MATH 31002 Linear Analysis

The University of Manchester, 2015

Solutions

Section A

A1. Let V be a normed vector space (with norm $\|\cdot\|$). Then we define a norm of a linear functional f by

$$\|f\| = \sup_{\|x\|=1} |f(x)|.$$

Alternatively,

$$\|f\| = \sup_{\|x\|\neq 0} \frac{|f(x)|}{\|x\|}.$$

(Either definition will do.)

[bookwork, 3 marks]

A2. Hölder's Inequality: if $p, q > 1$ satisfy $1/p + 1/q = 1$ then, for $a_i, b_i \in \mathbb{C}$, $i = 1, \ldots, n$,

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q},$$

with equality if and only if $|a_i|^p/|b_i|^q$ is constant.

[bookwork, 3 marks]

A3. T **is linear:** we have $T(\lambda f + \mu g)(x) = (\lambda f + \mu g)(1 - x^2) = \lambda f(1 - x^2) + \mu g(1 - x^2) = \lambda Tf(x) + \mu Tg(x)$.

[unseen, 2 marks]

T **is bounded:** Let f be such that $\|f\|_\infty = 1$. Then $|Tf(x)| = |f(1 - x^2)| \leq \|f\|_\infty = 1$. Hence T is bounded.

[unseen, 2 marks]

A4. We say that $\mathcal{A} \subset C(X, \mathbb{R})$ is an algebra if $\mathcal{A}$ is a linear subspace of $C(X, \mathbb{R})$ with the additional property that

$$f, g \in \mathcal{A} \implies fg \in \mathcal{A}.$$

[bookwork, 2 marks]

Stone-Weierstrass Theorem: Let X be a compact metric space. Let $\mathcal{A} \subset C(X, \mathbb{R})$ be an algebra such that

1. $\mathcal{A}$ contains a non-zero constant function;
2. $\mathcal{A}$ separates points (i.e., if $x, x' \in X$, $x \neq x'$, then there exists $f \in \mathcal{A}$ such that $f(x) \neq f(x')$).

Then $\mathcal{A}$ is uniformly dense in $C(X, \mathbb{R})$.

[bookwork, 3 marks]

A5. Let H be a Hilbert space. For $x, y \in H$, we have the identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

[bookwork, 2 marks]

A6. A Banach space X is called reflexive if the natural embedding of X into its second dual X^{**} is injective. $C[0,1], \ell^1$ and ℓ^∞ are examples of non-reflexive spaces.

[bookwork, 3 marks]

A7. Let H be a vector space over $\mathbb{R}$ or $\mathbb{C}$. An inner product is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ (or $\mathbb{C}$) such that, for all $x, y, z \in H$ and scalars λ, μ,

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (complex conjugation);
2. $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$; and
3. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

[bookwork, 3 marks]

Let $\langle \cdot, \cdot \rangle$ be an inner product on H. Then

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2},$$

for all $x, y \in H$.

[bookwork, 2 marks]

Section B

B8. (a) We say that two norms $\|\cdot\|$ and $\|\cdot\|'$ on X are equivalent if there exist $C_1, C_2 > 0$ such that

$$C_1\|x\|' \leq \|x\| \leq C_2\|x\|',$$

for all $x \in X$.

[bookwork, 3 marks]

(b) Let $\|\cdot\|_1$ be the 1-norm on $\mathbb{R}^n$ and let $\|\cdot\|$ be an arbitrary norm. We shall show that $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent.

As usual e_i is the basis vector with 1 in the ith place and 0 elsewhere. Write $x = (x_1, \ldots, x_n) = \sum_{i=1}^n x_i e_i$ and let $M = \max_{1\leq i\leq n} ||e_i||$. Then

$$\|x\| \leq \sum_{i=1}^{n} |x_i|\|e_i\| \leq M\sum_{i=1}^{n}|x_i| = M\|x\|_1.$$

Now we shall show that $\inf_{0\neq x\in\mathbb{R}^n} \|x\|/\|x\|_1$ is positive. If it isn't, then we can find a sequence x_i such that

$$\lim_{i\to+\infty} \|x_i\|/\|x_i\|_1 = 0.$$

Set $y_i = x_i/\|x_i\|_1$, so

$$y_i \in \{y \in \mathbb{R}^n : \|y\|_1 \leq 1\}.$$

This set is closed and bounded (hence compact), so y_i has a convergent subsequence y_{i_j} with limit y. In other words $\lim_{j\to+\infty} \|y_{i_j} - y\|_1 = 0$ and (since $\|y_{i_j}\|_1 = 1$) $\|y\|_1 \neq 0$. However, we also have

$$\big|\|y_{i_j}\| - \|y\|\big| \leq \|y_{i_j} - y\| \leq M\|y_{i_j} - y\|_1 \to 0, \text{ as } j \to +\infty,$$

so $\|y\| = \lim_{j\to+\infty}\|y_{i_j}\| = 0$. But $\|y\| = 0$ if and only if $y = 0$, giving a contradiction with $\|y\|_1 \neq 0$. Therefore, we can define

$$0 < m = \inf_{0\neq x\in\mathbb{R}^n} \frac{\|x\|}{\|x\|_1}.$$

Clearly, $\|x\| \geq m\|x\|_1$, as required.

[bookwork, 14 marks]

(c) Let $(a_i) \in \ell^2$. Then $a_i \to 0$, whence there exists $K > 0$ such that $|a_i| \leq K$ for all $i \geq 0$. On the other hand, $(1, 1, 1, \dots)$ is in ℓ^∞ but not in ℓ^2.

[similar to example sheets, 3 marks]

(d) Put

$$x_k^{(n)} = \begin{cases} 1, & 1 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

Then $\|x^{(n)}\|_2 = \sqrt{n}$, which tends to ∞, whilst $\|x^{(n)}\|_\infty = 1$. Therefore, these norms are not equivalent.

[unseen, 5 marks]

B9. (a) the nth Bernstein polynomial for f is defined as follows:

$$B_n(f;x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k(1-x)^{n-k}.$$

[bookwork, 2 marks]

(b) Suppose that $f \in C[0,1]$ and that $\varepsilon > 0$. Then there exists a polynomial $p(x)$ such that $||f-p||_\infty \leq \varepsilon$.

