \sim x/\log x$ if, and only if, $\theta(x) \sim x$ as $x \to \infty$.

ii) Prove that if $\pi(x) \sim x/\log x$ then the n-th prime p_n satisfies $p_n \sim n \log n$ as $n \to \infty$.

[30 marks]

3.

i) Prove that
$$3 + 4\cos\theta + \cos 2\theta \geq 0,$$
for all $\theta \in \mathbb{R}$.

ii) Assume that
$$\log|\zeta(s)| = \mathrm{Re}\sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}},$$
for $\mathrm{Re}s > 1$.

Prove that for $\sigma > 1$
$$|\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1,$$
for all $t \in \mathbb{R}$.

iii) Prove that the Riemann zeta function has no zeros on the line $\mathrm{Re}s = 1$.

(You may assume that $\zeta(s)$ is holomorphic in $\mathrm{Re}s > 0$ except for a pole at $s = 1$.)

[30 marks]

4.

i) Let f be the multiplicative function defined on prime powers as $f(1) = 1$ and
$$f(p^a) = \begin{cases} 1 & \text{if } a \equiv 1 \bmod 4 \\ -1 & \text{if } a \equiv 3 \bmod 4 \\ 0 & \text{otherwise,} \end{cases}$$
for p prime and $a \geq 1$. Show that the Dirichlet Series $D_f(s)$ for f satisfies
$$D_f(s) = \frac{\zeta(s)\zeta(4s)}{\zeta(2s)\zeta(3s)}$$
for $\mathrm{Re}s > 1$.

ii) The multiplicative function Q_3 is defined on prime powers by $Q_3(p^a) = 1$ if $a = 0, 1$ or 2 and 0 in all other cases. The multiplicative function μ_3 is defined on the integers n by $\mu_3(n) = \mu(m)$ if $n = m^3$, where μ is the Mobius function, while $\mu_3(n) = 0$ if n is not a cube.

a) Prove that $Q_3 = 1 * \mu_3$.

b) Prove that
$$\sum_{n \leq x} Q_3(n) = \frac{x}{\zeta(3)} + O\left(x^{1/3}\right).$$

You may assume $\sum_{n \geq y} 1/n^r = O\left(1/y^{r-1}\right)$.

[30 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

ALGEBRAIC TOPOLOGY

24 May 2016

9:45-11:45

Answer **ALL** FOUR questions in Section A (40 marks in total). Answer **THREE** of the FOUR questions in Section B (45 marks in total). If more than THREE questions from Section B are attempted then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** FOUR questions.

A1.

(a) Define what is meant by a *topological manifold*.

(b) State the classification theorem for connected compact topological surfaces.

(c) Give an example of two distinct surfaces from the classification theorem with the same Euler characteristic.

[10 marks]

A2.

(a) Define what it meant by a *geometric simplicial complex* K and its underlying space $|K|$. [The notions of geometric simplex and face of a simplex may be used without definition.]

(b) An abstract simplicial complex has vertices v_1, v_2, v_3, v_4, v_5 and simplices $\{v_1, v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_4, v_5\}$, $\{v_3, v_5\}$, $\{v_2, v_5\}$ and their faces. Draw a realisation K of this simplicial complex as a geometric simplicial complex in $\mathbb{R}^2$.

(c) Define the *Euler characteristic* of a simplicial complex and calculate the Euler characteristic of the simplicial complex in part (b).

(d) Draw the first barycentric subdivision K' of the geometric simplicial complex K in part (b).

(e) Find the Euler characteristic of K'.

[10 marks]

A3.

(a) Define what is meant by the *r-chain group* $C_r(K)$, the *r-cycle group* $Z_r(K)$, and the *r-boundary group* $B_r(K)$ of a simplicial complex K.

(b) Write down, without proof, generators for the groups $Z_1(K)$ and $B_1(K)$ of the simplicial complex K in Question A2(b). Hence, find the first homology group $H_1(K)$.

[10 marks]

A4.

(a) Let $\bar{\Delta}^6$ be be the simplicial complex consisting of all of the faces of the standard 6-simplex in $\mathbb{R}^7$ (including the 6-simplex itself). Consider the simplicial complex K obtained from starring in the barycentre of Δ^6. Write down the simplicial homology groups of K. Justify your answer.

(b) Let L be the 3-skeleton of K. Calculate the Euler characteristic of L and find its simplicial homology groups

[10 marks]

SECTION B

Answer **THREE** of the FOUR questions.

B5. Let e_i be the ith standard basis vector in $\mathbb{R}^8$, $1 \leqslant i \leqslant 8$. Consider the set K of sixteen triangles with vertices e_i , e_j and e_k where ijk runs over the following triples:

$$126, 236, 138, 148, 348, 146, 365, 345, 467, 675, 472, 751, 452, 152, 237, 137.$$

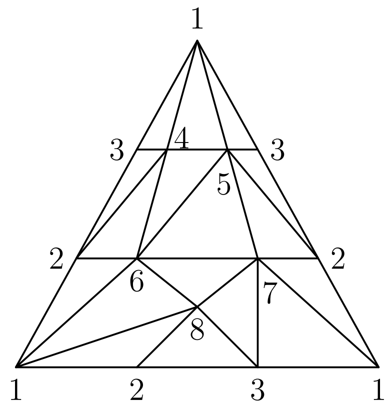
(a) Verify that K is a simplicial surface.
[For the link condition, you need only check the vertices e_1 and e_8 to illustrate the method.]

(b) Represent the underlying space of K as a polygon with edges identified in pairs, and hence represent $|K|$ by a symbol.

(c) Reduce the symbol to canonical form and hence determine the genus and orientability type of the surface $|K|$.

[15 marks]

B6. Consider the triangulation K of the *dunce hat* given by the following picture:



(a) Show that the simplicial homology groups of the dunce hat are given by

$$H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

You may use the fact, that every 1-cycle is homologous to one involving only edges on the boundary of the template and e.g. the following “internal” edges: $\langle 3,4\rangle$, $\langle 3,5\rangle$, $\langle 2,6\rangle$, $\langle 2,7\rangle$ and $\langle 1,8\rangle$.

(b) Use the classification theorem to show that the dunce hat is *not* a closed surface.

[15 marks]

B7.

(a) Define the rth *Betti number* β_r of a simplicial complex K

(b) Define the *Euler characteristic* $\chi(K)$ of a simplicial complex K.

(c) State and prove the relationship between the Betti numbers and the Euler characteristic of K.

(d) Give an application of this relationship.

[15 marks]

B8.

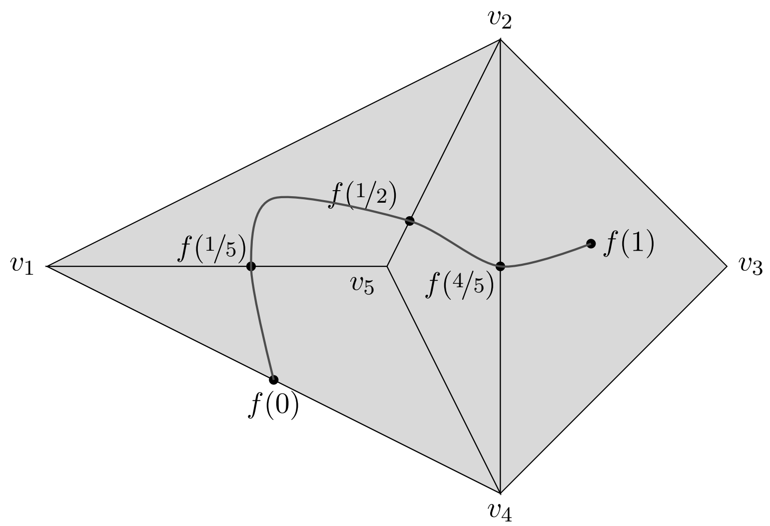
(a) Let K and L be simplicial complexes. Define what is meant by a *simplicial map* $|K| \to |L|$ (with respect to K and L). Define what is meant by a *simplicial approximation* to a continuous map $f\colon |K| \to |L|$ (with respect to K and L).

Prove that a vertex map s with $f(\mathrm{star}(v)) \subset \mathrm{star}(s(v))$ for all vertices v of K induces a simplicial approximation to f.

(b) Consider the simplicial complex L with vertices v_1, v_2, v_3, v_4, v_5, which is drawn below, and an injective continuous map $f\colon [0,1] \to |L|$ with

$$f(0) \in \langle v_1, v_4\rangle,\ f(1/5) \in \langle v_1, v_5\rangle,\ f(1/2) \in \langle v_2, v_5\rangle,\ f(4/5) \in \langle v_2, v_4\rangle,\ f(1) \in \langle v_2, v_3, v_4\rangle$$

and having the image indicated in the picture. Let K be the simplicial complex consisting just of the simplex $\langle 0, 1\rangle$ and its faces. Give a simplicial approximation to f on a sufficiently fine barycentric subdivision $K^{(m)}$ of K.



[15 marks]

END OF EXAMINATION PAPER

MATH 31082

Two hours

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY

20 May 2016
14:00—16:00

ANSWER THREE OF THE FOUR QUESTIONS
If four questions are answered credit will be given for the best three questions
All questions are worth 20 marks

Electronic calculators may not be used

P.T.O.

1.

(a) Explain what is meant by saying that G is a Riemannian metric on a manifold M.

Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with Riemannian metric $G = \sigma(x, y)(dx^2 + dy^2)$.
Explain why $\sigma(x, y) > 0$.

Let $\mathbf{A}$ and $\mathbf{B}$ be two arbitrary vectors attached at some point of this plane. Explain why the cosine of the angle between these vectors does not depend on a choice of a function $\sigma(x, y)$.

In the plane with the metric G as above, consider the circle C defined by the equation $x^2 + y^2 = R^2$ and calculate its length in the case $\sigma(x, y) = e^{-x^2-y^2}$.

[8 marks]

(b) Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with Riemannian metric $G = e^{-x^2-y^2}(dx^2 + dy^2)$.

Write down the formula for the area element in this metric in coordinates (x, y) and in polar coordinates (r, φ) $(x = r\cos\varphi, y = r\sin\varphi)$.
Calculate the area of the disc, $x^2 + y^2 \leq R^2$ (in the given metric).

In the case when $R = 1$ give an example of another metric $G' = \sigma(x, y)(dx^2 + dy^2)$ such that the area of the disc $x^2 + y^2 \leq 1$ will be the same as for the metric G.

[8 marks]

(c) Explain what is meant by saying that a Riemannian manifold is locally Euclidean.

Consider in Euclidean space $\mathbf{E}^3$ the surface

$$\begin{cases} x = 3h\cos\varphi \\ y = 3h\sin\varphi \\ z = h \end{cases}, \qquad h > 0\,, \quad 0 \leq \varphi < 2\pi\,,$$

(the upper sheet of the cone).
It is known that the induced Riemannian metric on this surface is given by the formula $G = 10dh^2 + 9h^2d\varphi^2$.
Show that this surface is locally Euclidean.

[4 marks]

P.T.O.

2.

(a) Explain what is meant by an affine connection on a manifold.

Let ∇ be an affine connection on a 2-dimensional manifold M in local coordinates (u, v). It is known that $\nabla_{\frac{\partial}{\partial u}} \left(u \frac{\partial}{\partial u}\right) = \frac{\partial}{\partial u} + u \frac{\partial}{\partial v}$.
Calculate the Christoffel symbols Γ^u_{uu} and Γ^v_{uu}.

[5 marks]

(b) Explain what is meant by the induced connection on a surface in Euclidean space.

Consider a sphere in $\mathbf{E}^3$:

$$\mathbf{r}(\theta, \varphi): \quad \begin{cases} x = R \sin\theta \cos\varphi \\ y = R \sin\theta \sin\varphi \\ z = R \cos\theta \end{cases} .$$

Let ∇ be the induced connection on the sphere.

Calculate the Christoffel symbols $\Gamma^\theta_{\theta\varphi}$, $\Gamma^\varphi_{\theta\varphi}$, $\Gamma^\theta_{\varphi\theta}$ and $\Gamma^\varphi_{\varphi\theta}$.

[6 marks]

(c) Give a detailed formulation of the Levi-Civita Theorem. In particular write down the expression for the Christoffel symbols Γ^i_{km} in terms of a Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

Consider the open disc $u^2 + v^2 < 1$ with the Riemannian metric

$$G = \frac{4(du^2 + dv^2)}{(1 - u^2 - v^2)^2},$$

(Poincaré disc).
Show that all Christoffel symbols of the Levi-Civita connection of this Riemannian manifold vanish at the point $u = v = 0$.

Let ∇' be a symmetric connection on the Poincaré disc such that all Christoffel symbols of this connection in coordinates (u, v) vanish identically (at all points).
Show that the connection ∇' does not preserve the metric of the Poincaré disc.
(You may wish to consider the vector field $\mathbf{A} = \frac{\partial}{\partial u}$.)

[9 marks]

P.T.O.

3.

(a) Let (M, G) be a Riemannian manifold.
Let C be a curve on M starting at the point $\mathbf{p}_1$ and ending at the point $\mathbf{p}_2$.

Explain what is meant by the parallel transport P_C along the curve C.

Explain why the parallel transport P_C is a linear orthogonal operator.

Let the points $\mathbf{p}_1$ and $\mathbf{p}_2$ coincide, so that C is a closed curve.
Let $\mathbf{a}$ be a tangent vector at the point $\mathbf{p}_1$, and $\mathbf{b} = P_C(\mathbf{a})$.
Suppose that $P_C(\mathbf{b}) = -\mathbf{a}$.
Show that vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other.

[8 marks]

(b) Write down the differential equation for geodesics of a Riemannian manifold in terms of Christoffel symbols.

Explain the relation between the Lagrangian of a free particle on a Riemannian manifold and the differential equations for geodesics.

Calculate the Christoffel symbols on the Lobachevsky plane.

(You may use the Lagrangian of a free particle on the Lobachevsky plane $L = \frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2}$.)

[6 marks]

(c) Explain why great circles are geodesics on a sphere in $\mathbf{E}^3$.

On the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbf{E}^3$ consider the curve C defined by the equation $\cos\theta - \sin\theta \sin\varphi = 0$ in spherical coordinates.

Show that in the process of parallel transport along the curve C an arbitrary tangent vector to the curve remains tangent to the curve.

[6 marks]

P.T.O.

4.

(a) Let M be a surface in the Euclidean space $\mathbf{E}^3$. Let $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface. Consider the derivation formula

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix},$$

where a, b and c are 1-forms on the surface M.
Express the mean curvature and the Gaussian curvature of M in terms of these 1-forms and vector fields.
Show that $da + b \wedge c = 0$.

[6 marks]

(b) Consider the surface of a saddle in Euclidean space $\mathbf{E}^3$,

$$\mathbf{r}(u, v)\colon \quad \begin{cases} x = u \\ y = v \\ z = kuv \end{cases} . \quad k \text{ is a parameter, } k \neq 0.$$

Find vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ defined at the points of the saddle such that they form an orthonormal basis at any point, the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface.
For the obtained orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ calculate the vector 1-forms $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$ and the 1-forms a, b and c at the point $\mathbf{p}$ with coordinates $u = v = 0$.
Deduce from these calculations the Gaussian curvature of the saddle at the point $\mathbf{p}$.

[8 marks]

(c) Consider a surface M in $\mathbf{E}^3$ with local coordinates (u, v) such that the induced metric of this surface is equal to $G = \sigma(u, v)(du^2 + dv^2)$.
Write down the formula expressing Gaussian curvature of this surface in terms of the function $\sigma(u, v)$. (You do not need to prove this formula.)

Let $\sigma(u, v) = \frac{1}{v^2}$. Calculate the Gaussian curvature of this surface and explain why this surface is not locally Euclidean.

[6 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

COMMUTATIVE ALGEBRA

19 May 2016

14:00 – 16:00

Answer ALL THREE questions in Section A (40 marks in total). Answer TWO of the THREE questions in Section B (40 marks in total). If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

Electronic calculators are permitted, provided they cannot store text.

SECTION A

Answer **ALL** questions in this section (40 marks in total)

A1. Let R be a commutative ring and let $M(X_1, \ldots, X_n)$ be the set of all monomials in $X_1, \ldots, X_n$.

(a) What is meant by a monomial ordering $\preccurlyeq$ on $M(X_1, \ldots, X_n)$?

(b) What is meant by the leading monomial, $\mathrm{lm}_{\preccurlyeq} f$, of a polynomial $f \in R[X_1, \ldots, X_n] \setminus \{0\}$? What is meant by saying that f is monic with respect to $\preccurlyeq$?