[bookwork, 2 marks]

Proof. Fix $\varepsilon > 0$. By uniform continuity of f, there exists $\delta > 0$ such that, for $x, y \in [0,1]$,

$$|x-y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Using the binomial formula, we have

$$f(x) - B_n(f;x) = \sum_{k=0}^{n} \left(f(x) - f\left(\frac{k}{n}\right)\right) \binom{n}{k} x^k(1-x)^{n-k}.$$

Thus

$$|f(x) - B_n(f;x)| \leq \sum_{k=0}^{n} \left|f(x) - f\left(\frac{k}{n}\right)\right| \binom{n}{k} x^k(1-x)^{n-k} = \Sigma_1(x) + \Sigma_2(x),$$

where

$$\Sigma_1(x) = \sum_{\substack{0 \leq k \leq n \\ k \,:\, \left|x - \frac{k}{n}\right| < \delta}} \left|f(x) - f\left(\frac{k}{n}\right)\right| \binom{n}{k} x^k(1-x)^{n-k} < \frac{\varepsilon}{2}$$

and

$$\Sigma_2(x) = \sum_{\substack{0\le k\le n \\ k\,:\,\left|x-\frac{k}{n}\right|\ge\delta}} \left|f(x) - f\left(\frac{k}{n}\right)\right| \binom{n}{k} x^k(1-x)^{n-k}$$

$$\le 2\|f\|_\infty \sum_{k\,:\,(k-nx)^2\ge n^2\delta^2} \binom{n}{k} x^k(1-x)^{n-k}$$

$$\le 2\|f\|_\infty \frac{1}{n^2\delta^2} \sum_{k=0}^{n} (k-nx)^2 \binom{n}{k} x^k(1-x)^{n-k}$$

$$= 2\|f\|_\infty \frac{nx(1-x)}{n^2\delta^2}$$

$$\le \frac{\|f\|_\infty}{2\delta^2 n},$$

where we have used the identity

$$\sum_{k=0}^{n} (k-nx)^2 \binom{n}{k} x^k(1-x)^{n-k} = nx(1-x).$$

Combining the estimates on $\Sigma_1(x)$ and $\Sigma_2(x)$, we obtain

$$|f(x) - B_n(f;x)| \le \frac{\varepsilon}{2} + \frac{\|f\|_\infty}{2\delta^2 n}.$$

Now choose N sufficiently large that

$$\frac{\|f\|_\infty}{2\delta^2 N} < \frac{\varepsilon}{2}.$$

(One may take $N = [\|f\|_\infty/\varepsilon\delta^2] + 1$.) Then

$$|f(x) - B_N(f;x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so $B_N(f;x)$ is a polynomial satisfying the conclusion of the theorem.

[bookwork, 15 marks]

(c)

$$|Tf(x)| \le \sum_{k=0}^{n} \binom{n}{k} \left|f\left(\frac{k}{n}\right)\right| x^k(1-x)^{n-k} \le \|f\|_\infty \cdot \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k}$$
$$= \|f\|_\infty,$$

whence $\|T\| \leq 1$. On the other hand, $T(1) = 1$, whence $\|T\| = 1$.

[unseen, 6 marks]

B10. (a) f is continuous if $\lim_{n\to+\infty} \|x_n - x\| = 0$ implies $\lim_{n\to+\infty} f(x_n) = f(x)$.

[bookwork, 2 marks]

(b) A linear functional f on a vector space X is called bounded if there exists $M \geq 0$ such that $|f(x)| \leq M\|x\|$ for any $x \in X$.

[bookwork, 2 marks]

(c)

Suppose that f is continuous. Assume (for a contradiction) that there is no $M \geq 0$ for which $|f(x)| \leq M\|x\|$, for all $x \in V$. Then we can choose a sequence $x_n \in V$, $n \geq 1$, such that $|f(x_n)| > n\|x_n\|$, so that

$$\left| f\left(\frac{1}{n}\frac{x_n}{\|x_n\|}\right)\right| = \frac{|f(x_n)|}{n\|x_n\|} > 1.$$

On the other hand,

$$\left\|\frac{1}{n}\frac{x_n}{\|x_n\|}\right\| \to 0, \text{ as } n \to +\infty,$$

so, by continuity at 0,

$$f\left(\frac{1}{n}\frac{x_n}{\|x_n\|}\right) \to f(0) = 0, \text{ as } n \to +\infty,$$

giving the required contradiction.

Suppose that f is bounded. Given $x \in X$ and $\varepsilon > 0$, we need to show that there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$. If $M = 0$ then $|f(x) - f(y)| = |f(x - y)| = 0$, so any δ will do. If $M > 0$, choose $\delta = \varepsilon/M$. Then, if $\|x - y\| < \delta$,

$$|f(x) - f(y)| = |f(x - y)| \leq M\|x - y\| < M\frac{\varepsilon}{M} = \varepsilon,$$

as required.

[bookwork, 15 marks]

(d) Let $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in \ell^2(\mathbb{R})$. Then

$$f(\lambda x + \mu y) = \sum_{n=0}^{\infty} \frac{\lambda x_n + \mu y_n}{n+1} = \lambda f(x) + \mu f(y),$$

i.e., f is linear. Furthermore,

$$|f(x)| \leq \sum_{n=1}^{\infty} \frac{|x_n|}{n} \leq \sum_{n=1}^{\infty} |x_n| = \|x\|_1,$$

whence f is bounded, and $\|f\| \leq 1$. Now take $x = (1, 0, 0, 0, \dots)$; we have $f(x) = 1$, whence $\|f\| = 1$.

[similar to example sheets, 6 marks]

B11. (a) The spectrum of T is the set of complex numbers

$$\text{spec}(T) = \{\lambda \in \mathbb{C} : (\lambda I - T) : X \to X \text{ is not invertible}\}.$$

[bookwork, 2 marks]

(b) λ is an eigenvalue of T is there exists $x \in X \setminus \{0\}$ such that $Tx = \lambda x$. It lies in spec(T) because if $\lambda I - T$ were invertible, then we would have

$$x = (\lambda I - T)^{-1}(\lambda I - T)x = (\lambda I - T)^{-1}(0) = 0,$$

a contradiction.