(c) Explain why, if R is a domain, $\mathrm{lm}_{\preccurlyeq}(fg) = \mathrm{lm}_{\preccurlyeq}(f)\,\mathrm{lm}_{\preccurlyeq}(g)$.

(d) Write down an example of a ring R, a monomial ordering $\preccurlyeq$ on $M(X, Y)$ and a polynomial $f \in R[X, Y]$ such that $\mathrm{lm}_{\preccurlyeq} f = \mathrm{lm}_{\preccurlyeq}(f^2) = XY$.

[14 marks]

A2. Consider the ideal $I = <X^2 + Y,\ Y^2 + Z,\ Z^2 + X>$ of $\mathbb{C}[X, Y, Z]$.

(a) Find a lexicographic $(X \succ Y \succ Z)$ Gröbner basis of I.

(b) Briefly explain why I is a zero-dimensional ideal of $\mathbb{C}[X, Y, Z]$.

[14 marks]

A3.

(a) Define the radical, $\sqrt{J}$, of an ideal J of a ring R. What is a radical ideal?

You are given that $\sqrt{J}$ is an ideal of R, and you do not have to prove this.

(b) Prove that $\sqrt{J}$ is a radical ideal.

(c) Let a domain R contain $a \neq 0$ such that $<a^2>$ is a radical ideal of R. Show that $<a^2> = R$.

(d) Is $<(X + 1)^4 + 4X^4>$ a radical ideal of $\mathbb{Q}[X]$? Justify your answer. Any facts from the course can be used without proof.

[12 marks]

SECTION B

Answer **TWO** of the three questions in this section (40 marks in total).

If more than TWO questions from this section are attempted, then credit will be given for the best TWO answers.

B4.

(a) Define what is meant by a quadratic integer.

(b) Prove that the set of all quadratic integers is not closed under addition.

(c) Prove: if α is a quadratic integer, then so is every element of $\mathbb{Z}[\alpha] = \{m + n\alpha \mid m, n \in \mathbb{Z}\}$.

In (d)–(f), you may use the fact that $\mathbb{Z}[\alpha]$ is a ring, and you do not have to prove it.

(d) Show that both $\dfrac{1+\sqrt{-67}}{2}$ and $1+\sqrt{-67}$ are quadratic integers.

(e) Give an example of a subset $T \subset \mathbb{Z}[1+\sqrt{-67}]$ such that the ideal $<T>$ of $\mathbb{Z}[1+\sqrt{-67}]$ is not principal. You do not have to justify your example.

(f) Let β be a quadratic integer such that $\mathbb{Z}[2\beta]$ is a principal ideal domain. Show that $\beta \in \mathbb{Z}$. You may use any facts from the course without particular comment.

[20 marks]

B5. Let R be a domain.

(a) What is meant by saying that an element p of R is a prime in R?

(b) Prove that if p is a prime in R, then p (viewed as a constant polynomial) is a prime in $R[X]$.

(c) What is meant by a primitive polynomial in $R[X]$?

(d) State without proof the Gauss Lemma on primitive polynomials.

(e) Does there exist a polynomial of the form $aX^{2016} + b$, with $a, b \in \mathbb{Z}$, $a > 1$, $b > 1$, which is:

i. irreducible and not primitive in $\mathbb{Z}[X]$?

ii. primitive and not irreducible in $\mathbb{Z}[X]$?

Justify your answers.

(f) Let p be a prime in R such that the polynomial $X^3 - p$ is irreducible in $R[X]$ but not irreducible in $F[X]$, where F is the field of fractions of R. Prove that R is not noetherian.

[20 marks]

B6.

(a) What is meant by a Gröbner basis of an ideal I of $\mathbb{Q}[X_1, \ldots, X_n]$ with respect to a monomial ordering $\preccurlyeq$?

(b) What is meant by saying that a Gröbner basis G is reduced?

(c) Prove that if G, H are reduced Gröbner bases of I with respect to $\preccurlyeq$, then $G = H$. You may use any other facts from the course without proof.

(d) Let G be a reduced Gröbner basis of an ideal I of $\mathbb{Q}[X, Y]$ with respect to some monomial ordering.

i. Is it possible that G contains the polynomial $X - X^2Y$?

ii. Is it possible that G contains both $X^2Y + 1$ and XY^2?

iii. Is it possible that G contains a subset $F \subset G$, $F \neq G$ such that $<F> = I$?

Justify your answer in each case.

[20 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO ALGEBRAIC GEOMETRY

Date: 25 May, 2016

Time: 14:00–16:00

Answer THREE of the FOUR questions.
If more than THREE questions are attempted, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text/transmit or receive information/display graphics.

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1. In this question K denotes an arbitrary field and unless specified otherwise, varieties are defined over K.

(a) (i) Define the *affine algebraic variety* $\mathcal{V}(J) \subseteq K^n$ determined by an ideal $J \lhd K[x_1, x_2, \ldots, x_n]$.

(ii) Define the *ideal* $\mathcal{I}(X)$ of a set $X \subseteq K^n$.

(iii) Show that if $X \subseteq K^n$ is an affine algebraic variety, then $\mathcal{V}(\mathcal{I}(X)) = X$.

(b) Let $J = \langle (x + y + z^2)^3, x^3 + xy + z^2 - xz^2 \rangle \lhd \mathbb{C}[x, y, z]$.
Show that $x + y + z^2 \in \mathcal{I}(\mathcal{V}(J))$, but $x + y + z^2 \notin J$.

(c) (i) Define what it means for an affine algebraic variety W to be *irreducible*.

(ii) Prove that every affine algebraic variety can be written as a union of finitely many irreducible affine algebraic varieties. (You may use any other results from the course that you need.)

(d) Find the irreducible components of the variety

$$W = \mathcal{V}(\langle (x - 1)yz, x^2 + y^2 - z^2 - 1 \rangle) \subset \mathbb{A}^3(\mathbb{C}).$$

(You should either identify geometrically what kind of variety each component is or prove directly that it is irreducible.)

[20 marks]

2. (a) Let K be a field. Let $V \subseteq \mathbb{A}^n(K)$ be an affine algebraic variety and let $P \in V$.

(i) Let $P \in V$ be a point and let $\mathbf{v} \in K^n$ be a non-zero vector. Let $\ell = \{P + t\mathbf{v} \mid t \in K\}$ be the line through P with direction vector $\mathbf{v}$. Explain what is meant by saying that ℓ is *tangent* to V at P and give the definition of the *tangent space* $T_P V$ to V at P.

(ii) Describe without proof how you can calculate $T_P V$ if you have a set of generators $f_1, f_2, \ldots, f_r$ for $\mathcal{I}(V)$ and explain why this implies that $T_P V$ is an affine subspace.

(b) This part of the question is over $\mathbb{C}$. Let x, y be co-ordinates on $\mathbb{A}^2(\mathbb{C})$ and let t be the co-ordinate on $\mathbb{A}^1(\mathbb{C})$. Let $C \subset \mathbb{A}^2(\mathbb{C})$ be the variety defined by the equation $x^3 - 5xy^3 - y^5 = 0$. (You may assume without proof that $x^3 - 5xy^3 - y^5$ is an irreducible polynomial, therefore C is irreducible and $\mathcal{I}(C) = \langle x^3 - 5xy^3 - y^5 \rangle$.)

(i) Find the singular points of C or show that it has none.

(ii) Show that $\phi(t) = (t^3(t+5)^2, t^2(t+5))$ is a morphism $\mathbb{A}^1 \to C$.

(iii) Show that $\dfrac{y^2}{x} + 5$ and $\dfrac{x^2}{y^3}$ represent the same element in the function field of C. Use this to prove that $\psi : C \dashrightarrow \mathbb{A}^1$, $\psi(x, y) = \dfrac{y^2}{x}$ is an inverse to ϕ considered as a rational map.

(iv) Prove that C is rational but it is not isomorphic to $\mathbb{A}^1$.

[20 marks]

3. In this question K denotes an arbitrary field.

(a) (i) Let I be an ideal in $K[X_0, X_1, \ldots, X_n]$. Explain what is meant by saying that I is *homogeneous*.

(ii) Let $I \lhd K[X_0, X_1, \ldots, X_n]$ be a homogenenous ideal. Give the definition of the projective variety $\mathcal{V}(I) \subseteq \mathbb{P}^n$.

(iii) Let $V \subseteq \mathbb{P}^n$ be a projective variety. Define $\mathcal{I}(V)$, the homogeneous ideal of V, and $K[V]$, the homogeneous co-ordinate ring of V.

(b) Let m, n be positive integers. Let $(X_0 : X_1 : \ldots : X_m)$ be homogeneous co-ordinates on $\mathbb{P}^m(K)$ and let let $(Y_0 : Y_1 : \ldots : Y_n)$ be homogeneous co-ordinates on $\mathbb{P}^n(K)$. Let $a_{ij} \in K$ for $0 \leq i \leq n$ and $0 \leq j \leq m$. For $0 \leq i \leq n$, let $\Phi_i(X_0, X_1, \ldots, X_m) = \sum_{j=0}^{m} a_{ij} X_j$ be homogeneous polynomials of degree 1 and assume that they are not all scalar multiples of a single homogeneous polynomial of degree 1. Let $\Phi : \mathbb{P}^m(K) \dashrightarrow \mathbb{P}^n(K)$ be the rational map $\Phi = (\Phi_0 : \Phi_1 : \ldots : \Phi_n)$.

(i) Show that if $m > n$ then Φ is not a morphism.

(ii) Show that if $m = n$ and Φ is a morphism then it has an inverse morphism.

(c) In this part of the question we identify $\mathbb{P}^1(K)$ with $K \cup \{\infty\}$ by identifying $(X_0 : X_1)$ with $X_0/X_1 \in K$ if $X_1 \neq 0$ and $(X_0 : 0)$ with ∞ as usual.
Recall that if z_1, z_2, z_3, $z_4 \in K \cup \{\infty\}$ with at least 3 of them pairwise distinct, their cross ratio is defined to be

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \in K \cup \{\infty\}$$

with appropriate rules for arithmetic involving ∞ and for division by 0.

Let z_1, z_2, z_3, $z_4 \in K \cup \{\infty\}$ with at least 3 of them distinct. Show that $(z_1, z_2; z_3, z_4) = (\phi(z_1), \phi(z_2); \phi(z_3), \phi(z_4))$ for any projective transformation $\phi : \mathbb{P}^1 \to \mathbb{P}^1$.

(c) Find $a, b, c, d \in \mathbb{C}$ such that $\phi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, $\phi(z) = \dfrac{az + b}{cz + d}$ satisfies $\phi(2) = 1$, $\phi(3) = 2$ and $\phi(5) = 7$.

[20 marks]

4. (a) Let K be a field. Let $E \subset \mathbb{P}^2(K)$ be a non-singular cubic curve and let O be a point of E. (You should not assume that the equation of E is in a special form.)

(i) Describe without proof how one can define the addition operation on the points of E to obtain a commutative group in which O is the zero element.

(ii) Assume now that O is an inflection point of E. Show that $P \in E$ satisfies $3P = O$ if and only if P is an inflection point.

(b) Let F be the elliptic curve given by the affine equation

$$y^2 = x^3 - 15x + 22$$

over $\mathbb{C}$ together with the point $O = (0 : 1 : 0)$ in homogeneous co-ordinates, which is taken to be the zero element of the group law as usual.

(i) Let $A = (-1, 6)$, $B = (2, 0)$. Calculate $A + B$ and $2A$.

(ii) Calculate the Hessian of F and use it to show that the points $(3, 2)$ and $(3, -2)$ have order 3 in the group structure.

[20 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

MATH 32112 Lie Algebras

23 May 2016

9:45–11:45

Electronic calculators may be used, provided that they cannot store text.

Answer the **ONE** question in **SECTION A** (26 marks)
and
TWO of the **THREE** questions in **SECTION B** (54 marks in total)

P.T.O

SECTION A

Answer the **ONE** question.

A1. Let $\Bbbk$ be a field.

(i) Let A be an algebra over $\Bbbk$. When is A called *associative*? When is A called *anti-commutative*? When is A called a *Lie algebra*? For an associative algebra A over $\Bbbk$ define the Lie algebra $A^{(-)}$. Proving the Jacobi identity in $A^{(-)}$ is not required.

[3 marks]

(ii) Let A be an associative, anti-commutative algebra over $\Bbbk$ with bilinear operation $(x, y) \mapsto x \cdot y$ where $x, y \in A$. Show that if A satisfies the Jacobi identity then $a \cdot b \cdot c = 0$ for all $a, b, c \in A$.

[3 marks]

From now on assume that L is a Lie algebra over $\Bbbk$.

(iii) When is a subspace I of L called an *ideal* of L? Prove that if I and J are ideals of L then so is $I + J$.

[2 marks]

(iv) Define the *derived series* of L. State the *Lemma on Two Ideals* and use it to show that all members of the derived series of L are ideals of L. When is L called *solvable*?

[5 marks]

(v) Define the *lower central series* of L and show that all members of that series are ideals of L. When is L called *nilpotent*? Give an example of a two-dimensional solvable Lie algebra over $\Bbbk$ which is not nilpotent.

[4 marks]

(vi) Let V be a finite dimensional vector space over $\Bbbk$. What is the *general linear Lie algebra* $gl(V)$? Given $x \in L$ define the *adjoint endomorphism* $\operatorname{ad} x \in gl(L)$ and show that the map $\operatorname{ad}\colon L \to gl(L)$ is a representation of L. When is a representation $\rho\colon L \to gl(V)$ called *irreducible*? Explain what is meant by saying "L is simple". Prove that L is simple if and only if L is nonabelian and the adjoint representation of L is irreducible.

[6 marks]

(vii) When is an endomorphism $A \in gl(V)$ called *nilpotent*? Show that if L is a nilpotent Lie algebra, then all endomorphisms $\operatorname{ad} x$ with $x \in L$ are nilpotent.

[3 marks]

P.T.O

SECTION B

Answer **TWO** of the **THREE** questions.

B2. Let $\mathbb{k}$ be an algebraically closed field and let L be a finite dimensional Lie algebra over $\mathbb{k}$.

(i) State *Engel's theorem* for Lie subalgebras of $gl(V)$ and use it to prove that if all adjoint endomorphisms $\operatorname{ad} x$, where $x \in L$, are nilpotent, then L is a nilpotent Lie algebra.

[4 marks]

(ii) Define the *radical*, $\operatorname{rad}(L)$, of the Lie algebra L and explain what is meant by "L is *semisimple*". When is a Lie algebra S called *simple*? Explain why the direct sum $S_1 \oplus \cdots \oplus S_r$ of finite dimensional simple Lie algebras $S_1, \ldots, S_r$ is semisimple. Hint: use the canonical projections $\pi_i \colon S_1 \oplus \cdots \oplus S_r \to S_i$.

[4 marks]

(iii) State *Cartan's criterion* for solvability of a Lie subalgebra of $gl(V)$, where V is a finite dimensional vector space over $\mathbb{k}$.

[3 marks]

(iv) Suppose in this part that $\mathbb{k}$ is a field of characteristic 2 and let L be a Lie algebra over $\mathbb{k}$ with basis $\{a, b, c\}$ and Lie bracket $[a, b] = c$, $[b, c] = a$, $[c, a] = b$. Show that L is not solvable. Show, by computing the Killing form of L, that L demonstrates the failure of Cartan's criterion for a field of nonzero characteristic.

[5 marks]

(v) Let R be the radical of L. State the *Isomorphism Theorem* for Lie algebras and explain how one can use it to prove that the factor-algebra L/R is semismple. What is the radical of a nilpotent Lie algebra?

[6 marks]

(vi) State *Lie's theorem* on solvable Lie algebras in the form involving flags of subspaces and use it to show that if L is solvable and $\operatorname{char}(\mathbb{k}) = 0$ then the derived subalgebra $L^{(1)} = [L, L]$ is nilpotent.

[5 marks]

P.T.O

B3. Let L be a Lie algebra over an algebraically closed field $\Bbbk$.

(i) Define the *Killing form* κ of L and show that κ is L-invariant. Explain what is meant by $\operatorname{Rad}\kappa$, the *radical* of κ. Prove that the Killing form of a nilpotent Lie algebra is zero.

[5 marks]

(ii) Let I be an ideal of L. Show that the Killing form of the Lie algebra I coincides with the restriction of κ to $I \times I$.

[4 marks]

(iii) Prove that if L is not semisimple then the Killing form κ of L is degenerate.