[bookwork, 3 marks]

(c) Suppose P has degree n. For a fixed $\lambda \in \mathbb{C}$, we can write

$$\lambda - P(z) = a(\beta_1 - z)(\beta_2 - z) \cdots (\beta_n - z), \qquad (*)$$

where $\beta_1, \dots, \beta_n \in \mathbb{C}$ are the roots of the polynomial $z \mapsto \lambda - P(z)$. We can then write

$$\lambda I - P(T) = a(\beta_1 I - T)(\beta_2 I - T) \cdots (\beta_n I - T).$$

If $\lambda \in \operatorname{spec}(P(T))$ then $\lambda I - P(T)$ is not invertible, so $(\beta_i I - T)$ is not invertible for some i, giving $\beta_i \in \operatorname{spec}(T)$. Substituting $z = \beta_i$ in $(*)$, we have $\lambda = P(\beta_i)$. This shows that $\operatorname{spec}(P(T)) \subset \{P(\lambda) : \lambda \in \operatorname{spec}(T)\}$.

Now suppose that $\lambda \notin \operatorname{spec}(P(T))$. Then $(\lambda I - P(T))$ is invertible, so $(\beta_i I - T)$ is invertible for all $i = 1, \dots, n$, i.e., $\{\beta_1, \dots, \beta_n\} \cap \operatorname{spec}(T) = \varnothing$. Since the equation $\lambda - P(z) = 0$ has no other solutions, this shows that $\operatorname{spec}(P(T))^c \cap \{P(\lambda) : \lambda \in \operatorname{spec}(T)\} = \varnothing$. This completes the proof.

[bookwork, 10 marks]

(d) We have $T^2(x_1, x_2, x_3, x_4, \dots) = (x_1, x_2, x_3, \dots)$, i.e., $T^2 = I$.

[unseen, 2 marks]

(e) We have $T(1,0,1,0,1,0,\dots) = (1,0,1,0,1,0,\dots)$ and $T(0,1,0,1,0,\dots) = (0,-1,0,-1,0,\dots)$, whence both ± 1 are eigenvalues of T.

[unseen, 4 marks]

(f) By (c) and (d), $\{0\} = \operatorname{spec}(T^2 - I) = \{\lambda^2 - 1 : \lambda \in \operatorname{spec}(T)\}$. Hence $\operatorname{spec}(T) \subset \{-1, 1\}$, and by (b) and (e), $\operatorname{spec}(T) = \{-1, 1\}$.

[unseen, 4 marks]

MATH3\4\61022 Exam and Solutions, 2014-15

Throughout the paper you may assume that the Dirichlet Convolution of two multiplicative functions is multiplicative.

SECTION A

1. i. Show, by estimating integrals or otherwise, that

$$\int_1^N \frac{du}{u^\sigma} + \frac{1}{N^\sigma} \leq \sum_{n=1}^N \frac{1}{n^\sigma} < 1 + \int_1^N \frac{du}{u^\sigma},$$

for real $\sigma > 0$.

Deduce that the series defining $\zeta(\sigma)$ diverges for $\sigma \leq 1$, converges for $\sigma > 1$ and satisfies

$$\frac{1}{\sigma - 1} \leq \zeta(\sigma) \leq 1 + \frac{1}{\sigma - 1}$$

for $\sigma > 1$.

ii. Explain why

$$\sum_{n \in \mathcal{N}} \frac{1}{n^\sigma} = \prod_{p \leq N} \left(1 - \frac{1}{p^\sigma}\right)^{-1},$$

for $N \geq 1$, where $\mathcal{N} = \{n : p|n \Rightarrow p \leq N\}$.

iii Prove that

$$\log \zeta(\sigma) \leq 1 + \sum_p \frac{1}{p^\sigma},$$

for $\sigma > 1$.

Deduce that there are infinitely many primes.

You may assume that $\sum_p (-\log(1 - 1/p^\sigma) - 1/p^\sigma) \leq 1$ *for* $\sigma \geq 1$.

[30 marks]

Solution i Use

$$\int_n^{n+1} \frac{dt}{t^\sigma} \leq \frac{1}{n^\sigma} \int_n^{n+1} dt = \frac{1}{n^\sigma}.$$

Sum over $n = 1, 2, ..., N-1$ to get

$$\int_1^N \frac{dt}{t^\sigma} + \frac{1}{N^\sigma} \le \sum_{1\le n\le N} \frac{1}{n^\sigma}.$$

Use

$$\int_{n-1}^n \frac{dt}{t^\sigma} \ge \frac{1}{n^\sigma}\int_{n-1}^n dt = \frac{1}{n^\sigma}.$$

Sum over $n = 2, ..., N$ to get

$$\sum_{1\le n\le N} \frac{1}{n^\sigma} \le \frac{1}{1^\sigma} + \int_1^N \frac{dt}{t^\sigma}.$$

Combine to get stated result. Bookwork [9 marks]

It is possible to prove this by Partial Summation which I will accept.

If $\sigma = 1$ the left hand side gives

$$\sum_{1\le n\le N} \frac{1}{n^\sigma} \ge \frac{1}{N^\sigma} + \log N,$$

which $\to \infty$ as $N \to \infty$ in which case the series defining $\zeta(s)$ diverges. [1 mark]

If $\sigma \ne 1$ then integrating gives

$$\frac{N^{1-\sigma}-1}{1-\sigma} + \frac{1}{N^\sigma} \le \sum_{n=1}^N \frac{1}{n^\sigma} \le 1 + \frac{N^{1-\sigma}-1}{1-\sigma}.$$

If $\sigma < 1$ then $1-\sigma > 0$ and so $N^{1-\sigma} \to \infty$ as $N \to \infty$ in which case, by the left hand inequality, the series defining $\zeta(\sigma)$ diverges.

If $\sigma > 1$ then $1-\sigma < 0$ and so $N^{1-\sigma} \to 0$ as $N \to \infty$ in which case, by the right hand inequality, the series defining $\zeta(\sigma)$ converges. We also get in the limit

$$-\frac{1}{1-\sigma} \le \sum_{n=1}^\infty \frac{1}{n^\sigma} \le 1 - \frac{1}{1-\sigma},$$

equivalent to stated result. [5 marks]

ii

$$\prod_{p\le N}\left(1-\frac{1}{p^\sigma}\right)^{-1} = \prod_{p\le N}\left(1+\frac{1}{p^\sigma}+\frac{1}{p^{2\sigma}}+\frac{1}{p^{3\sigma}}+\frac{1}{p^{4\sigma}}+...\right).$$

On multiplying out we get terms $1/n^\sigma$ with integers n composed only of primes $\leq N$, i.e. $n \in \mathcal{N}$. Every integer in $\mathcal{N}$ will arise by *the factorisation of integers into primes* and every integer in $\mathcal{N}$ will occur only once by the *unique* factorisation on integers into primes. Hence stated result.

Bookwork [5 marks]

iii. Take the logarithm of part ii to get

$$\begin{aligned}
\log\left(\sum_{1\leq n\leq N}\frac{1}{n^\sigma}\right) &\leq \log\left(\prod_{p\leq N}\left(1-\frac{1}{p^\sigma}\right)^{-1}\right)\\
&= \sum_{p\leq N} -\log\left(1-\frac{1}{p^\sigma}\right)\\
&= \sum_{p\leq N}\frac{1}{p^\sigma}+\sum_{p\leq N}\left(-\log\left(1-\frac{1}{p^\sigma}\right)-\frac{1}{p^\sigma}\right).
\end{aligned}$$

Since $\sigma > 1$ we can let $N \to \infty$ to get stated result, having used the assumption in the question.

Bookwork [7 marks]

Combining parts i and iii gives

$$\sum_p \frac{1}{p^\sigma} \geq \log\left(\frac{1}{\sigma-1}\right) - 1,$$

for $\sigma > 1$. Let $\sigma \to 1+$. The right hand side diverges as thus must the series on the left hand side. Yet all terms in the series remain finite so there must be infinitely many terms, i.e. infinitely many primes. Bookwork [3 marks]

2. i. By Partial Summation prove that for $s \neq 1$ we have

$$\sum_{1 \leq n \leq N} \frac{1}{n^s} = 1 + \frac{1}{s-1} + \frac{N^{1-s}}{1-s} - s \int_1^N \{u\} \frac{du}{u^{s+1}}$$

for any **integer** $N \geq 1$.

ii. Deduce that

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{1+s}} du, \tag{1}$$

for $\mathrm{Re}\, s > 1$.

Explain why (1) can be used to define $\zeta(s)$ for complex s with $\mathrm{Re}\, s > 0$, $s \neq 1$.

iii) Using parts i and ii prove that for all complex s with $\mathrm{Re}\, s > 0$, $s \neq 1$, and all integers $N \geq 1$,

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + O\left(\frac{|s|}{\sigma N^\sigma}\right).$$

Deduce that

$$\sum_{n=1}^\infty \frac{1}{n^{1+it}}$$

diverges for all real $t > 0$.

[30 marks]

Solution i. By Partial Summation

$$\begin{aligned}
\sum_{1 \leq n \leq N} \frac{1}{n^s} &= \sum_{1 \leq n \leq N} \left(\frac{1}{N^s} - \left(\frac{1}{N^s} - \frac{1}{n^s}\right)\right) \\
&= \frac{N}{N^s} - \sum_{1 \leq n \leq N} \int_n^N (-s) \frac{du}{u^{s+1}} \\
&= \frac{N}{N^s} + s \int_1^N \left(\sum_{1 \leq n \leq u} 1\right) \frac{du}{u^{s+1}} \\
&= \frac{N}{N^s} + s \int_1^N [u] \frac{du}{u^{s+1}}.
\end{aligned}$$

$$\begin{aligned}\sum_{1\le n\le N}\frac{1}{n^s} &= \frac{N}{N^s}+s\int_1^N u\frac{du}{u^{s+1}}-s\int_1^N (u-[u])\frac{du}{u^{s+1}}\\ &= \frac{N}{N^s}+\frac{s}{1-s}\left(N^{1-s}-1\right)-s\int_1^N \{u\}\frac{du}{u^{s+1}},\end{aligned}$$

which rearranges to stated result. Bookwork [11 marks]

ii We have $\operatorname{Re} s>1$ so $1-\sigma<0$. Thus

$$\left|\frac{N^{1-s}}{1-s}\right|=\frac{N^{1-\sigma}}{|1-s|}\to 0 \text{ as } N\to\infty.$$

Also the resulting integral satisfies

$$\int_1^\infty \left|\frac{\{u\}}{u^{s+1}}\right|du\le\int_1^\infty\frac{du}{u^{\sigma+1}}=\frac{1}{\sigma}, \tag{2}$$

i.e. it converges (absolutely). So we can let $N\to\infty$ to get the stated result

$$\zeta(s)=1+\frac{1}{s-1}-s\int_1^\infty\frac{\{u\}}{u^{1+s}}du, \tag{3}$$

for $\operatorname{Re} s>1$. Bookwork [5 marks]

Looking at (2) we see that the integral in fact converges for $\sigma>0$. This is why the right hand side of (3) can be used to define a function on $\operatorname{Re} s>0$ which agrees with the series definition of $\zeta(s)$ on $\operatorname{Re} s>1$.

Bookwork [3 marks]

I require no discussion on analytic continuation or uniqueness.

iii Subtract the last two results to get, for $\operatorname{Re} s>0$

$$\zeta(s)-\sum_{1\le n\le N}\frac{1}{n^s}=-\frac{N^{1-s}}{1-s}+s\int_1^N\{u\}\frac{du}{u^{s+1}}-s\int_1^\infty\frac{\{u\}}{u^{1+s}}du,$$

i.e.

$$\zeta(s)=\sum_{1\le n\le N}\frac{1}{n^s}+\frac{N^{1-s}}{s-1}-s\int_N^\infty\frac{\{u\}}{u^{1+s}}du.$$

The integral here is estimated as

$$\left| s \int_N^\infty \frac{\{u\}}{u^{1+s}} du \right| \leq |s| \left| \int_N^\infty \frac{du}{u^{1+\sigma}} \right| = \frac{|s|}{\sigma N^\sigma}.$$

Bookwork [7 marks]

With $s = 1 + it$, $t > 0$, the last result rearranges to

$$\sum_{1 \leq n \leq N} \frac{1}{n^{1+it}} = \zeta\left(1 + it\right) + \frac{e^{i(t \log N + \pi/2)}}{t} + O\left(\frac{|t|}{N}\right).$$

As $N \to \infty$ we get a sequence of partial sums that get ever closer to a circle, centre $\zeta\left(1 + it\right)$ and radius $1/t$, and keep going round the circle without end. Hence we do not have convergence to a point and so must have divergence.