[4 marks]

(iv) Let $\mathfrak{s}$ be a 2-dimensional non-abelian Lie algebra over $\Bbbk$. Show that $\mathfrak{s}$ has a basis $\{u, v\}$ such that $[u, v] = v$. Compute the radical of the Killing form κ of $\mathfrak{s}$. Is it true that $\operatorname{Rad}\kappa = \operatorname{rad}(\mathfrak{s})$?

[5 marks]

(v) Let $S = \{e, h, f\}$ be the standard basis of $L = sl(2, \Bbbk)$. Compute $\kappa(h, h)$. Use the L-invariance of κ to compute $\kappa(f, h)$, $\kappa(h, e)$ and $\kappa(e, f)$. Determine $\kappa(f + 2h, e + 3h)$.

[4 marks]

(vi) Suppose $\operatorname{char}(\Bbbk) \neq 2, 3$ and let L be a 4-dimensional Lie algebra over $\Bbbk$ with basis $\{h, e_1, e_2, e_3\}$ such that

$$[h, e_1] = e_1,\ [h, e_2] = 2e_2,\ [h, e_3] = 3e_3,\ [e_1, e_2] = e_3,\ [e_i, e_j] = 0 \text{ for } i + j \geq 4.$$

Determine the lower central series and the derived series of L.

[5 marks]

P.T.O

B4. Let L be a finite dimensional Lie algebra over an algebraically closed field $\Bbbk$ and let V be a finite dimensional vector space over $\Bbbk$.

(i) Let a be an endomorphism of V. When is a called *semisimple*? When is a called *nilpotent*? Explain why 0 is the only eigenvalue of a nilpotent endomorphism a of V and use the Cayley–Hamilton theorem to show that $a^n = 0$ where $n = \dim V$.

[5 marks]

(ii) Let $x \in gl(V)$. Define a *Jordan decomposition* $x = x_s + x_n$ of x and explain why such a decomposition is unique. When is a Lie subalgebra L of $gl(V)$ called *separating*?

[6 marks]

(iii) When is V called an L-*module*? When is an L-module called *completely reducible*? State *Weyl's theorem* on finite dimensional representations of semisimple Lie algebras.

[5 marks]

(iv) Let A be a finite dimensional algebra over $\Bbbk$ (not necessarily associative or Lie). When is an endomorphism D of A called a *derivation* of A?

[1 marks]

(v) Suppose $\mathrm{char}(\Bbbk) = 3$ and let $A = \Bbbk[t]/(t^3)$, a truncated polynomial ring in one variable over $\Bbbk$. Then A is a commutative $\Bbbk$-algebra with basis $\{1, x, x^2\}$ where x is the coset of t in A. Find a basis of the Lie algebra $\mathrm{Der}(A)$ of all derivations of A and show that $\mathrm{Der}(A) \cong sl(2, \Bbbk)$ as Lie algebras. Hint: use the fact that $f(x)\frac{d}{dx} \in \mathrm{Der}(A)$ for all $f(x) \in A$.

[6 marks]

(vi) Suppose that $\mathrm{char}(\Bbbk) \neq 2$. Prove that the Lie algebra $sl(2, \Bbbk)$ is simple.

[4 marks]

END OF EXAM PAPER

UNIVERSITY OF MANCHESTER

GREEN'S FUNCTIONS, INTEGRAL EQUATIONS AND THE CALCULUS OF VARIATIONS

6 June 2016

14:00-16:00

Answer ALL six questions (100 marks in total)

Electronic calculators may be used, provided that they cannot store text.

Given equations:

- For functions f and g defined in either two or three dimensions, and a domain D with smooth boundary in either two or three dimensions

$$\int_D \left(f(\mathbf{x})\nabla^2 g(\mathbf{x}) - g(\mathbf{x})\nabla^2 f(\mathbf{x})\right)\, d\mathbf{x} = \int_{\partial D} \left(f(\mathbf{x})\nabla g(\mathbf{x}) - g(\mathbf{x})\nabla f(\mathbf{x})\right)\cdot \mathbf{n}(\mathbf{x})\, ds \tag{1}$$

  where the boundary ∂D is oriented by the outward pointing unit normal $\mathbf{n}(\mathbf{x})$ at each point $\mathbf{x}$ in ∂D.

1. You do not need to provide any proofs on the various parts of this problem. Just give the answers.

(a) What is the general form of a formally self-adjoint linear second order differential operator with real coefficients in one dimension?

[2 marks]

(b) Suppose that

$$\begin{cases} \mathcal{L}u(x) = f(x), \quad x \in (a, b) \\ \mathcal{B} \end{cases}$$

is a boundary value problem where $\mathcal{L}$ is a second order linear differential operator and $\mathcal{B}$ is a set of boundary conditions, and the adjoint problem is

$$\begin{cases} \mathcal{L}^* v(x) = f(x), \quad x \in (a, b) \\ \mathcal{B}^*. \end{cases}$$

What does the Fredholm alternative say about these two problems?

[6 marks]

(c) Define the Heaviside function $H(x)$, and give the derivative $H'(x)$.

[2 marks]

(d) Give the free-space Green's function for the Laplacian in three dimensions.

[2 marks]

(e) Give the general forms of Fredholm integral equations of the first and second kinds.

[4 marks]

2. Consider the following boundary value problem:

$$\begin{cases} u''(x) - 2u'(x) + 2u(x) = f(x), \quad x \in (0,1) \\ u(0) = 0, \quad u(\pi/2) = 0. \end{cases}$$

Find the Green's function for this problem, or show that no Green's function exists.

[10 marks]

3. This question concerns a general Sturm-Liouville boundary value problem:

$$\text{(BVP)} \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right) + q(x)u(x) = f(x), & x \in (a,b) \\ \alpha_1 u(a) + \alpha_2 u'(a) = 0, & \beta_1 u(b) + \beta_2 u'(b) = 0. \end{cases}$$

(a) Write down an associated Sturm-Liouville eigenvalue problem.

[2 marks]

(b) What conditions are required for the problem from part (a) to be a *regular* Sturm-Liouville eigenvalue problem? For the rest of this question, assume that this is a regular Sturm-Liouville eigenvalue problem.

[4 marks]

(c) What condition on the eigenvalues of the problem in part (a) is equivalent to the existence of a Green's function for (BVP)? You do not need to prove the condition is correct, just state it.

[1 marks]

(d) If the condition from part (c) is satisfied, derive a formula for the Green's function in terms of the eigenvalues and eigenfunctions.

[8 marks]

4. This question concerns the Helmholtz equation modelling time harmonic waves on a string:

$$u''(x) + k^2 u(x) = f(x).$$

We take the time dependent waves to be given by $U(x,t) = e^{-i\omega t}u(x)$, and the physical motion, i.e. the displacement of the string from a stretched equilibrium position, is the real part of this.

(a) Suppose that we have a finite string of length L whose ends are held fixed. What boundary conditions should be added to model this situation?

[2 marks]

(b) For the boundary value problem you found in part (a), determine what condition is required on k and L for there to be a unique solution u for any continuous function f.

[5 marks]

(c) If the condition from part (b) is not satisfied, what condition on f will guarantee there is a solution to the boundary value problem from part (a)?

[5 marks]

(d) Give a physical interpretation for the condition considered in part (b). How is it possible for there to be no solution to the boundary value problem from part (a) given that it is supposed to be modelling the motion of a string?

[3 marks]

5. This question concerns the integral equation

$$u(x) = \lambda \int_0^{\sqrt{\pi/2}} x \sin(y^2) u(y) \, \mathrm{d}y + f(x). \tag{2}$$

(a) For what values of $\lambda \in \mathbb{C}$ is there a unique solution $u(x)$ to (2) for any continuous function f?

[3 marks]

(b) For the values of λ found in part (a), find a formula for the solution $u(x)$ of (2).

[7 marks]

(c) When λ is not one of the values found in part (a) find a condition on f so that there is a solution for (2). Also write a general solution for (2) in this case.

[4 marks]

(d) Determine the first three terms in the Neumann series solution for (2). Simplify as much as possible.

[5 marks]

6. The *free-space Green's function* for the Laplacian operator $\mathcal{L} = \nabla^2$ in two dimensions is

$$G_{2\infty}(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \ln\left(|\mathbf{x} - \mathbf{x}_0|\right) = \frac{1}{2\pi} \ln\left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right).$$

(a) Show that $\nabla^2_{\mathbf{x}} G_{2\infty}(\mathbf{x}, \mathbf{x}_0) = 0$ when $\mathbf{x} \neq \mathbf{x}_0$.

[6 marks]

(b) Use the method of images to find a Green's function $G(\mathbf{x}, \mathbf{x}_0)$ for the two dimensional Laplacian $\mathcal{L} = \nabla^2$ on the domain $D = \{(x, y) \in \mathbb{R}^2 \ : \ y \geq 0\}$ such that $G(\mathbf{x}, \mathbf{x_0}) = 0$ for $\mathbf{x}_0$ on the boundary of this domain.

[6 marks]

(c) Ignoring the contribution from the integral at infinity, use the answer to part (b) to show that a solution of the boundary value problem

$$\begin{cases} \nabla^2 u(x, y) = 0, & y > 0, \\ u(x, 0) = h(x), & x \in \mathbb{R} \end{cases} \tag{3}$$

is given by

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} h(s) \frac{y}{y^2 + (x - s)^2} \, ds. \tag{4}$$

[7 marks]

(d) Use equation (4) to show that when $h(x) = xH(x-1) - xH(x-2)$, where $H(x)$ is the Heaviside function, a solution of (3) is

$$u(x, y) = \frac{y}{2\pi} \ln\left|\frac{y^2 + (2 - x)^2}{y^2 + (1 - x)^2}\right| + \frac{x}{\pi}\left(\text{atan}\left(\frac{2 - x}{y}\right) - \text{atan}\left(\frac{1 - x}{y}\right)\right)$$

[6 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

WAVE MOTION

26 May 2016

14:00 – 16:00

Answer **ALL** six questions.

Electronic calculators are permitted, provided they cannot store text.

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1. You are given that a shallow layer of fluid (of mean depth h) is governed by the equations

$$\begin{aligned} \frac{\partial u}{\partial t} - \Omega v &= -g\frac{\partial \eta}{\partial x}, \\ \frac{\partial v}{\partial t} + \Omega u &= -g\frac{\partial \eta}{\partial y}, \\ \frac{\partial \eta}{\partial t} &= -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \end{aligned}$$

where (u, v) are velocity components in the directions of increasing (x, y) respectively, η is a free surface deflection, t is time and h, g, Ω are positive constants.

(i) By considering solutions that are independent of the coordinate y, show that these equations can be reduced to a single equation for u.

(ii) Hence obtain the dispersion relation for harmonic travelling wave solutions in the usual form,

$$u(x, t) = A\exp[i(Kx - \omega t)]\,.$$

(iii) Find the phase speed and group velocity for these waves. In a sentence, explain the significance of the limit

$$\frac{\Omega}{K\sqrt{gh}} \ll 1\,,$$

to the evolution of these waves.

[10 marks]

2. A plane wave signal, f, takes the form:

$$f(x, t) = A\exp\Big\{i\Big(Kx - \omega(K)t\Big)\Big\}\,,$$

in the usual notation.

Consider two such plane waves of equal amplitude A, but with slightly different wavenumbers $K = k + \Delta$ and $K = k - \Delta$, where Δ is small.

(i) Show that the sum of these two waves can be viewed as a slow amplitude modulation of a plane wave of wavenumber k. Define the 'envelope' and 'carrier' waves in the expression that results from this sum.

(ii) Give the speed of propagation of the envelope of this amplitude modulated wave.

(iii) Sketch the (real part of the) signal at a fixed instant in time, indicating the direction of propagation for $\omega > 0$.

[10 marks]

3. A standing wave exists on an inviscid, incompressible fluid of infinite depth. The wave (of amplitude A) has a velocity potential

$$\phi(x, z, t) = \frac{gA}{\omega} e^{-Kz} \cos(Kx) \cos(\omega t),$$

where $K = \omega^2/g$, with g being the local acceleration due to gravity. The Cartesian coordinates (x, z) have z measured downwards from the mean location of the free surface.

(i) In the motion resulting from the potential ϕ above, a particle whose initial position is (x_0, z_0) moves to $(x(t), z(t))$. Write down the equations of motion for this particle.

(ii) For small amplitude waves, find the leading-order approximations for $x(t)$ and $z(t)$. Sketch the shape of the particle paths.

[10 marks]

4. Consider an impermeable sphere centred at the origin, surrounded by an inviscid, compressible fluid of sound speed c. The radius $a(t)$ of the sphere changes, and this generates sound waves in the fluid; these waves are to be described in terms of a potential $\phi(r, t)$, where r is the distance from the origin.

(i) State the boundary conditions to be satisfied by $\phi(r, t)$.

(ii) Show that the wave equation (given below) can be satisfied by a velocity potential of

$$\phi(r, t) = -\frac{1}{4\pi r} m(t - r/c)\,,$$

where m is a (twice differentiable) function.

(iii) For small deflections of the sphere radius about a mean value a_0, show that

$$m(t) + \frac{a_0}{c} m'(t) = V'(t + a_0/c)\,,$$

where $V(t)$ is the instantaneous volume of the sphere.

You may use: The wave equation for a spherically symmetric velocity potential is

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) .$$

[10 marks]

5. Consider a wave motion at the interface of two inviscid, incompressible fluids of density ρ_1 (in $\eta \leq z < \infty$) and ρ_2 (in $-\infty < z < \eta$) with $\rho_2 < \rho_1$. Here $z = \eta(x,t)$ defines an interface between the fluids and local gravitational acceleration is acting in the direction of increasing z.

(i) Let ϕ_1 and ϕ_2 be the velocity potentials in the fluids of density ρ_1 and ρ_2 respectively. Assume solutions of the form

$$\phi_1 = \Re\{X(x)Z_1(z)\exp(i\omega t)\}\,,$$

$$\phi_2 = \Re\{X(x)Z_2(z)\exp(i\omega t)\}\,,$$

and use the method of separation of variables to determine $X(x)$, $Z_1(z)$ and $Z_2(z)$. You may assume that the wave motion decays away from the interface.

(ii) The linearised conditions to be satisfied at $z = 0$ are

$$\frac{\partial \phi_i}{\partial t} + \frac{p}{\rho_i} - g\eta = 0\,,$$

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi_i}{\partial z} = 0\,,$$

with $i = 1, 2$, where p is the pressure at the interface and g the local gravitational acceleration.

Hence obtain the dispersion relation for this wave system. Simplify the dispersion relation in the limit of $\rho_2/\rho_1 \ll 1$.

[20 marks]

6. Consider the propagation of small amplitude acoustic waves in an inviscid compressible fluid of sound speed c.

(i) State the two-dimensional wave equation and verify that the velocity potential

$$\phi_i(x, y, t) = A \exp\{im(x \cos\alpha - y \sin\alpha - ct)\},$$

is a solution, in a Cartesian coordinate system (x, y) where m is a wavenumber and α a constant.

(ii) For a flat, rigid, impermeable, boundary along $y = 0$, state the boundary condition to be satisfied. Show that this boundary condition and the wave equation can be satisfied by a velocity potential in the form

$$\phi_{flat}(x, y, t) = \phi_i(x, y, t) + \phi_r(x, y, t),$$

where ϕ_r is to be given as part of the answer.

(iii) Consider a small perturbation to the rigid boundary, such that its location is now given by $y = \epsilon f(x)$, where $|\epsilon| \ll 1$ and $f \to 0$ as $x \to \pm\infty$.

By seeking a solution in the form

$$\phi(x, y, t) = \phi_{flat}(x, y, t) + \epsilon\hat{\phi}(x, y)e^{-imct},$$

obtain the linearised condition for $\hat{\phi}$ applied at the boundary. (You may assume without proof that the linearised condition applies at $y = 0$.)

(iv) Using the definition of the Fourier transform (defined below), give the corresponding transformed version of this boundary condition for $\hat{\phi}$.

You may use: The Fourier Transform of a function $g(x)$ is defined by

$$\mathcal{F}(g(x)) = \int_{-\infty}^{\infty} g(x)e^{ikx}\,\mathrm{d}x.$$

[20 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL BIOLOGY

27 May 2016

9:45–11:45

Answer **ALL THREE** questions in Section A (30 marks in total).
Answer **TWO** of the THREE questions in Section B (50 marks in total).
If all THREE questions from Section B are attempted,
credit will be given for the TWO best answers.

University approved calculators may be used

SECTION A

Answer **ALL** 3 questions

A1. The following differential equation models a fish population with self-limiting growth that is subject to harvesting:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - hN$$

Here the parameters r, K and h are all positive real numbers.

(a) Explain briefly what each of the parameters mean and what their units are.

(b) What range of values for the parameter h give rise to a sustainable fishery? That is, for which values of h is there a non-zero steady-state population?

(c) Take K and r to be fixed and sketch a graph of the steady-state yield of the fishery as a function of h. What value of h gives rise to maximal steady-state yield?

[10 marks]

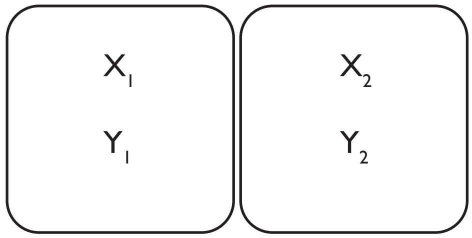
A2. Explain what is meant by the following terms:

- a *motif* in a genetic regulatory network
- an *incoherent 3-node feed-forward loop.* Give two examples of such loops.

In a randomly-assembled regulatory network with $N = 6$ genes and $E = 12$ regulatory interactions, what is the probability of finding exactly 3 autoregulatory loops?

[10 marks]

A3. The first example in Turing's famous paper on morphogenesis involves two adjacent cells, each containing two morphogens, as illustrated below:



The concentrations of these morphogens obey the coupled ODEs

$$\begin{aligned} \frac{dX_1}{dt} &= 5X_1 - 6Y_1 + 1 + \gamma_1(X_2 - X_1) \\ \frac{dY_1}{dt} &= 6X_1 - 7Y_1 + 1 + \gamma_2(Y_2 - Y_1) \\ \frac{dX_2}{dt} &= 5X_2 - 6Y_2 + 1 + \gamma_1(X_1 - X_2) \\ \frac{dY_2}{dt} &= 6X_2 - 7Y_2 + 1 + \gamma_2(Y_1 - Y_2). \end{aligned} \tag{A3.1}$$

(a) Show that the system (A3.1) has an equilibrium in which

$$X_1 = X_2 = X_\star \qquad \text{and} \qquad Y_1 = Y_2 = Y_\star \tag{A3.2}$$

and that $X_\star$ and $Y_\star$ do not depend on γ_1 and γ_2.

(b) Consider the case where no diffusion occurs—that is, when $\gamma_1 = \gamma_2 = 0$. Analyse the linearization of (A3.1) near the equilibrium (A3.2) and show that this equilibrium is stable.

(c) The example Turing discussed has $\gamma_1 = 0.5$ and $\gamma_2 = 4.5$. In this case the linearization of (A3.1) near the equilibrium (A3.2) has characteristic polynomial

$$p(\lambda) = \lambda^4 + 14\lambda^3 - 3\lambda^2 - 44\lambda - 28.$$

Demonstrate that this polynomial has a positive root and use this observation to determine the stability of the equilibrium.

[10 marks]

SECTION B

Answer **2** of the 3 questions

B4. Consider the following system of chemical reactions, which is a variant of the Michaelis-Menten system in which product formation is reversible:

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} E + P. \tag{B4.1}$$

Here the k_j are rate constants and E, S, C and P represent, respectively, molecules of enzyme, substrate, complex and product.

(a) Write down a stoichiometric matrix N for the system of reactions (B4.3) and explain how you constructed it.

(b) By performing Gaussian elimination on N, or by any other means, find two conserved quantities for the system (B4.3).

(c) Find the rank r of the stoichiometric matrix N and decompose N as a product $N = LN_R$ where N_R, the reduced stoichiometry matrix, has full rank and L is of the form

$$L = \begin{bmatrix} I_r \\ L_0 \end{bmatrix},$$

where I_r is the $r \times r$ identity matrix. Your answer should include expressions for L, L_0 and N_R.

(d) Using the Law of Mass Action, write down a system of ODEs that describes the temporal evolution of the concentrations of the chemical species in (B4.3). Use the symbols symbols e, s, c and p for the concentrations of, respectively, enzyme, substrate, complex and product.

(e) Show that the system of ODEs from part (d) has an equilibrium $(e^\star, s^\star, c^\star, p^\star)$ in which

$$\frac{p^\star}{s^\star} = \frac{k_1 k_2}{k_{-1} k_{-2}}.$$

[25 marks]

B5. Consider the following system of differential equations, which have been proposed as a model of competition between Neanderthals N and Early Modern Man E.

$$\begin{aligned}
\frac{dN}{dt} &= N\left(a - c(N+E) - b\right) \\
\frac{dE}{dt} &= E\left(a - c(N+E) - \theta b\right)
\end{aligned} \qquad \text{(B5.1)}$$

Here a, b and c are strictly positive constants with $a > b$, while $0 < \theta \leq 1$ measures a difference in mortality between the two kinds of hominids.

(a) Explain briefly the nature of the interaction and describe the behaviour of the population of each group when left to develop in isolation.

(b) Consider the population of an isolated group of Neanderthals (that is, a place where $E(t) = 0$). Show that $N(t)$ obeys a differential equation of the form

$$\frac{dN}{dt} = r_N N \left(1 - \frac{N}{K_N}\right)$$

and obtain expressions for r_N and K_N in terms of the parameters a, b, c and θ of (B5.1).

(c) Show that a suitable change of variables reduces the system above to the form

$$\frac{du}{d\tau} = u\left(1 - u - \rho v\right) \qquad \text{and} \qquad \frac{dv}{d\tau} = \rho v \left(1 - v - \frac{u}{\rho}\right) \qquad \text{(B5.2)}$$

with $\rho \geq 1$. Your answer should include expressions for τ, u, v and the new parameter ρ in terms of t, N, E and the parameters of (B5.1).

(d) Assume that $\rho > 1$ and sketch the null clines of (B5.2), then find all the equilibria and determine their stabilities.

(e) Find conditions on the parameters of (B5.1) that would permit Early Modern Man and Neanderthals to coexist indefinitely.

[25 marks]

B6. Consider the following differential delay equation for a self-regulating genetic system whose protein product, x is a translational repressor that suppresses its own production.

$$\frac{dx(t)}{dt} = \frac{\beta K^\nu}{K^\nu + [x(t-T)]^\nu} - \alpha x(t) \tag{B6.1}$$

Here the delay T and the parameters α, β and K are positive real numbers while ν is a positive integer.

(a) Show that a suitable choice of variables allows one to reduce the system (B6.1) to the following dimensionless form:

$$\frac{du(\tau)}{d\tau} = \frac{1}{1 + [u(\tau - \hat{T})]^\nu} - \lambda u(\tau). \tag{B6.2}$$

Your answer should include formulae for u, τ, $\hat{T}$ and λ in terms of x, t, T and the parameters of the original problem.

(b) Show that if $\lambda < 1/2$ the system has a constant solution $u(\tau) = u_\star$ with $u_\star > 1$.

(c) Write (B6.2) in the form

$$\frac{du(\tau)}{d\tau} = f(u(\tau - \hat{T}), u(\tau))$$

and then, by linearizing the function f, show that small perturbations of the form $u(\tau) = u_\star + \eta(\tau)$ evolve according to the linearized dynamics

$$\frac{d\eta(\tau)}{d\tau} = -\nu\lambda^2 u_\star^{\nu+1} \eta(\tau - \hat{T}) - \lambda\eta(\tau) \tag{B6.3}$$

(d) Consider perturbations of the form $\eta(t) = \eta_0 e^{(\mu + i\omega)t}$ evolving according to (B6.3) and prove that if $\nu = 1$ all such perturbations die away exponentially.

[25 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

SYMMETRY IN NATURE

20 May 2016

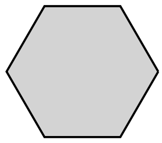
14:00 – 16:00

Answer **ALL FOUR** questions

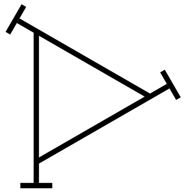
Electronic calculators may be used, provided that they cannot store text,
transmit or receive information, or display graphics

1.

(a) Name the symmetry groups of the following two planar shapes, and state in each case the order of the group and whether the group is Abelian.



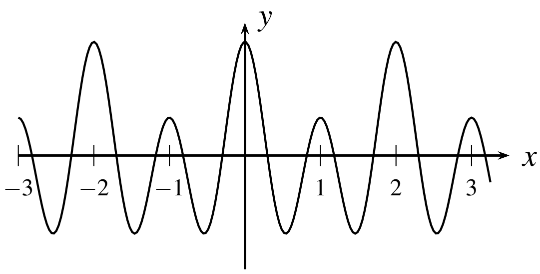
(i)



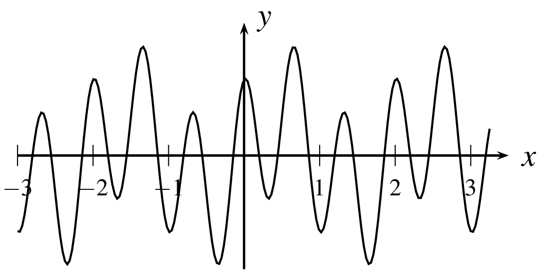
(ii)

[6 marks]

(b) Consider the two functions f and g of one variable whose graphs are shown below.



$y = f(x)$



$y = g(x)$

Describe the full symmetry group of each of the functions f and g, including possible transformations that reverse the sign of the function. (It is sufficient to give generators of the groups, making clear which if any reverse the sign of the function.)

[8 marks]

2.

(a) Suppose a finite group G acts on a set X.

(i) Define the orbit and stabilizer of a point $x \in X$. Show that the stabilizer of a point is always a subgroup of G.

(ii) Let $h \in G$ and $y = h \cdot x$. Show that the subset

$$G_{x,y} = \{g \in G \mid g \cdot x = y\}$$

is a left coset of G_x.

(iii) Deduce that there is a bijection between the orbit $G \cdot x$ and the set G/G_x of left cosets of G_x.

[10 marks]

(b) Let D_3 be the standard dihedral subgroup of $\mathsf{O}(2)$ of order 6, generated by r_0 and $R_{2\pi/3}$. Let $f(x,y) = (x^2 - y^2,\ -2xy)$. Show f is a D_3-equivariant map of the plane to itself. [Hint: You may like to use complex numbers.]

[6 marks]

3.

(a) Define the action on $\mathbb{R}^2$ of an element $(\mathbf{v},A) \in \mathsf{E}(2)$, where $\mathbf{v} \in \mathbb{R}^2$ and $A \in \mathsf{O}(2)$, and deduce that multiplication in the group is given by

$$(\mathbf{v},A)(\mathbf{u},B) = (\mathbf{v}+A\mathbf{u},AB).$$

[4 marks]

(b) Consider the following lattice in the plane:

$$L = \left\{(x,y) \in \mathbb{R}^2 \mid x \in \mathbb{Z},\ y \in 2\mathbb{Z},\ x+\tfrac{1}{2}y \in 2\mathbb{Z}\right\}.$$

(i) Show that the two vectors $\mathbf{u}_1 = (2,0)$ and $\mathbf{u}_2 = (0,4)$ both belong to L, and sketch a diagram showing all the points of L that lie in the rectangle $0 \le x \le 6$ and $0 \le y \le 8$. Deduce that $L \neq \mathbb{Z}\{\mathbf{u}_1,\mathbf{u}_2\}$.

(ii) Find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $L = \mathbb{Z}\{\mathbf{a},\mathbf{b}\}$, proving carefully that this is the case.

(iii) Show that the point group of the lattice L has order 4 by finding appropriate elements of its symmetry group $\mathscr{W}_L < \mathsf{E}(2)$, expressed in the form $(\mathbf{v},A)$, and state why it is Abelian.

[16 marks]

4. Consider a system of 3 coupled cells governed by the system of differential equations

$$\left\{\begin{array}{rcl} \dot{x} &=& x-y^2 \\ \dot{y} &=& y-z^2 \\ \dot{z} &=& z-x^2. \end{array}\right. \qquad (*)$$

(a) Show that the system $(*)$ has $\mathbb{Z}_3$ symmetry, and draw a cell diagram illustrating the system. Explain briefly why the system does not have S_3 symmetry. [6 marks]

(b) Deduce that the diagonal, where $x=y=z$, is invariant under the evolution of the system, stating carefully any results used. [4 marks]

(c) Find all the equilibrium points that lie on the diagonal. [4 marks]

(d) Find the unique solution to this system with initial value $(x,y,z)=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$. [6 marks]

(e) Suppose $\gamma(t)$ is a periodic solution with fundamental period T of a differential equation with symmetry group G. Define the symmetry group of the periodic solution. If the system $(*)$ has any periodic orbits with non-trivial symmetry, show that the only possible symmetry types are $\mathbb{Z}_3$ and $\widetilde{\mathbb{Z}}_3$, making explicit what each means for the orbit. [10 marks]

END OF EXAMINATION PAPER

Two Hours

THE UNIVERSITY OF MANCHESTER

NUMERICAL ANALYSIS 2

7th June 2016

14:00–16:00

Answer **ALL** five questions in **SECTION A** (40 marks in total)
and
Answer **TWO** of the three questions in **SECTION B** (20 marks each)
If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** five questions

A1.

(a) Let $r_{km}(x) = p_{km}(x)/q_{km}(x)$ be a rational function. State all conditions that must be satisified for r_{km} to be a $[k/m]$ Padé approximant of f on $[a, b]$ (where f is sufficiently differentiable). Give **one reason** why Padé approximants may be more suitable for approximation than single polynomials.

[4 marks]

(b) Consider

$$r_n(x) = \cfrac{c_1 x}{1 + \cfrac{c_2 x}{1 + \cfrac{c_3 x}{1 + \cdots + \cfrac{c_{n-2} x}{1 + \cfrac{c_{n-1} x}{1 + c_n x}}}}}$$

with coefficients

$$c_1 = 1, \qquad c_{2j} = \frac{j}{2(2j-1)}, \quad c_{2j+1} = \frac{j}{2(2j+1)}, \qquad j = 1, 2, \ldots$$

Verify that $r_2(x)$ is the $[1/1]$ Padé approximant of $f(x) = \log(1 + x)$. In addition, explain briefly why it is beneficial to consider continued fraction representations of Padé approximants.

[8 marks]

A2. Derive the two-point Gauss–Hermite quadrature rule

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx \approx w_1 f(x_1) + w_2 f(x_2)$$

using the method of undetermined coefficients. That is, determine the choice of w_1, w_2 and x_1, x_2 to maximize the degree of precision. You may use the fact that $\int_{-\infty}^{\infty} \exp(-x^2)\, dx = \sqrt{\pi}$ without proof.

HINT: You may find it useful to define a quadratic polynomial $\Phi(x) = x^2 + ax + b$ whose roots coincide with x_1 and x_2 and find a and b.

[9 marks]

A3. Use the composite Trapezium rule $T(h)$ with $h = \pi/3$ to estimate

$$\int_0^{\pi} \sin^2(x)\, dx.$$

Compare your answer with the exact value of the integral. What do you observe and why?

HINT: You might find it useful to consider the Euler–Maclaurin summation formula.

[8 marks]

A4. Consider the initial value problem (IVP)

$$y'(x) = f(x, y), \qquad 0 \le x \le 1, \quad y(0) = y_0,$$

where $f(x, y)$ is continuous for $x \in [0, 1]$ and for all $y \in \mathbb{R}$.

Prove that if $f(x, y)$ has a continuous derivative with respect to y that is bounded for $x \in [0, 1]$ and for all $y \in \mathbb{R}$, then it satisfies a Lipschitz condition. State the Lipschitz constant in this case.

[5 marks]

A5. The backward Euler method for the initial value problem $y'(x) = f(x, y(x))$, $y(x_0) = y_0$ is written

$$y_{n+1} = y_n + h f_{n+1},$$

where h is the chosen stepsize, $f_{n+1} = f(x_{n+1}, y_{n+1})$ and y_n is the approximation to $y(x_n)$ at the grid point x_n for $n = 0, 1, \ldots$.

(a) By analysing the local truncation error $\tau(h)$, determine the order of the method.

(b) Write down the standard Euler method. What is the order of this method?

(c) State an important advantage that the Backward Euler method has over the standard Euler method.

[6 marks]

SECTION B

Answer **TWO** of the three questions

B6. Let $f \in C[a,b]$ and let $\{\phi_i(x)\}_{i=0}^n$ be a given set of linearly independent functions.

(a) Let $w(x)$ be a "weight function" on $[a,b]$. Define the norm $\|\cdot\|_{2,w}$, making sure to state any conditions that $w(x)$ should satisfy for a well-defined norm. In addition, explain what is meant by the best weighted L_2 approximation g to f from a set of functions G.