Bookwork [4 marks]

3. i. Write down the Euler product for the Riemann zeta function $\zeta(s)$.

ii Let $\omega(n)$ denote the number of distinct prime factors of n. By looking at the Euler product of the Dirichlet Series on the left hand side of the identity below, prove that

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)},$$

for $\operatorname{Re} s > 1$.

You may assume that 2^ω is multiplicative.

iii Let λ be Liouville's function, defined as $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime divisors of n counted with multiplicity. By looking at the Euler product of the Dirichlet Series on the left hand side of the identity below, prove that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)},$$

for $\operatorname{Re} s > 1$.

You may assume that λ is multiplicative.

iv. Explain why parts ii and iii suggest that

$$2^\omega * \lambda = 1.$$

Prove this by showing equality on prime powers.

[30 marks]

Solution. i

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\operatorname{Re} s > 1$. Bookwork [2 marks]

ii. Using the fact that 2^ω is multiplicative and $2^{\omega(p^a)} = 2$ for all primes p and $a \geq 1$,

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \prod_p \left(1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \frac{2}{p^{3s}} + \frac{2}{p^{4s}} + \ldots\right).$$

Sum the series

$$1+2y+2y^2+2y^3+... = 1+2y\left(1+y+y^2+...\right) = 1+\frac{2y}{1-y}$$
$$= \frac{1+y}{1-y} = \frac{1-y^2}{(1-y)^2}.$$

Applying this with $y = 1/p^s$ gives

$$\prod_p\left(1+\frac{2}{p^s}+\frac{2}{p^{2s}}+\frac{2}{p^{3s}}+...\right) = \prod_p\frac{1-\frac{1}{p^{2s}}}{\left(1-\frac{1}{p^s}\right)^2} = \frac{\zeta^2(s)}{\zeta(2s)},$$

by part i. Bookwork [8 marks]

iii Using the fact that λ is multiplicative and $\lambda(p^a) = (-1)^a$ for all primes p and $a \geq 1$,

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} = \prod_p\left(1-\frac{1}{p^s}+\frac{1}{p^{2s}}-\frac{1}{p^{3s}}+...\right).$$

This time each sum is a geometric series,

$$1-y+y^2-y^3+... = \frac{1}{1-(-y)} = \frac{1}{1+y} = \frac{1-y}{1-y^2}.$$

Hence

$$\prod_p\left(1-\frac{1}{p^s}+\frac{1}{p^{2s}}-\frac{1}{p^{3s}}+...\right) = \prod_p\frac{1-\frac{1}{p^s}}{1-\frac{1}{p^{2s}}} = \frac{\zeta(2s)}{\zeta(s)}.$$

Bookwork [8 marks]

iv With $D_f(s) := \sum_{n=1}^{\infty} f(n)\, n^{-s}$, parts ii and iii give

$$D_{2^\omega*\lambda}(s) = D_{2^\omega}(s)\, D_\lambda(s) = \frac{\zeta^2(s)}{\zeta(2s)}\frac{\zeta(2s)}{\zeta(s)} = \zeta(s) = D_1(s)$$

for $\operatorname{Re} s > 1$. This suggests $2^\omega * \lambda = 1$ (only suggests for we have not proved that $D_f(s) = D_g(s)$ for an appropriate set of s implies $f = g$.)

Bookwork [3 marks]

Since 2^ω, λ and thus $2^\omega * \lambda$ are multiplicative

$$2^\omega * \lambda(n) = 2^\omega * \lambda\left(\prod_{p^r \| n} p^r\right) = \prod_{p^r \| n} 2^\omega * \lambda(p^r).$$

Yet, by the definition of Dirichlet Convolution,

$$2^{\omega} * \lambda\left(p^{r}\right) = \sum_{\substack{a+b=r\\ a,b\geq 0}} 2^{\omega\left(p^{a}\right)}\lambda\left(p^{b}\right).$$

We take out the $a=0$ separately for $2^{\omega\left(p^{0}\right)} = 2^{0} = 1$, so

$$\begin{aligned}\sum_{\substack{a+b=r\\ a,b\geq 0}} 2^{\omega\left(p^{a}\right)}\lambda\left(p^{b}\right) &= \lambda\left(p^{r}\right) + 2\sum_{0\leq b\leq r-1}\lambda\left(p^{b}\right)\\ &= (-1)^{r} + 2\sum_{0\leq b\leq r-1}(-1)^{b}.\end{aligned}$$

This sum is a finite geometric sum with common ratio -1. Thus

$$2^{\omega} * \lambda\left(p^{r}\right) = (-1)^{r} + 2\frac{1-(-1)^{r}}{1-(-1)} = 1,$$

as required. Bookwork [9 marks]

4) i. State, without proof, Möbius Inversion, not forgetting to define all terms.

ii a. Define Euler's phi function, ϕ.

b. Using Mobius Inversion or otherwise prove that $\phi = \mu * j$, i.e.

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d},$$

for all $n \geq 1$. Here j is the identity function, $j(n) = n$ for all n.

Deduce that

c.

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

for all $n \geq 1$.

d.

$$\sum_{d|n} \phi(d) = n,$$

for all $n \geq 1$.

iii Prove that

$$\sum_{n \leq x} \frac{\phi(n)}{n} = \frac{1}{\zeta(2)} x + O(\log x).$$

You may assume that $\sum_{n>x} 1/n^2 = O(1/x)$ *and* $\sum_{n \leq x} 1/n = O(\log x)$.

[30 marks]

Solution i Möbius Inversion states that

$$\mu * 1 = \delta \qquad \text{or equivalently,} \qquad \sum_{d|n} \mu(d) = \delta(n).$$

Here $1(n) = 1$ for all $n \geq 1$ while $\delta(n) = 1$ if $n = 1$, $\delta(n) = 0$ for all $n \geq 2$. If $n = \prod_{i=1}^r p_i^{a_i}$ is a factorization into distinct primes then the Möbius function is

$$\mu(n) = \begin{cases} (-1)^r & \text{if } a_1 = a_2 = a_3 = \ldots = 1, \\ 0 & \text{if some } a_i \geq 2. \end{cases}$$

Bookwork [5 marks]

ii.a. Euler's phi function is

$$\phi(n) = \sum_{\substack{1 \le r \le n \\ \gcd(r,n)=1}} 1.$$

Bookwork [1 mark]

b. Rewrite the condition $\gcd(r, n) = 1$ in terms of δ as

$$\phi(n) = \sum_{1 \le r \le n} \delta(\gcd(r, n)) = \sum_{1 \le r \le n} \sum_{d | \gcd(r,n)} \mu(d),$$

by Möbius Inversion. Note that $d | \gcd(r, n)$ if, and only if, $d|r$ and $d|n$. Interchange summations to get

$$\phi(n) = \sum_{d|n} \mu(d) \sum_{\substack{1 \le r \le n \\ d|r}} 1.$$

In the inner sum we can write $r = sd$, $n = md$ and we are counting the number of integers $s \le m$, of which there are $m = n/d$, hence

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = \sum_{d|n} \mu(d) j\left(\frac{n}{d}\right) = (\mu * j)(n).$$

Bookwork [6 marks]

c) Since μ and j are multiplicative then so is ϕ. So it suffices to consider, with $r \ge 1$ and prime p,

$$\phi(p^r) = (\mu * j)(p^r) = \sum_{\substack{a+b=r \\ a,b \ge 0}} \mu(p^a) j\left(p^b\right) = \mu\left(p^0\right) j(p^r) + \mu\left(p^1\right) j\left(p^{r-1}\right),$$

since $\mu(p^a) = 0$ if $a \ge 2$. Thus

$$\phi(p^r) = p^r - p^{r-1} = p^r \left(1 - \frac{1}{p}\right).$$

Multiply together to get stated result. Bookwork [4 marks]

d) By definition of Dirichlet Convolution

$$\sum_{d|n} \phi(d) = (1 * \phi)(n).$$

Yet

$$\begin{aligned} 1 * \phi &= 1 * (\mu * j) \qquad \text{by part b} \\ &= (1 * \mu) * j \\ &= \delta * j \qquad \text{by Mobius Inversion} \\ &= j \qquad \text{since } \delta \text{ is the identity under } * \end{aligned}$$

Hence

$$\sum_{d|n} \phi(d) = (1 * \phi)(n) = j(n) = n.$$

Problem Sheet [5 marks]

I'll accept other proofs, i.e. partitioning integers r depending on $\gcd(r,n)$

iii By Part ii b,

$$\sum_{n \le x} \frac{\phi(n)}{n} = \sum_{n \le x} \sum_{d|n} \frac{\mu(d)}{d} = \sum_{d \le x} \frac{\mu(d)}{d} \sum_{\substack{n \le x \\ d|n}} 1,$$

on interchanging summations. Continuing

$$= \sum_{d \le x} \frac{\mu(d)}{d} \left[\frac{x}{d}\right] = \sum_{d \le x} \frac{\mu(d)}{d} \left(\frac{x}{d} + O(1)\right) = x \sum_{d \le x} \frac{\mu(d)}{d^2} + O\left(\sum_{d \le x} \frac{1}{d}\right).$$

The error here is $O(\log x)$ by assumption in question. In the main term complete the sum up to infinity

$$\sum_{d \le x} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d > x} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} + O\left(\sum_{d > x} \frac{1}{d^2}\right).$$

The error here is $O(1/x)$ by assumption in question. Combining

$$\sum_{n \le x} \frac{\phi(n)}{n} = x\left(\frac{1}{\zeta(2)} + O\left(\frac{1}{x}\right)\right) + O(\log x).$$

Keeping only the dominant error term we get the stated result.

Bookwork [9 marks]

SECTION B

This Section is Compulsory, answer all parts.

5. i. a) Prove that there exists a constant γ such that

$$\sum_{n\leq x}\frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right),$$

for real $x>1$.

b) Explain why this error term is best possible for real x.

c) Prove that

$$\sum_{n\leq N}\frac{1}{n}=\log N+\gamma+\frac{1}{2N}+O\left(\frac{1}{N^2}\right),$$

for integer $N\geq 1$.

You may assume that $\psi_2(x):=\int_0^x(\{t\}-1/2)\,dt$ *is periodic in* x*, with period 1.*

ii. The Bernoulli polynomials and numbers are defined iteratively by

$$P_k(x)=k\int_0^x P_{k-1}(t)\,dt+B_k \qquad \text{for } k\geq 2,$$

where each B_k is chosen so that

$$\int_0^1 P_k(t)\,dt=0,$$

along with $P_1(x)=\{x\}-1/2$ when $x\notin\mathbb{Z}$, 0 when $x\in\mathbb{Z}$.

a) Find the Fourier Series for $P_k(x)$, $k\geq 2$,

You may assume that every $P_k(x)$ *is periodic with period 1 and* $P_1(x)$ *has Fourier Series* $-\sum_{n\neq 0}e^{2\pi inx}/(2in\pi)$.

b) Deduce that

$$\zeta(2\ell)=\frac{(-1)^{\ell+1}(2\pi)^{2\ell}}{2(2\ell)!}B_{2\ell},$$

for all $\ell\geq 1$. [45 marks]

Solution i. a From either Partial Summation or, as here, from Euler Summation with $f(x) = 1/x$ we have

$$\sum_{n \le x} \frac{1}{n} = \int_1^x \frac{dt}{t} + 1 - \frac{\{x\}}{x} - \int_1^x \frac{\{t\}}{t^2} dt. \tag{4}$$

The second integral converges absolutely since

$$\int_1^\infty |\{t\}| \frac{dt}{t^2} \ll \int_1^\infty \frac{dt}{t^2} \ll 1.$$

Thus we can *complete the integral up to* ∞, the error in doing so is

$$\le \int_x^\infty |\{t\}| \frac{dt}{t^2} \ll \int_x^\infty \frac{dt}{t^2} \ll \frac{1}{x}.$$

Combining,

$$\sum_{n \le x} \frac{1}{n} = \log x + 1 + O\left(\frac{1}{x}\right) - \int_1^\infty \{t\} \frac{dt}{t^2}.$$