[4 marks]

(b) Suppose we approximate f by a function of the form

$$g(x) = \sum_{i=0}^{n} c_i \phi_i(x), \qquad (1)$$

with coefficients $c_i \in \mathbb{R}$. Prove that there is a unique choice of coefficients $c_0, \dots, c_n$ such that g is a best weighted L_2 approximation to f on $[a,b]$.

HINT: Show that the desired vector of coefficients $\underline{c} = [c_0, \dots, c_n]^T$ satisfies a linear system of the form $A\underline{c} = \underline{f}$ and show that the matrix A is non-singular.

[12 marks]

(c) Suppose we want to construct a best weighted L_2 approximation to f, with respect to the weight function $w(x) = (1 - x^2)^{-1/2}$. State a good choice for the functions ϕ_i in part (b) and justify your answer. Write down the corresponding choice of coefficients c_i in (1) that yields a best weighted L_2 approximation.

[4 marks]

B7. Consider the integral

$$I(f) = \int_a^b w(x) f(x)\, dx,$$

where $w(x) \geq 0$ is a given "weight function" on $[a, b]$.

(a) Explain the strategy used to find the weights $\{w_i\}_{i=1}^n$ for the n-point Gauss rule

$$G_n(f) := \sum_{i=1}^n w_i f(x_i),$$

and show that w_i can be expressed as an integral involving a Lagrange interpolating polynomial $l_i(x)$. In addition, explain how the points x_i are chosen and state the degree of precision of the rule.

[5 marks]

(b) Give one advantage and one disadvantage of the n-point Newton–Cotes rule when compared to the n-point Gauss quadrature rule for computing $I(f)$.

[2 marks]

(c) Starting from $\phi_0(x) = 1$, $\phi_1(x) = x - \alpha_0$, and using the three-term recurrence,

$$\phi_{i+1}(x) = (x - \alpha_i)\phi_i(x) - \beta_i \phi_{i-1}(x), \qquad i \geq 1,$$

where α_i and β_i are coefficients that you should fully define, explain how we can construct a set of polynomials $\{\phi_i(x)\}$ which are useful in the construction of Gauss rules.

[5 marks]

(d) Using part (c), construct an appropriate cubic polynomial ϕ_3 on $[-1, 1]$ and derive the nodes x_1, x_2 and x_3 needed to construct the 3-point Gauss–Legendre rule for

$$G_3(x) := \sum_{i=1}^3 w_i f(x_i) \approx \int_{-1}^1 f(x)\, dx.$$

[8 marks]

B8. An ℓ-step linear multistep method for the initial value problem $y'(x) = f(x, y(x))$, $x \geq x_0$, $y(x_0) = y_0$ has the form

$$y_{n+1} = \sum_{i=1}^{\ell} a_i y_{n+1-i} + h \sum_{i=0}^{\ell} b_i f_{n+1-i},$$

where a_i, b_i are given constants, h is the stepsize, and $f_n = f(x_n, y_n)$. You may assume, without proof, that the one-step Adams–Bashforth method coincides with Euler's method.

(a) By replacing the integrand in

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))\, dx,$$

with a suitable interpolating polynomial $p_1(x)$ of degree one, derive the one-step Adams–Moulton method

$$y_{n+1} = y_n + \frac{h}{2}\left(f_n + f_{n+1}\right).$$

[8 marks]

(b) Define the PE(CE)^m method using the one-step Adams–Bashforth method as the predictor and the one-step Adams–Moulton method as the corrector.

[5 marks]

(c) Show that the method in part (b) obtained with $m = 1$ can be interpreted as a two stage Runge–Kutta method.

[3 marks]

(d) By examining the test problem $y'(x) = \lambda y(x)$ where $\lambda \in \mathbb{C}$, derive a quadratic polynomial $p(\lambda h)$ such that the condition $|p(\lambda h)| < 1$ provides the region of absolute stability for the method in part (b) obtained with $m = 1$. If λ is real and negative, what choice of h guarantees a stable solution?

[4 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

MARKOV PROCESSES

27 May 2016

14:00-16:00

Answer 3 of the 4 questions. If more than 3 questions are attempted, credit will be given for the best 3 answers. Each question is worth 20 marks.

Electronic calculators may be used, provided that they cannot store text.

Answer 3 questions

1.
a)

(i) Define the terms "recurrent state" and "transient state" in the context of discrete time Markov chains.

[2 marks]

(ii) Define the stationary distribution of a discrete time Markov chain.

[2 marks]

(iii) Give an example of a Markov chain which has more than one stationary distribution.

[2 marks]

(iv) Prove that if a Markov chain has more than one stationary distribution, it has an infinite number of stationary distributions.

[2 marks]

b) Consider the discrete time Markov chain $\{X_n : n \geq 0\}$ with state space {1,2,3,4,5} and transition probability matrix

$$\begin{bmatrix} 0.2 & 0 & 0 & 0.8 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0.3 & 0.2 & 0.5 & 0 & 0 \\ 0.3 & 0 & 0 & 0.7 & 0 \\ 0 & 0.4 & 0 & 0 & 0.6 \end{bmatrix}$$

(i) Prove that state 3 is transient.

[2 marks]

(ii) Determine $\lim_{n\to\infty} p_{11}(n)$.

[3 marks]

(iii) For $n \geq 1$ let $f_{ij}(n) = P(X_n = j,\, X_k \neq j \quad for \quad 1 \leq k < n \mid X_0 = i)$. Calculate $f_{31}(n)$. Deduce that $P(X_n = 1 \text{for some } n \geq 1 \mid X_0 = 3) = \frac{3}{5}$.

[3 marks]

(iv) Determine $\lim_{n\to\infty} p_{31}(n)$.

[2 marks]

(v) Calculate the expected time of first visit to state 2 given $X_0 = 2$.

[2 marks]

2. a) Let $\{X_n, n \geq 0\}$ be a Markov chain, with state space the set of integers and transition probabilities given by

$$p_{i,i+1} = p, \quad p_{i,i-1} = 1 - p = q \quad \textit{where } 0 < p < 1.$$

(i) Show that, for n odd,

$$P(X_n = 0 \mid X_0 = 0) = 0$$

and that, for n even,

$$P(X_n = 0 \mid X_0 = 0) = \binom{n}{\frac{n}{2}} (pq)^{\frac{n}{2}}.$$

[5 marks]

(ii) Show that the Markov chain is transient when $p \neq q$ and null recurrent when $p = q$

[You may assume the asymptotic result $n! \sim n^n e^{-n}(2\pi n)^{1/2}$ as $n \longrightarrow \infty$; and use without proof that the sequence $p_{0,0}(n) = \binom{n}{\frac{n}{2}} (pq)^{\frac{n}{2}}$ for n even and $p_{0,0}(n) = 0$ for n odd has generating function $P_0(s) = (1 - 4pqs^2)^{\frac{-1}{2}}$.]

[7 marks]

b) $\{Y_n\}$ is a Markov chain whose state space is the set of non-negative integers. The transition probabilities are

$$\begin{aligned} p_{0,0} &= 0, \quad p_{0,1} = 1 \\ p_{i,i+1} &= \frac{1}{i+1} \quad i \geq 1 \\ p_{i,0} &= 1 - \frac{1}{i+1} \quad i \geq 1. \end{aligned}$$

(i) Show that zero is a positive recurrent state.

[4 marks]

(ii) Derive the stationary distribution of $\{Y_n\}$.

[4 marks]

3. a) Consider the continuous time Markov chain $\{X(t), t \geq 0\}$ with state space $\{0, 1\}$, where the holding times for states 0 and 1 are independent exponential random variables with parameters λ and μ respectively. Upon leaving one state, the process enters the other.

(i) Write down the system of forward equations $P'(t) = P(t)Q$ for this Markov chain.

[3 marks]

(ii) With the initial condition $p_{00}(0) = 1$, use this system to show that, for $t \geq 0$,

$$p'_{00}(t) = -(\lambda + \mu)p_{00}(t) + \mu$$

and solve this differential equation to obtain

$$p_{00}(t) = \frac{1}{\lambda + \mu}\left\{\mu + \lambda \exp(-[\lambda + \mu]t)\right\}.$$

[6 marks]

(iii) Determine the stationary distribution of the Markov chain.

[3 marks]

b) A city guide, stationed at a kiosk, can be hired to give city tours. If the guide is free, an interested tourist negotiates with her for an exponential, parameter μ_1 distributed time. The tourist then takes a tour with probability $1 - \alpha$, or decides against it with probability α. The duration of a tour is exponentially distributed with parameter μ_2. Tourists arrive at the guide's kiosk according to a Poisson process with rate λ and request service only if the guide is present, but unoccupied in negotiation; otherwise they leave. All the negotiation, tour and inter-arrival times are independent.

By defining a 3 state continuous time Markov Chain, find the proportion of time the guide spends in negotiation.

[8 marks]

4. a) Consider the following $M/M/s$ queueing model. Customers arrive at the queue in a Poisson process with rate λ. There are s servers. Service times are independent, independent of the arrival process and exponentially distributed with parameter μ. Arriving customers finding no server free, turn away.

Stating carefully any results to which you appeal, calculate the stationary distribution of the process, stating for which values of λ and μ it exists.You may quote without proof the expression for the stationary distribution of a birth-death process.

[8 marks]

b) Assuming that $s = 4$ and $\lambda = \mu = 1$ in the queueing model described in part (a), determine for what proportion of time three or more servers are occupied.

[4 marks]

(c) Consider the queueing model described in part (a), but modified so that there are unlimited servers available.

(i) Determine the stationary distribution and hence the mean number of customers in the system when it is in equilibrium.

[6 marks]

(ii) You are told that customers who have been served leave this queue in a Poisson process, rate α. Suggest, giving a reason, the value you think α must take.

[2 marks]

END OF EXAMINATION PAPER

Statistical Tables to be provided

Two hours

THE UNIVERSITY OF MANCHESTER

Time Series Analysis

31 May 2016
09:45 – 11:45

Electronic calculators are permitted, provided they cannot store text.

Answer **ALL** Four questions in Section A (32 marks in all)

and

Two of the Three questions in Section B (24 marks each).

If more than Two questions from Section B are attempted, then credit will be given for the best Two answers.

SECTION A
Answer ALL four questions

A1. Let $\{X_t\}$ be the moving average process

$$X_t = (1 + 0.8\mathbf{B})(1 + 0.4\mathbf{B}^{12})\varepsilon_t,$$

where $\{\varepsilon_t\}$ is a white noise with variance $\sigma^2 = 1$. Calculate the autocorrelations of $\{X_t\}$ for lags $k = 0$ and $k = 12$. [8 marks]

A2. Let $X_t = \varepsilon_t + 2\varepsilon_{t-1}$, where $\{\varepsilon_t\}$ is a white noise with $\text{Var}(\varepsilon_t) = 1$.

a) Check if the model is invertible.

b) Find $\text{Pred}(X_t|X_{t-1})$, the best linear predictor of X_t given X_{t-1}.

[8 marks]

A3. Consider the ARIMA(1,0,1) model

$$(1 - \phi\mathbf{B})X_t = (1 - \theta\mathbf{B})\varepsilon_t,$$

where $\text{Var}(\varepsilon_t) = \sigma^2$ and ε_t are normal (Gaussian) random variables. The variance of the h step ahead prediction error is

$$V(h) \equiv \text{Var}(X_{t+h} - \text{Pred}(X_{t+h}|X_t, X_{t-1}, \dots)) = \sigma^2 \left(1 + (\phi - \theta)^2 \left(\frac{1 - \phi^{2h-2}}{1 - \phi^2}\right)\right),$$

where $\text{Pred}(X_{t+h}|X_t, X_{t-1}, \dots)$ is the best linear predictor of X_{t+h} at time t. Suppose that $\{x_t\}$, $t = 1, \dots, 80$, is a sample path of an ARIMA(1,0,1) process with $\phi = 0.8$, $\theta = -0.2$ and $\sigma^2 = 0.01$. Assuming that $x_{79} = 0.6$, $x_{80} = 0.9$ and $\varepsilon_{80} = 0.05$, give point predictions (at time $t = 80$) and 95% prediction intervals for X_{81} and X_{82}.

[8 marks]

A4. The following model was considered for a monthly time series:

$$(1 - \mathbf{B})(1 - \mathbf{B}^{12})(1 - \phi_1\mathbf{B})X_t = (1 + \theta_1\mathbf{B})(1 + \Theta_1\mathbf{B}^{12})\varepsilon_t.$$

a) Identify the model as a seasonal ARIMA$(p, d, q)(p_s, d_s, q_s)_s$ model (i.e. replace p, d, etc. with the appropriate numbers).

b) The model was fitted and parameter estimates and their standard errors (s.e.) are given below.

	$\hat{\phi}_1$	$\hat{\theta}_1$	$\hat{\Theta}_1$
estimate	0.1527	-0.8393	-0.1102
s.e.	0.0392	0.0689	0.2717

Assuming that the estimators of the parameters are approximately normally distributed, give approximate 95% confidence intervals for the parameters.

Is there evidence to consider a simplified model? If so, what alternative model would you fit?

[8 marks]

SECTION B
Answer 2 of the 3 questions

B5.

a) Suppose that $\{X_t\}$ is a stationary AR(3) process with mean $\mathrm{E}\, X_t = \mu$,

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + \phi_2(X_{t-2} - \mu) + \phi_3(X_{t-3} - \mu) + \varepsilon_t,$$

where $\{\varepsilon_t\}$ is a Gaussian white noise with variance σ^2 such that $\mathrm{E}\, \varepsilon_t X_{t-k} = 0$ when $k > 0$.

i) Which property of the model is referred to by the expression *"the model is causal"* (or, equivalently, *"the model satisfies the causality condition"*)?

ii) Write down the Yule-Walker system relating the autocovariances (or the autocorrelations) of the process $\{X_t\}$ to the prediction coefficients for the predictor $\mathrm{Pred}(X_t|1, X_{t-1} - \mu, X_{t-2} - \mu)$.

b) Further to the above, assume that

$$X_t = 1.25 + 0.50X_{t-1} - 0.25X_{t-2} + 0.125X_{t-3} + \varepsilon_t, \tag{1}$$

where $\mathrm{E}\, \varepsilon_t^2 = 1$. The variance of X_t is $\gamma_0 = 1.254902$. The first two autocorrelations of $\{X_t\}$ are $\rho_1 = 0.4$, $\rho_2 = 0$,

i) Find the mean of X_t.

ii) Write model (1) in mean-corrected form.

iii) Find γ_3, the lag 3 autocovariance coefficient of $\{X_t\}$.

iv) Find $\mathrm{E}\, X_t\varepsilon_t$.

v) Find $\mathrm{E}\, X_t\varepsilon_{t-1}$.

vi) Find $\mathrm{Pred}(X_t|1, X_{t-1} - \mu)$.

vii) Find $\mathrm{Pred}(X_t|1, X_{t-1} - \mu, X_{t-2} - \mu)$.

viii) Obtain the partial autocorrelations of $\{X_t\}$ for lags 1, 2 and 3.

ix) Let $Y_t = X_t - \mu$. Write down the model for $\{Y_t\}$ in state space form using a multivariate AR(1) model.

[24 marks]

B6.

a) i) Let $\{X_t\}$ be a stationary process with $\mathrm{E}\, X_t = \mu$ and autocovariances γ_k, $k = 0, 1, 2, \ldots$. Define another process, $\{Y_t\}$, by the equation

$$Y_t = X_t - \phi_1 X_{t-1} - \cdots - \phi_m X_{t-m},$$

where m is a positive integer and $\phi_1, \ldots, \phi_m$, are arbitrary constants. Show that the process $\{Y_t\}$ is stationary.

ii) Consider the process $\{U_t\}$ satisfying

$$\begin{aligned} (1 - \mathbf{B}^4)U_t &= \varepsilon_t && \text{for } t \geq 1, \\ U_t &= 0 && \text{for } t = 0, -1, -2, -3, \end{aligned}$$

where $\{\varepsilon_t\}$ is white noise.
Find the variances of U_1, U_5, $\ldots$, $U_{4k+1}, \ldots$ (for integer k). Is the process $\{U_t\}$ stationary?

iii) Let m be a positive integer and $\phi(z) = 1 - \phi_1 z - \cdots - \phi_m z^m$ be a polynomial whose roots have moduli larger than one. Consider the process $\{Z_t\}$ defined by the following equation

$$(1 - \mathbf{B}^4)\phi(\mathbf{B})Z_t = \varepsilon_t.$$

Write down the model for the process $V_t = \phi(\mathbf{B})Z_t$.
Show that the process $\{Z_t\}$ cannot be stationary.

b) Consider a random walk defined by

$$\begin{aligned} X_1 &= \varepsilon_1, \\ X_t &= X_{t-1} + \varepsilon_t && \text{for } t \geq 2, \end{aligned}$$

where $\{\varepsilon_t\}$ is white noise with variance σ^2. Let $\gamma(t, t-k) = \mathrm{E}\, X_t X_{t-k}$.

i) Determine $\gamma(t, t-k)$ for all $t \geq 1$ and $k \geq 0$.
Describe the limiting behaviour of $\gamma(t, t-k)$ when $t \to \infty$ and k is held fixed.

ii) Show that the correlation between X_t and X_{t-k} is $\mathrm{Corr}(X_t, X_{t-k}) = \sqrt{(t-k)/t}$.