Hence the result follows with

$$\gamma = 1 - \int_1^\infty \{t\} \frac{dt}{t^2}.$$

Bookwork [8 marks]

b. The error is best possible in that as x moves from $N-$ to $N+$ (where N is an integer) we gain a term $1/N$ on the left hand side, whereas because of continuity, the main terms on the right hand side vary by almost nothing. Hence the error term has to accommodate, be no less than, the $1/N$. i.e. approximately $1/x$

Bookwork [3 marks]

c. Return to (4) with $x = N$, and integer. Then

$$\sum_{n \le N} \frac{1}{n} = \int_1^N \frac{dt}{t} + 1 - \int_1^N \frac{\{t\}}{t^2} dt.$$

Write

$$\int_1^N \frac{\{t\}}{t^2} dt = \frac{1}{2} \int_1^N \frac{dt}{t^2} + \int_1^N \frac{\{t\} - 1/2}{t^2} dt$$

The first integral equals

$$\frac{1}{2}\left(1-\frac{1}{N}\right).$$

For the second integral, integration by parts gives

$$\int_1^N \frac{\{t\}-1/2}{t^2}dt = \left[\frac{\psi_2(t)}{t^2}\right]_1^N + 2\int_1^N \frac{\psi_2(t)}{t^3}dt = 2\int_1^N \frac{\psi_2(t)}{t^3}dt, \qquad (5)$$

since $\psi_2(t)$ is periodic, period 1 so $\psi_2(N) = \psi_2(0) = 0$ for all integers N.

Since ψ_2 is periodic and continuous (being defined by an integral) it is bounded. Thus the integral in (5) converges, so complete to infinity and bound the tail end as

$$\int_N^\infty \frac{\psi_2(t)}{t^3}dt \ll \int_N^\infty \frac{dt}{t^3} \ll \frac{1}{N^2}.$$

Hence

$$\sum_{n\le N}\frac{1}{n} = \log N + C + \frac{1}{2N} + O\left(\frac{1}{N^2}\right), \qquad (6)$$

for some constant C. Bookwork [12 marks]

From (6)

$$\begin{aligned} C &= \lim_{N\to\infty}\left(\sum_{n\le N}\frac{1}{n} - \log N - \frac{1}{2N}\right) \\ &= \lim_{N\to\infty}\left(\sum_{n\le N}\frac{1}{n} - \log N\right) = \gamma \end{aligned}$$

by Part i.a Bookwork & Problem Sheet [3 marks]

ii. a Since the Bernoulli functions $P_k(x)$ are periodic with period 1, they have a Fourier Series

$$\sum_{n=-\infty}^{\infty} c_n(k)\, e^{2\pi i n x} \quad \text{where} \quad c_n(k) = \int_0^1 P_k(x)\, e^{-2\pi i n x}dx.$$

From the definition of P_k we have $c_0(k) = 0$ for all $k \ge 1$.

Assume $n \neq 0$. From the definition we have $P_k'(x) = kP_{k-1}(x)$ so integration by parts gives

$$\begin{aligned} c_n(k) &= \int_0^1 P_k(x) e^{-2\pi inx} dx \\ &= \left[-P_k(x) \frac{e^{-2\pi inx}}{2\pi in} \right]_0^1 + \frac{k}{2\pi in} \int_0^1 P_{k-1}(x) e^{-2\pi inx} dx \\ &= \frac{k}{2\pi in} c_n(k-1). \end{aligned}$$

Continue,

$$c_n(k) = k! \left(\frac{1}{2\pi in} \right)^{k-1} c_n(1).$$

Next, by the given assumption,

$$P_1(x) = -\sum_{n \neq 0} \frac{e^{2\pi inx}}{2in\pi} \quad \text{so} \quad c_n(1) = -\frac{1}{2in\pi}.$$

Hence

$$c_n(k) = -k! \left(\frac{1}{2\pi in} \right)^k.$$

Thus, for $k \geq 1$,

$$P_k(x) = -\frac{k!}{(2\pi i)^k} \sum_{n \neq 0} \frac{e^{2\pi inx}}{n^k}.$$

Bookwork [13 marks]

ii. b. If we set $x = 0$ and recall $P_k(0) = B_k$ for $k \geq 2$, we get

$$B_k = -\frac{k!}{(2\pi i)^k} \sum_{n \neq 0} \frac{1}{n^k}.$$

In the sum we group n and $-n$ together. For each such pair $n > 0$ and $-n$, we have

$$\frac{1}{n^k} + \frac{1}{(-n)^k} = \frac{2}{n^k} \quad \text{if } k \text{ even, } 0 \text{ if } k \text{ is odd.}$$

Hence,

$$B_{2\ell} = -\frac{(2\ell)!}{(2\pi i)^{2\ell}} 2 \sum_{n=1}^{\infty} \frac{1}{n^{2\ell}} = (-1)^{\ell+1} 2 \frac{(2\ell)!}{(2\pi)^{2\ell}} \zeta(2\ell).$$

This rearranges to the stated result. Bookwork [6 marks]

Spring Semester 2014–2015

MATH31072/MATH41072/MATH61072 *Algebraic Topology*

May/June Examination Solutions

A1. (a) A *geometric simplicial surface* is a finite set K of triangles in some $\mathbb{R}^n$ satisfying the following properties.

(i) The intersection condition: Two triangles in K are either (i) disjoint, (ii) intersect in a common vertex, or (iii) intersect in a common edge.

(ii) The connectivity condition: For each pair of vertices there is a path along edges from one to the other.

(iii) The link condition: For each vertex v, the link of the vertex, i.e. the set of edges opposite v in the triangles containing v, form a simple closed polygon.

[5 marks, bookwork]

(b) An *orientation* of a triangle is a cyclic ordering of the vertices. Two triangles with a common edge are *coherently oriented* if the orientations induced on the common edge are opposite. A simplicial surface is *orientable* if all of the triangles can be oriented so that each pair of triangles with a common edge are coherently oriented.

[3 marks, bookwork]

(c) The statement that this is a topological property means that, given two simplicial complexes K_1 and K_2, if the underlying spaces $|K_1|$ and $|K_2|$ are homeomorphic, then K_1 is orientable if and only if K_2 is orientable.