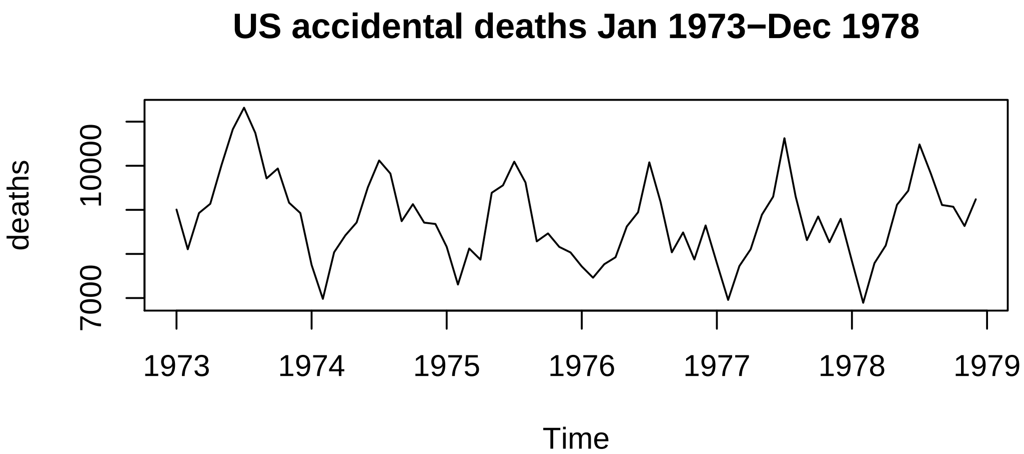
iii) Find the limit of $\mathrm{Corr}(X_t, X_{t-k})$ as $t \to \infty$ and k fixed.

[24 marks]

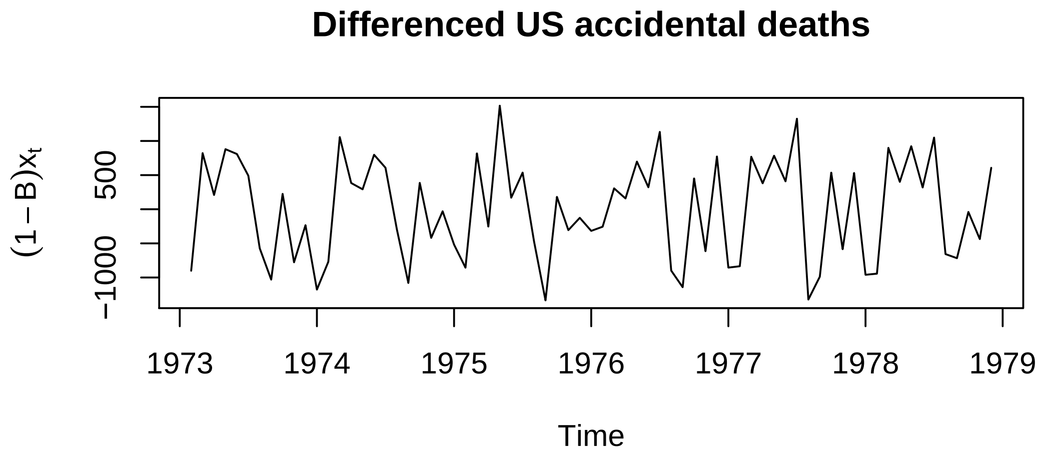
B7.
The following dataset gives the number of accidental deaths occurring monthly in the United States during 1973-1978.

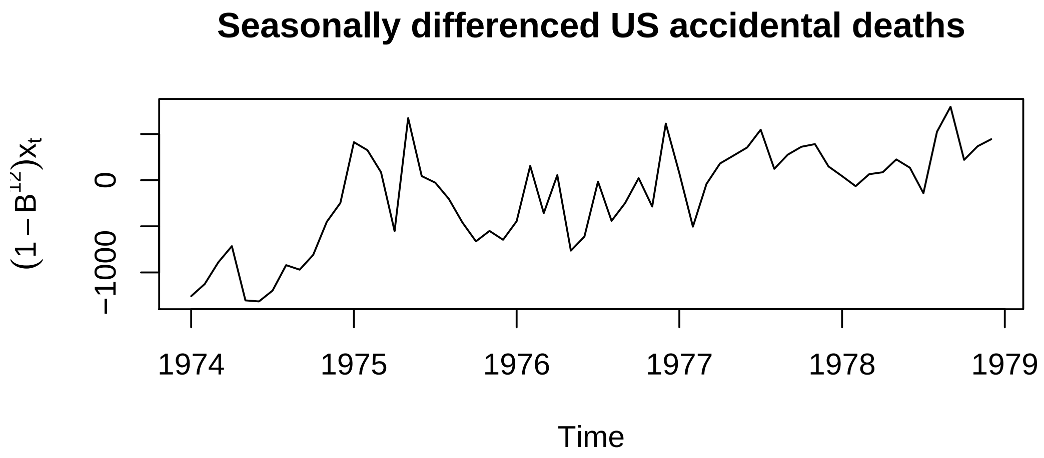
	1973	1974	1975	1976	1977	1978
Jan	9007	7750	8162	7717	7792	7836
Feb	8106	6981	7306	7461	6957	6892
Mar	8928	8038	8124	7767	7726	7791
Apr	9137	8422	7870	7925	8106	8192
May	10017	8714	9387	8623	8890	9115
Jun	10826	9512	9556	8945	9299	9434
Jul	11317	10120	10093	10078	10625	10484
Aug	10744	9823	9620	9179	9302	9827
Sep	9713	8743	8285	8037	8314	9110
Oct	9938	9129	8466	8488	8850	9070
Nov	9161	8710	8160	7874	8265	8633
Dec	8927	8680	8034	8647	8796	9240

a) In order to investigate possible non-stationarity of the series, the following three plots are produced from the data. What kind of effects do the plots show and how would these effects influence the analysis of the series?

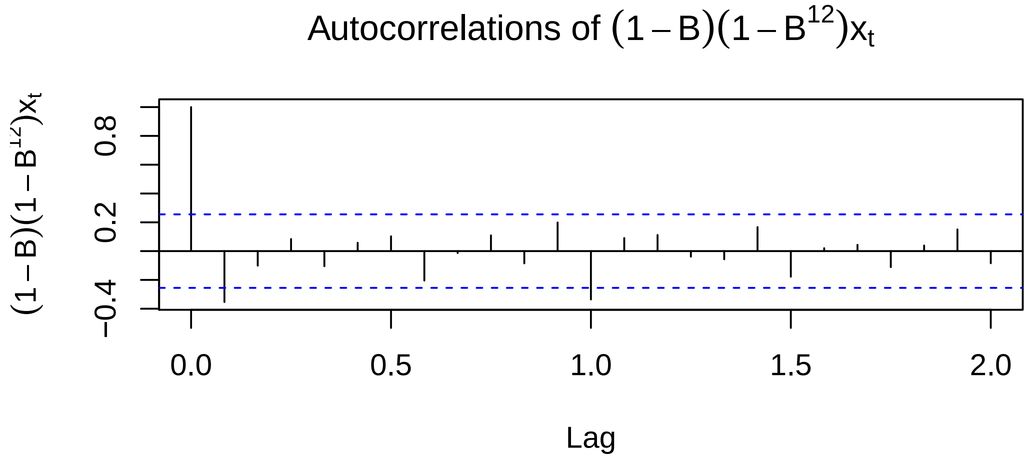


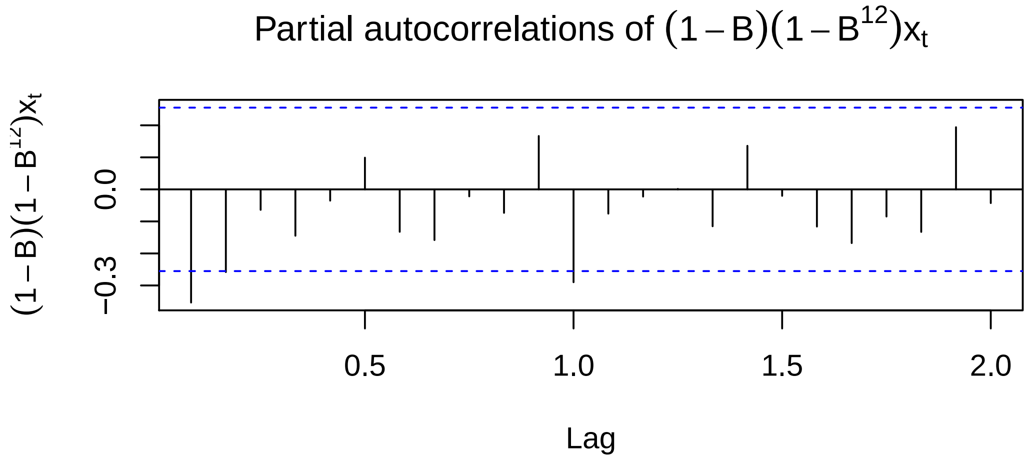
 P.T.O.


Differenced US accidental deaths
(1 − B)x_t
−1000
500
1973
1974
1975
1976
1977
1978
1979
Time


Seasonally differenced US accidental deaths
(1 − B^12)x_t
−1000
0
1974
1975
1976
1977
1978
1979
Time

b) The following graphs show the sample autocorrelations and partial autocorrelations of $y_t = (1-B)(1-B^{12})x_t$, where x_t is the accident time series. Explain what kind of model for y_t is indicated by these graphs.



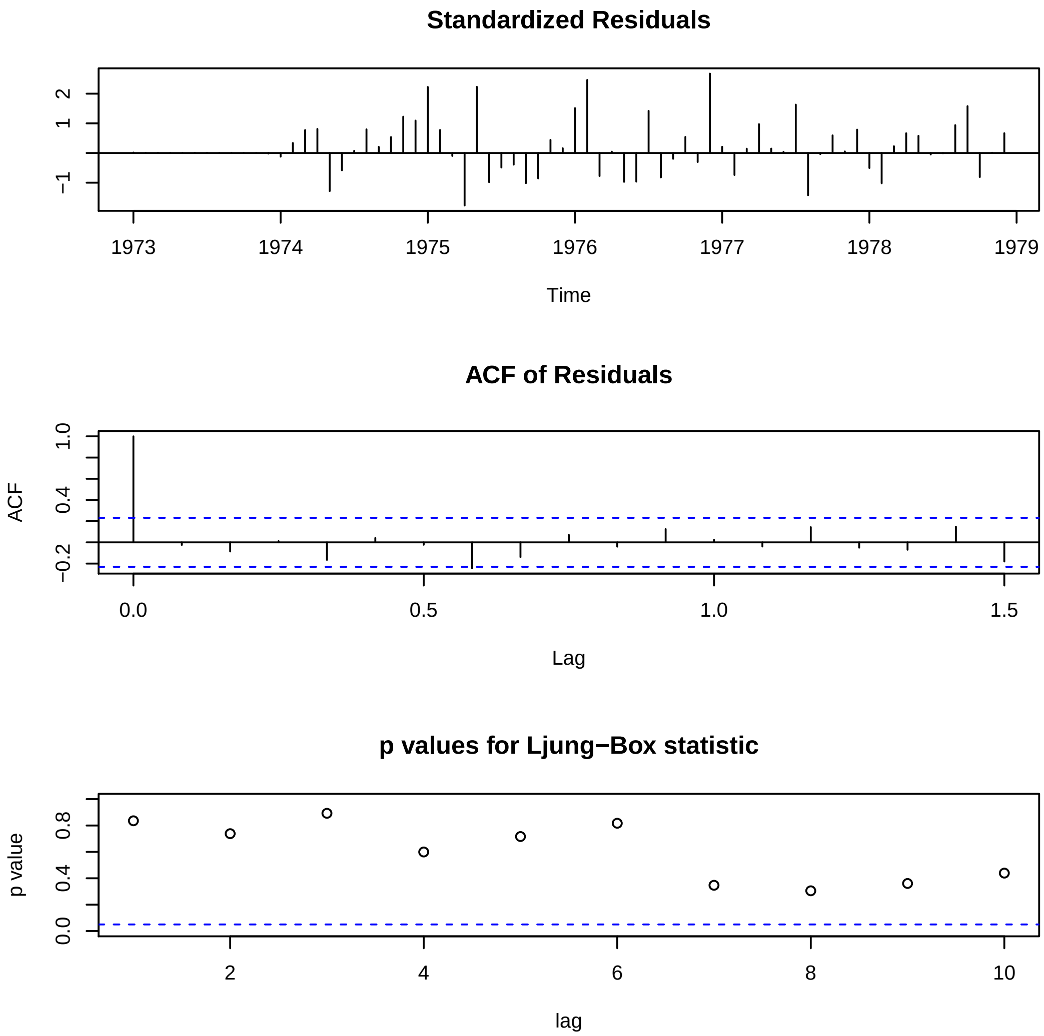


c) A time series model is fitted to the deaths series. A summary of the fitted model and some diagnostic plots are given below.

```
Call:
arima(x = USAccDeaths, order = c(0, 1, 1), seasonal = c(0, 1, 1))

Coefficients:
          ma1     sma1
      -0.4303  -0.5528
s.e.   0.1228   0.1784

sigma^2 estimated as 99347:  log likelihood = -425.44,  aic = 856.88
```



i) Using operator notation, write down the complete model fitted to the accident data series.

ii) Are there redundant parameters? Does the model fit well? Discuss the relevant parts of the supplied numerical and graphical information.

[24 marks]

END OF EXAMINATION PAPER

Two Hours

Mathematical formula books and statistical tables are to be provided

THE UNIVERSITY OF MANCHESTER

GENERALISED LINEAR MODELS

26 May 2016

14:00 – 16:00

Answer ALL TWO questions in Section A.
Answer TWO of the THREE questions in Section B.

If more than TWO questions from section B are attempted
then credit will be given to the best TWO answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** of the two questions

A1.

(a) Write down the general form of a distribution from the exponential family and give expressions for the mean and variance using derivatives of a certain function $b(\theta)$ of the canonical parameter θ.

[4 marks]

(b) Show that the Negative Binomial distribution NB(3, p) with probability mass function

$$P(Y = y) = \frac{(y+2)(y+1)}{2} p^3 (1-p)^y, \;\; y = 0, 1, 2, \ldots$$

belongs to the exponential family, where $p \in [0, 1]$ is an unknown probability value.

[4 marks]

(c) Derive the mean and variance of the distribution in (b) using your answers in (a).

[4 marks]

(d) Explain how a generalised linear model is more general than a linear model with normal errors.

[4 marks]

(e) Find the canonical link for a generalised linear model when the response variable follows the distribution in (b).

[4 marks]

[20 marks in total for this question.]

A2.

(a) Write down the log linear regression model with Poisson responses $y_i \sim \text{Pois}(\lambda_i)$ on a single explanatory variable x_i with intercept, $i = 1, \ldots, n$, stating any further assumptions on $y_1, \ldots, y_n$.

[3 marks]

(b) Find the Fisher information matrix $I(\underset{\sim}{\beta})$ based on $y_1, \ldots, y_n$ following the model in (a), where $\underset{\sim}{\beta}$ is the vector of regression coefficients.

[5 marks]

(c) Explain what the updating equation

$$\underset{\sim}{\beta}^{(k+1)} = (X'W^{(k)}X)^{-1}X'W^{(k)}\underset{\sim}{\xi}^{(k)}, \ k = 0, 1, 2, \ldots$$

does and give in particular for the model in (a) expressions for

(i) the matrix $W^{(k)}$ and

(ii) the vector $\underset{\sim}{\xi}^{(k)}$

in terms of $\underset{\sim}{\beta}^{(k)}$ from iteration k.

[6 marks]

(d) Derive the deviance of the model in (a) when fitted to data and state without proof its asymptotic distribution.

[4 marks]

(e) Explain the connection between Poisson regression and logistic regression with binomial responses.

[2 marks]

[20 marks in total for this question.]

SECTION B

Answer **TWO** of the three questions

B1.

(a) Write down the linear regression model as a GLM for $y_i \sim N(\mu_i, \sigma^2)$ on a single explanatory variable x with values x_i, $i = 1, 2, \ldots, n$ and state any further assumption on $y_1, y_2, \ldots, y_n$. An intercept or constant term should be included in the model.

[3 marks]

(b) Derive the log-likelihood function for the model in (a).

[3 marks]

(c) Obtain a matrix equation for the maximum likelihood estimates of the regression coefficients in model (a) and explain whether an iterative algorithm is required for its solution.

[6 marks]

(d) Find the Fisher information matrix for the regression coefficients in model (a).

[3 marks]

(e) Derive the scaled deviance of the model in (a) and state without proof its distribution. Is this distribution exact or approximate?

[3+2 marks]

[20 marks in total for this question.]

B2. A random sample of 433 men were observed for hypertension, obesity and snoring and the data are as follows.

Obese	Snoring	Total (n)	Hypertension (y)
no	no	77	7
yes	no	10	1
no	yes	272	48
yes	yes	74	23

For example, out of 77 men who were not obese and not snoring, 7 were found to have hypertension.

(a) A full model and an additive effects model were fitted.

```
> fit1 <- glm(cbind(y, n-y) ~ obese + snoring, family = binomial)
> fit2 <- glm(cbind(y, n-y) ~ obese * snoring, family = binomial)
```

Write down the two models formally in mathematical/statistical terms.

[4 marks]

(b) The following analysis of deviance table was obtained.

```
Analysis of Deviance Table

Model 1: cbind(y, n - y) ~ obese + snoring
Model 2: cbind(y, n - y) ~ obese * snoring
  Resid. Df Resid. Dev Df Deviance P(>|Chi|)
1         1    0.33422
2         0    0.00000  1  0.33422    0.5632
```

Is the interaction between obese and snoring significant? Give your reason and significance level used.

[4 marks]

(c) Another analysis of deviance table was also obtained.

```
> anova(fit1, test='Chisq')
Analysis of Deviance Table
Terms added sequentially (first to last)

        Df Deviance Resid. Df Resid. Dev P(>|Chi|)
NULL                        3    12.7820
obese    1   6.8260         2     5.9560  0.008984 **
snoring  1   5.6218         1     0.3342  0.017738 *
```

Is the effect of snoring significant in the presence of obese? Give your reason and significance level used.

[4 marks]

(d) A summary of the additive model is given below.

```
> summary(fit1)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  -2.3921     0.3757  -6.366 1.94e-10 ***
obeseyes      0.6954     0.2851   2.440   0.0147 *
snoringyes    0.8655     0.3967   2.182   0.0291 *

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 12.78200  on 3  degrees of freedom
Residual deviance:  0.33422  on 1  degrees of freedom
```

Is this model adequate for the data? Can it be simplified further? Give reasons and significance levels used.

[4 marks]

(e) The variance-covariance matrix of the parameter estimates is as follows.

```
> vcov(fit1)
            (Intercept)     obeseyes   snoringyes
(Intercept)  0.14118359 -0.014734530 -0.137033225
obeseyes    -0.01473453  0.081261419 -0.008154885
snoringyes  -0.13703322 -0.008154885  0.157340786
```

Estimate the probability of having hypertension for a man who snores but is not obese and give an approximate 95% confidence interval for this probability.

[4 marks]

[20 marks in total for this question.]

B3. A random sample of 227 males were classified according to hair colour (black, brown or fair) and eye colour (brown, grey or blue). A generalised linear model was fitted in R with the following output, where Hair and Eye are factors.

```
> fit<-glm(y ~ Hair * Eye, family=poisson)
> summary(fit)
       Coefficients:
                            Estimate   Std. Error  z value   Pr(>|z|)
      (Intercept)            2.30259     0.31623    7.281    3.30e-13
       HairBrown             0.47000     0.40311    1.166    0.244
       HairFair             -0.69315     0.54772   -1.266    0.206
       EyeGrey               0.87547     0.37639    2.326    0.020
       EyeBlue              -0.22314     0.47434   -0.470    0.638
       HairBrown:EyeGrey     0.06551     0.47808    0.137    0.891
       HairBrown:EyeBlue     0.70865     0.57093    1.241    0.215
       HairFair:EyeGrey      0.98083     0.61067    1.606    0.108
       HairFair:EyeBlue      2.78809     0.66361    4.201    2.65e-05

       Null deviance:      1.0971e+02 on 8 degrees of freedom
       Residual deviance: -3.5527e-15 on 0 degrees of freedom

> anova(fit,test="Chisq")
  Analysis of Deviance Table
  Terms added sequentially (first to last)
                  Df  Deviance Resid.  Df   Resid. Dev   P(>|Chi|)
        NULL                           8     109.711
        Hair       2      26.829       6      82.882     1.493e-06
        Eye        2      46.078       4      36.804     9.872e-11
        Hair:Eye   4      36.804       0    -3.553e-15   1.977e-07
```

(a) Let y_{ij} be the number of people in cell (i, j), i=1,2,3 (hair), j=1,2,3 (eye). Name the joint distribution of $\{y_{ij}\}$ and write down the log-likelihood.

[4 marks]

(b) What is the model that was fitted above? What does its deviance say about this model?

[4 marks]

(c) Give three reasons why model (b) was fitted when the data follow model (a).

[4 marks]

(d) What conclusion can be made from the analysis of deviance table regarding interaction and what further information does the HairFair:EyeBlue line in the summary provide?

[4 marks]

(e) Which category of Hair:Eye does the estimated intercept 2.30259 correspond to? Use this value to compute the fitted Poisson mean and cell probability for this category.

[4 marks]

[20 marks in total for this question.]

END OF EXAMINATION PAPER

TWO hours

Electronic calculators may be used, provided that they cannot store text.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL PROGRAMMING

23 May 2016

14.00 – 16.00

Answer **THREE** of the **FOUR** questions. If more than THREE questions are attempted, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

1.

A toy manufacturer produces three types of wooden toys: soldiers, boats and trains. The manufacturing process consists of two parts, carpentry and finishing. The time required to manufacture a single item of each product and the profit per item are given in the following table

	Soldier	Boat	Train	Available hrs.
Carpentry time (hrs)	1	1	1	50
Finishing time (hrs)	2	3	1	100
Profit per unit ($)	3	7	5	

(a) Formulate an LP problem to determine how much of each product should be manufactured.

[2 marks]

(b) Use the simplex algorithm to determine the manufacturer's best manufacturing policy (the product mix) and corresponding profit.

[6 marks]

(c) For what range of values for the unit profit on soldiers does the current final basis remain optimal? If the profit on a soldier were increased to $7 what would be the manufacturer's new optimal product mix and corresponding profit?

[4 marks]

(d) For what range of values of unit profit on boats does the current final basis remain optimal? If the profit on a boat were increased to $13 what would be the manufacturer's new optimal product mix and corresponding profit?

[4 marks]

(e) For what values of available carpentry hours does the current basis remain optimal? If the availability of carpentry hours is increased to 60 what is the new optimal manufacturing policy and corresponding profit? Confirm your result by a shadow price calculation.

[4 marks]

[20 marks in total for this question]

2.

(a) Write down the dual of the following LP problem in canonical form

$$\begin{array}{rr} \text{Minimize} & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

where $\mathbf{x} \in \mathbb{R}^n$; $\mathbf{A} \in \mathbb{R}^{m\times n}$, and $\mathbf{b}, \mathbf{c}$ are known real constants.

[2 marks]

Beginning with this form of the primal-dual pair of problems, state and prove the dual of the standard form LP in which the constraints $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ are replaced by $\mathbf{A}\mathbf{x} = \mathbf{b}$.

[6 marks]

(b) Obtain the dual of the following LP

$$\begin{array}{rc} \text{Minimize} & z = 4x_1 + x_2 \\ \text{s.t.} & 4x_1 + 3x_2 \geq 6 \\ & x_1 + 2x_2 \leq 3 \\ & 3x_1 + x_2 = 3 \\ & x_1, x_2 \geq 0 \end{array}$$

[4 marks]

(i) Find the optimal solution to the dual problem through simplex iterations.

[4 marks]

(ii) Use the final tableau to deduce the optimal solution to the primal and confirm that it is the optimal solution using the duality theorem.

[4 marks]

[20 marks in total for this question]

3.

(a) State the two properties required of a cutting plane and prove that they apply in the case of Gomory's cutting plane for a pure integer programming problem (Problem IP):

$$s - \sum_{j \in J} f_{ij} x_j = -f_{i0}$$

where J indexes a set of non-basic variables. It may be assumed that

$x_i + \sum_{j \in J} y_{ij} x_j = y_{i0}$ is row i of the optimal tableau for the LP relaxation of Problem IP, and f_{ij} is the fractional part of y_{ij} $(j \in J \cup \{0\})$ with $f_{i0} > 0$.

[8 marks]

(b) Using Gomory's cutting plane algorithm,

$$\begin{array}{ll} \text{maximize} & z = -x_1 + 3x_2 \\ \text{subject to} & x_1 - x_2 \leq 2 \\ & 2x_1 + 4x_2 \leq 15 \\ & x_1, x_2 \geq 0 \text{ and integer.} \end{array}$$

[8 marks]

Find the equation(s) of the cutting plane(s) and illustrate your solution graphically.

[4 marks]

[20 marks in total for this question]

4.

Three electrical power plants supply the needs of four cities. Each plant can supply the following amounts of electricity (in million KWh)

Plant no.	1	2	3
Supply	90	20	90

The peak power demand at these cities, occuring at the same time of day (2 pm.) are given (million KWh.), in the following table:

City	A	B	C	D
Demand	20	80	20	40

The costs ($) of sending 1 million KWh. of electricity from a plant to a city depends on the distance between them and are as follows:

		City			
		A	B	C	D
	1:	5	3	1	9
Plant	2:	1	2	3	2
	3:	3	2	1	8

(a) Formulate a LP problem to mimimize the total cost of meeting the peak demand for electricity at each city.

[3 marks]

(b) How many routes do you expect to have to use in an optimal strategy for electricity supply (for a balanced problem)? Explain why an integer solution should be expected to the LP.

[3 marks]

(c) Observing that total power available at the plants exceeds the total peak demand, what steps need to be taken to be able to apply the transportation algorithm? Hence find an optimal supply strategy. Discuss briefly the number of routes used.

[10 marks]

(d) Find the cost of the optimal strategy. Confirm that the strategy is optimal using the duality theorem. [4 marks]

[20 marks in total for this question]

END OF EXAMINATION PAPER

2 hours

Tables of the cumulative distribution function are provided.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL MODELLING IN FINANCE

1 June, 2016

2:00pm – 4:00pm

Answer **all** 4 questions in **Section A** (60 marks in all) and
2 of the 3 questions in **Section B** (20 marks each). If all three questions from Section B are attempted then credit will be given for the two best answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** 4 questions

A1.

Consider a general stochastic process with the dynamics

$$dS = A(S,t)dt + B(S,t)dW,$$

where W is a standard Brownian motion.

(i) Let the function $f(S,t)$ have a continuous second derivative with respect to S, and a continuous first derivative with respect to t. Show that

$$df = \left[A(S,t)\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}B^2(S,t)\frac{\partial^2 f}{\partial S^2}\right]dt + B(S,t)\frac{\partial f}{\partial S}dW.$$

(ii) Increments in S, a share price, are described by the following variant on geometric Brownian motion:

$$dS = \mu S dt + \sigma S^{\beta} dW,$$

where μ is the drift (constant) and σ is the volatility (constant), and β is also constant. Using the result from part (i), find the dynamics dV followed by the option price $V(S,t)$.

(iii) Consider a portfolio comprising one option $V(S,t)$ and $-\Delta$ of the underlying asset, S, which follows the dynamics given in (ii) above. Show that by choosing Δ (held constant during a time step dt) such that

$$\Delta = \frac{\partial V}{\partial S},$$

and using arbitrage arguments, leads to the following variant of the Black-Scholes partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\beta}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where r is the risk-free interest rate, assumed constant.

[15 marks]

A2.

Consider an *Employee Stock Ownership Plan* (ESOP) contract that has a payoff at time $t = T$ of

$$W(S,T) = S - \beta \min(S,X)$$

for $0 < \beta < 1$ (where S is the price of a company stock).

(i) Sketch the payoff diagram for $W(S,T)$.

(ii) Show that the value of an ESOP contract at time $t < T$ is given by

$$W(S,t) = (1 - \beta S) + \beta C(S,t),$$

where $C(S,t)$ denotes the value of a vanilla call option (with strike price X and expiry $t = T$).

[15 marks]

A3.

Suppose that the share price in Duck Mega Corp. is currently \$10, and that in six months time it will be \$12 with probability q or \$8 with probability $1-q$. A put option with value P has exercise price \$10. Set up a simple delta-hedged portfolio comprising one put and $-\Delta$ shares, and hence find the value of P; ignore interest rates. What do you conclude about the effect of q?

[15 marks]

A4.

Consider a new class of option that allows the holder to chose at time T_1 to receive at expiry T_2 ($> T_1$) either a call option with strike X and exercise date T_2 ($C(S,t;T_2,X)$) or a corresponding put option with strike X and exercise date T_2 ($P(S,t;T_2;X)$).

Using put-call parity, namely

$$C(S,t;T;X) = P(S,t;T;X) + S - Xe^{-r(T-t)},$$

show that at time t ($\leq T_1$), the arbitrage-free price of this option is

$$\Pi(S,t) = C(S,t;T_2;X) + P(S,t;T_1;Xe^{-r(T_2-T_1)}).$$

[15 marks]

SECTION B

Answer **2** of the 3 questions

B5.

Consider an asset whose price 6 months from the expiration of an option is \$25, the risk-free interest rate is 2% per annum (fixed) and the volatility (constant) is 20% per $(\text{annum})^{\frac{1}{2}}$. Consider a portfolio whose payoff is given by

$$\Pi = \$30 - S, \quad 0 < S < \$25,$$
$$\Pi = \$5, \quad S > \$25,$$

where S is the value of the asset. By regarding the portfolio as comprising long one European call, long one European put and short one European call (with different strike prices if necessary, but all with the same expiration date), determine the value of this portfolio at the start of the contract ($t = 0$). Give your answer correct to 3 figures.

You may assume (in the usual notation) that the solution of the Black-Scholes equation for a European call option (paying no dividends) is given by

$$C(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2),$$

and for a European put option (paying no dividends) is given by

$$P(S,t) = Xe^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where

$$d_1 = \frac{\log(S/X) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\log(S/X) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and $N(x)$ should be determined by interpolation from the tables of the cumulative distribution function, as provided. You may assume put-call parity for vanilla options, i.e.

$$S + P - C = Xe^{-r(T-t)}$$

without proof.

[20 marks]

B6.

(i) Draw the expiry payoff diagrams for each of the following portfolios:

(a) Short one share, long two calls, all with exercise price X.

(b) Long one put with exercise price X_1 and short one call with exercise price X_2. Consider the cases $X_1 > X_2$, $X_1 = X_2$, $X_1 < X_2$.

(c) As (b) but also short one call and short one put, both with exercise price X. (Consider just the case $X_1 < X < X_2$.)

(ii) The value of an asset today is \$100. An investor expects the value of this asset to be within 10% of this value in six months time. Using a combination of buying and writing four call options (each with a different exercise price), construct a simple portfolio which has constant value payoff if the asset value at expiry is within this range; if the asset value is below \$85 or above \$115 at expiry, the value of the portfolio is to be zero. What is the value of the portfolio at expiry if the asset price remains at \$100?

[20 marks]

B7.

Consider the Black-Scholes equation for a call option, C, where the underlying asset pays a continuous known constant dividend yield, D, namely

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S\frac{\partial C}{\partial S} - rC = 0,$$

in the usual notation, where r and σ are both constants.

(i) By neglecting the time derivative in the above equation, seek solutions of the form

$$C(S) = AS^{\alpha},$$

where A is a constant. Show that the two possible values of α are

$$\alpha = \frac{1}{2}\left\{-\frac{2}{\sigma^2}(r - D - \frac{1}{2}\sigma^2) \pm \left[\frac{4}{\sigma^4}(r - D - \frac{1}{2}\sigma^2)^2 + \frac{8r}{\sigma^2}\right]^{\frac{1}{2}}\right\}.$$

(ii) Consider a *perpetual* American call option (on an asset paying a continuous known constant dividend yield, D), i.e. an option with no expiry date but where it is possible at any point in time to exercise and receive $S - X$, where X is the exercise price. This can be valued using the time-independent solutions in (i) above. If the exercise boundary is denoted by S_f, write down the two conditions to be imposed on S_f; justify your answer.

(iii) State why only one of the solutions (say α_1) in (i) is retained, and which one it is.

(iv) Use the conditions at $S = S_f$ to determine both the value of the constant A and S_f itself; hence determine the value of the perpetual call option $C(S)$ (you may leave α_1 as part of your solution)

(v) What happens to S_f as $D \to 0$; explain your answer.

[20 marks]

END OF EXAMINATION PAPER

2 hours

UNIVERSITY OF MANCHESTER

CONTINGENCIES 2 **(ONLINE)**

3 June 2016

09:45 - 11:45

Answer ALL EIGHT questions
The total number of marks in the paper is 100.

University approved calculators may be used.

Please note the following before you get started

- You can find a formula sheet on the last page of this exam paper.
- The marking is based on your answer booklet only, so make sure you write down your working there.

1. You are given the following extract from a mortality table :

male age x	l_x	female age y	l_y
30	10,000	30	10,000
31	9,950	31	9,975
32	9,900	32	9,950
33	9,850	33	9,925
34	9,800	34	9,900
35	9,750	35	9,875

Calculate the following where the first life is the male life and the second the female life. Assume the mortality distributions for both lives are independent of each other and an interest rate of 3% p.a.

a) ${}_2p_{32:31}$

[2 marks]

b) ${}_{2|}q_{\overline{32:31}}$

[5 marks]

c) $\ddot{a}_{30:30:\overline{3}|}$

[3 marks]

[Total 10 marks]

2. An insurance company issues a joint whole life assurance policy to two lives, a male aged 40 and a female aged 35. The policy pays a sum assured of £30,000 at the end of the year in which the second death of the two lives occurs. Level premiums are payable annually in advance until the first death of the two lives occurs.

a) Calculate the net premium for the policy assuming an interest rate of 4% p.a. and mortality of PMA92C20 for the male life and PFA92C20 for the female life.

[5 marks]

b) The insurer offers another basis on which premiums are paid. An initial level of premium is paid whilst the male life is alive but, in the event that the male life dies first, the level of premium is reduced to 50% of the original rate.

Calculate the revised premium for the policy using the same assumptions as in part a).

[5 marks]

[Total 10 marks]

3. An insurance company provides a sickness policy. The sickness policy pays £10,000 p.a. so long as the policyholder remains ill.

a) Using the standard 3 state model for sickness benefits and defining the probabilities that you use, write in integral form the expected present value of the policy to a healthy life aged x.

[2 marks]

b) The insurance company decides to change the policy so that it pays £10,000 p.a. during the first period of sickness and £5,000 p.a during any subsequent period of sickness.

Write in integral form the expected present value of the benefits payable during the first period of sickness to a healthy life aged x, again defining the probabilities that you use.

[4 marks]

c) Assume that you have calculated the expected present values in part a) and b). State how you would calculate the expected present value for benefits payable on the second and subsequent periods of sickness.

[2 marks]

[Total 8 marks]

4. You are provided with the following multiple decrement table which has been used to calculate the EPVs for an endowment assurance policy which, as well as paying out lump sum benefits on death or survival, also pays out benefits on surrender. The numbers dying or surrendering when aged x are shown under the columns $(ad)_x^d$ and $(ad)_x^w$ respectively. The number of lives aged x exact is shown as $(al)_x$.

age x	$(al)_x$	$(ad)_x^d$	$(ad)_x^w$
55	10,000	43.80	396.12
56	9,560.08	47.16	346.16
57	9,166.76	-	-

You are informed that the independent probability of death used in the construction of this table was AM92 Ultimate. It is also assumed that the forces of transition are constant from age x to $x+1$ for $x = 55$ and 56.

a) Show that the independent probabilities of surrender at ages 55 and 56 are 0.0397 and 0.0363 respectively.

[10 marks]

b) You are asked to change the independent mortality assumption to PMA92C20. Construct a new multiple decrement table based on the revised independent mortality rate and the existing surrender assumption. Show your entries to the Multiple Decrement Table to 2 decimal places.

[8 marks]

c) State whether the dependent probabilities of death and surrender have increased or decreased relative to those in the original multiple decrement table. Give an explanation for these changes.

[3 marks]

[Total 21 marks]

5. A pension scheme provides an ill heath pension (whole of life annuity paid continuously to a person who retires due to ill health) which is calculated as:

$$\frac{1}{60} \times \text{Final Pensionable Salary} \times (\text{AS} + 0.5\ \text{PFS})$$

where AS is actual service completed up to retirement and PFS is potential future service from retirement date to Normal Pension Age of 65.

The definition of Final Pensionable Salary is the average of salary over the three years prior to retirement.Ill health retirements are only permitted once a person has completed 5 years of service.

A person has just joined the pension scheme at age 50 exact. Her salary over the next 12 months is expected to be £20,000.

In the calculations that follow, use the standard service table provided and assume 4% p.a..

a) When is the earliest age at which the person will be allowed to retire on grounds of ill health?

[1 marks]

b) Calculate an estimate of the pension that would be paid if retirement occurs during the first year where ill heath retirement is permitted.

[4 marks]

c) Calculate a benefit vector showing the annual amount of annuity available at all possible ages of ill health retirement.

[6 marks]

d) Use this to calculate the expected present value of the ill health benefits for this individual.

[4 marks]

e) Recalculate this expected present value if there is a maximum value of £22,500 for the Final Pensionable Salary that can be used to calculate pension in any year.

[4 marks]

f) The pension scheme is considering improving the ill health benefit by giving full credit rather than 50% credit for potential service. You are asked to consider whether this would affect the decrement for ill health retirement. Comment briefly on whether you expect the incidence of ill health retirement to change.

[2 marks]

[Total 21 marks]

6. An insurer writes a 20 year unit linked endowment policy for a life aged 40. Level premiums are paid yearly in advance. Benefits are paid at the end of the year.

a) Construct a table setting out the development of the unit fund for the first two years and then a table showing the cash flows for the first two years for the non unit fund. Show all workings so that the derivation of each item in the tables can be seen.

Details are as follows:

- Premium = £1,000 p.a.
- Allocation is 80% in year 1 and 100% in all subsequent years
- Bid offer spread = nil
- Management charges 1.5% deducted from the unit fund at the end of the year, before any claim payments are made.
- Initial expenses are £ 300 on receipt of the first premium.
- Renewal expenses are £ 50 on payment of each subsequent premium.
- Surrender is possible at each year end. 100% of the unit fund value is paid on surrender.
- Assumed rate of return of 5% p.a. on the unit fund and 3% p.a. on the non-unit fund.

[9 marks]

b) i) Write down an iterative formula that links the value of the unit fund at time $t+1$ (assuming it remains in force at $t+1$) with the value at t for t= 1,2,3,4....,19. For the non unit fund, write down a formula that shows the value of the non unit fund at time t, for t= 2,3,4....,20.

ii) Hence calculate the profit vector for the whole term of the policy and determine which is the first year that shows a positive non unit fund cash flow.

[9 marks]

c) Comment on the risks of having a lengthy period of negative cash flows in the non unit fund and suggest two ways in which the policy might be redesigned to reduce the initial negative cash flows.

[2 marks]

[Total 20 marks]

7. An insurer has issued an insurance policy which has a cash flow vector with elements CF_t as shown below. CF_t is the cash flow for the policy at the end of the t^{th} year given that the policy is in force at the start of the t^{th} year.

(-200, 400, -100, 200, 300)

Produce a profit vector on the basis that the company sets up reserves to zeroise future expected negative cash flows. Assume the policyholder is aged 50 exact, mortality is in accordance with AM92 Ultimate and the interest rate is 3% p.a.

[Total 5 marks]

8. You have the following information for a sub group of people compared to that for the whole population.

age	Number of lives in the subgroup	Number of deaths in the sub group	Number of lives in the whole population	Number of deaths in the whole population
60	8,000	75	800,000	6,800
61	10,000	100	700,000	7,400
62	12,000	125	500,000	5,600

Calculate the crude mortality rate for both the sub group and the whole population and the directly standardised mortality rate for the sub group.

[Total 5 marks]

Formula sheet Contingencies 2

1. Compound interest

i is the interest rate per annum, $v = (1+i)^{-1}$ and $\delta = \log(1+i)$

$a_{\overline{n}|} = v + v^2 + \ldots + v^n$ is the present value of an annuity certain in arrears with term n years

2. Mortality tables and laws

l_x is the number of lives at age x

$d_x = l_x - l_{x+1}$ is the number of lives who die between ages x and $x+1$

${}_tp_x$ is the probability that a life aged x will survive at least t years and ${}_tq_x$ is the probability that a life aged x will die within t years

${}_{m|n}q_x$ is the probability that a life aged x will die between ages $x+m$ and $x+m+n$

K_x is the curtate future lifetime for a life aged x; $e_x = \mathbb{E}[K_x]$

T_x is the complete (exact) future lifetime for a life aged x; $\mathring{e}_x = \mathbb{E}[T_x]$

μ_x is the force of mortality at age x:

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s} ds\right)$$

3. Assurances

A_x is the EPV of a whole life assurance that pays at the end of the year of death of a life aged x

$\bar{A}_x$ is the EPV of a whole life assurance that pays at the moment of death of a life aged x

$$\mathbb{E}\left[v^{K_x+1}\right] = A_x;\ \mathrm{Var}\left(v^{K_x+1}\right) = A_x^{i^*} - (A_x)^2$$

$$\mathbb{E}\left[v^{T_x}\right] = \bar{A}_x;\ \mathrm{Var}\left(v^{T_x}\right) = \bar{A}_x^{i^*} - (\bar{A}_x)^2$$

(where $i^* = (1+i)^2 - 1$. Similar relationships hold for other assurances.

A_{xy} is the EPV of a whole life assurance that pays at the end of the year in which the first death of two lives aged x and y occurs.

$A_{\overline{xy}}$ is the EPV of a whole life assurance that pays at the end of the year in which the second death of two lives aged x and y occurs.

A^1_{xy} is the EPV of a whole life assurance that pays at the end of the year in which a life aged x dies assuming that a life aged y remains alive at that time.

A^2_{xy} is the EPV of a whole life assurance that pays at the end of the year in which a life aged x dies assuming that a life aged y is already dead at that time.

4. Annuities

a_x is the EPV of an annuity for life in arrears for a life aged x

$\ddot{a}_x = 1 + a_x$ is the EPV of an annuity for life in advance for a life aged x

$\bar{a}_x$ is the EPV of a continuously paying annuity for life for a life aged x

$$\mathbb{E}[\ddot{a}_{\overline{K_x+1}|}] = \ddot{a}_x;\ \mathrm{Var}(\ddot{a}_{\overline{K_x+1}|}) = \frac{A_x^{i^*} - (A_x)^2}{(1-v)^2}$$

$$\mathbb{E}[\bar{a}_{\overline{T_x}|}] = \bar{a}_x;\ \mathrm{Var}(\bar{a}_{\overline{T_x}|}) = \frac{\bar{A}_x^{i^*} - (\bar{A}_x)^2}{\log(v)^2}$$

(where $i^* = (1+i)^2 - 1$. Similar relationships hold for temporary annuities with status ${}_{x:\overline{n}|}$)

a_{xy} is the EPV of an annuity for the joint lifetime of two lives aged x and y, paid in arrears.

$a_{\overline{xy}}$ is the EPV of an annuity payable in arrears so long as at least one of two lives aged x and y is alive.

$a_{x|y}$ is the EPV of a reversionary annuity payable to a life aged y after the death of a life aged x, paid in arrears.

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m};$$

$$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - v^n\, {}_np_x\right).$$

$$A_x = 1 - (1-v)\ddot{a}_x; \quad \bar{A}_x = 1 - \delta\bar{a}_x$$

END OF EXAMINATION PAPER

Two hours

UNIVERSITY OF MANCHESTER

RISK THEORY

23 May 2016

14:00 – 16:00

Answer ALL SIX questions
The total number of marks in the paper is 90.

University approved calculators may be used

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1. Consider the Cramér-Lundberg model in which the capital process U for an insurance company is given by

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i \quad \text{for all } t \geq 0,$$

where $u \geq 0$ is the initial capital, $c > 0$ the premium rate, N a Poisson process with intensity $\lambda > 0$ and $\{X_i\}_{i \geq 1}$ a sequence of i.i.d. positive random variables independent of N representing the claim sizes.

(a) For any $t \geq 0$, compute the probability that no claims occur in the time interval $[0, t]$.

[3 marks]

(b) Show that the mean of U_1 is equal to $u + c - \lambda \mathbb{E}[X_1]$.

[5 marks]

(c) State the Net Profit Condition.

[2 marks]

(d) State whether the ruin probability will increase or decrease if one of the following changes is made to the model. Assume that the Net Profit Condition is satisfied throughout.

(i) u is increased.

(ii) c is increased.

(iii) λ is increased.

[4 marks]

[Total 14 marks]

2. An insurance company models the evolution of their capital by means of the Cramér-Lundberg model as in question 1. Their premium rate is $c = 1$ and the intensity is $\lambda = 1$. Furthermore the common pdf of the claim sizes is given by

$$f_X(x) = \begin{cases} 12(e^{-3x} - e^{-4x}) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the Laplace exponent ξ for U and state its domain.

[5 marks]

(b) Determine the Lundberg coefficient R for U.

[7 marks]

(c) Show that if the initial capital is 2, then the ruin probability is bounded above by 0.14.

[4 marks]

[Total 16 marks]

3. The insurance company from question 2 is considering to buy reinsurance. The available reinsurance agreement depends on a parameter $y \in [0, 1]$ and entails the following:

- the company pays a premium to the reinsurance company, with premium rate y,
- of each claim amount, the reinsurance company pays $100y\%$ of the part that exceeds 1 (if any).

For example, suppose that $y = 0.3$ is agreed. Then the company pays premium at a rate 0.3 to the reinsurance company. If a claim of 0.8 arrives then this is fully paid by the insurance company since it is less than 1. If a claim of 5 arrives, then the reinsurance company pays 30% of $5 - 1$ i.e. 1.2, while the rest i.e. 3.8 is still paid by the insurance company.

Note that choosing $y = 0$ is the same as not buying reinsurance.

(a) Show that the Net Profit Condition is satisfied if and only if

$$0 \leq y < \frac{5}{12 - 16e^{-3} + 9e^{-4}}.$$

[10 marks]

(b) Assume that there is no initial capital i.e. $u = 0$. Show that if the insurance company aims to minimise the ruin probability, they should not buy reinsurance at all.

[7 marks]

[Total 17 marks]

4. Let X be a risk following a uniform distribution on $[0, 100]$.

(a) Show that the premium for X based on the variance principle with loading factor 0.5 amounts to $1400/3$.

[4 marks]

(b) Why could a premium of $1400/3$ for X be considered unreasonable?

[3 marks]

(c) Compute both the Value at Risk and the Tail Value at Risk for X at confidence level 0.9.

[6 marks]

(d) Show that the exponential premium principle is additive.

[6 marks]

[Total 19 marks]

5. Recall that a geometric distribution with parameter $a \in [0, 1]$ has a pmf (mass function) p given by

$$p(k) = (1-a)^{k-1}a \quad \text{for } k = 1, 2, \ldots.$$

Suppose that given $\Theta = \theta$, X follows a geometric distribution with parameter θ. The prior distribution is

$$f_\Theta(\theta) = \begin{cases} 2\theta & \text{for } \theta \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that a sample x_1 from X is observed.

(a) Show that the posterior distribution of Θ is $\text{Beta}(3, x_1)$.

[4 marks]

(b) Find the Bayes estimate for θ under the squared error loss function.

[3 marks]

(c) Suppose that a second sample x_2 from X is observed. Show that the Bayes estimate for θ now becomes $4/(x_1 + x_2 + 3)$.

[7 marks]

[Total 14 marks]

6. An insurance company models the number of bicycle accidents an insured individual in Manchester has per year by means of a Poisson(θ) distribution, where the parameter $\theta > 0$ is a sample from an exponential distribution with parameter 1/2. Lecturers at the University of Manchester form a particular sub group of individuals with equal risk profile.

The number of insured lecturers and the average number of accidents per lecturer over the past few years are as follows:

Year	Insured lecturers	Average nr of accidents per insured lecturer
2012	34	1.20
2013	28	0.70
2014	30	0.80
2015	35	1.50

Using the Bühlmann-Straub model, estimate the number of accidents for an insured lecturer in Manchester in 2016 based on the data for the years 2012–2015.

[Total 10 marks]

END OF EXAMINATION PAPER