[2 marks, bookwork]
[**Total: 10 marks**]

A2. A *geometric simplicial complex* is a non-empty finite set K of simplices in some Euclidean space $\mathbb{R}^n$ such that

(i) **the face condition:** if $\sigma \in K$ and $\tau \prec \sigma$ then $\tau \in K$;

(ii) **the intersection condition:** if σ_1 and $\sigma_2 \in K$ then $\sigma_1 \cap \sigma_2 \in K$ and $\sigma_1 \cap \sigma_2 \prec \sigma_1$, $\sigma_1 \cap \sigma_2 \prec \sigma_2$.

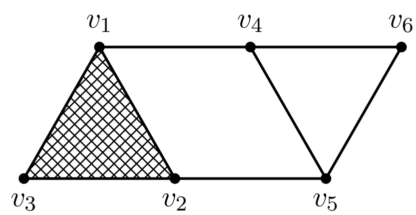
[2 marks, bookwork]

The *underlying space* $|K|$ of a simplicial complex K is given by

$$|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n$$

with the subspace topology. [1 mark, bookwork]

A realization of the given abstract complex as a geometric complex is as follows.



[2 marks, similar to question set]

The *Euler characteristic* of a simplicial complex K is given by the alternating sum

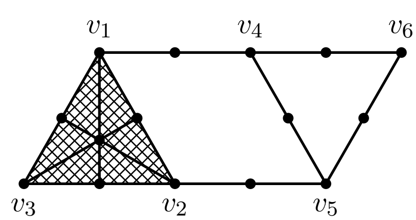
$$\chi(K) = \sum_{r=0}^{\infty} (-1)^r n_r$$

where n_r is the number of simplices of dimension r. [1 mark, bookwork]

In this case, $n_0 = 6$, $n_1 = 8$ and $n_2 = 1$ and so $\chi(K) = 6 - 8 + 1 = -1$.

[1 mark, similar to question set]

The first barycentric subdivision is as follows.



[2 marks, similar to question set]

This also has Euler characteristic -1 since the Euler characteristic is unchanged by barycentric subdivision (or because it is a topological invariant and the underlying space is unchanged) [It can also be found by counting simplices.]. [1 mark, simple application]

[**Total: 10 marks**]

A3. For $r \in \mathbb{Z}$. the r*-chain group of* K, denoted $C_r(K)$, is the free abelian group generated by K_r, the set of (non-empty) oriented r-simplices of K subject to the relation $\sigma + \tau = 0$ whenever σ and τ are the same simplex with the opposite orientations. [2 marks, bookwork]

For each $r \in \mathbb{Z}$ we define the *boundary homomorphism* $d_r \colon C_r(K) \to C_{r-1}(K)$ on the generators of $C_r(K)$ by

$$d_r(\langle v_0, v_1, \ldots, v_r \rangle) = \sum_{i=0}^{r} (-1)^i \langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_r \rangle$$

and then extend linearly. Here $\hat{v}_i$ indicates that this vertex should be omitted.

[2 marks, bookwork]

The kernel of the boundary homomorphism $d_r \colon C_r(K) \to C_{r-1}(K)$ is called the r*-cycle group* of K and is denoted $Z_r(K)$. Thus

$$Z_r(K) = \{ x \in C_r(K) \mid d_r(x) = 0 \}.$$

[1 mark, bookwork]

The image of the boundary homomorphism $d_{r+1} \colon C_{r+1}(K) \to C_r(K)$ is called the r*-boundary group* of K and is denoted $B_r(K)$. Thus

$$B_r(K) = \{ x \in C_r(K) \mid x = d_{r+1}(y) \text{ for some } y \in C_{r+1}(K) \}.$$

[1 mark, bookwork]

In the case of K in Question A.2 we can see that

- $Z_1(K)$ is generated by $x_1 = \langle v_1, v_2 \rangle - \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$, $x_2 = \langle v_1, v_2 \rangle - \langle v_1, v_4 \rangle + \langle v_2, v_5 \rangle - \langle v_4, v_5 \rangle$ and $x_3 = \langle v_4, v_5 \rangle - \langle v_4, v_6 \rangle + \langle v_5, v_6 \rangle$.
- $B_1(K)$ is generated by x_1.

[2 marks, similar to questions set]

The kernel of the homomorphism $Z_1(K) \to \mathbb{Z}^2$ defined by $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \mapsto (\lambda_2, \lambda_3)$ is generated by x_1 and so is $B_1(K)$. Hence by the First Isomorphism Theorem this induces an isomorphism $H_1(K) = Z_1(K)/B_1(K) \cong \mathbb{Z}^2$. [2 marks, similar to questions set]

[**Total: 10 marks**]

A4. (a) The underlying space of $K = \bar{\Delta}^8$ is the 8-simplex Δ^8 which is a convex subset of $\mathbb{R}^9$ and so is contractible. Hence it has the same homology groups as a point:

$$H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

[3 marks, standard example]

(b) For subcomplex L of K, $n_0 = 9$, $n_1 = \binom{9}{2} = 36$, $n_2 = \binom{9}{3} = 84$ and $n_3 = \binom{9}{4} = 126$ and so the Euler characteristic $\chi(L) = 9 - 36 + 84 - 126 = -69$.

[2 marks, similar to example set]

Now L is 3-dimensional and so and so has trivial homology groups in dimensions above 3. In dimensions $0 \leqslant i \leqslant 3$, $C_i(L) = C_i(K)$ with the same boundary homomorphims between these groups. Hence in dimensions $0 \leqslant i \leqslant 2$, $H_i(L) = H_i(K)$. However, in dimension 3, $B_3(L) = 0$ since $C_4(L) = 0$ and so $H_3(L) = Z_3(L)$ a free group of rank β_3, the third Betti number of L.

Now using the formula $\chi(L) = \sum_{i=0}^{3}(-1)^i \beta_i(L)$ we see that $-69 = 1 - \beta_3(L)$ (since $\beta_1(L) = \beta_2(L) = 0$) and so $\beta_3(L) = 70$. Hence

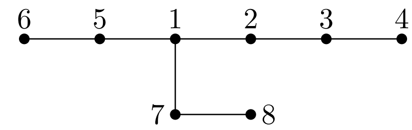
$$H_i(L) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}^{70} & \text{for } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

[5 marks, similar to example set]

[**Total: 10 marks**]

B5. The intersection condition is satisfied automatically since the vertices are linearly independent. [1 mark]

The connectivity condition is satisfied because (for example) the following edges link all of the vertices. [1 mark]

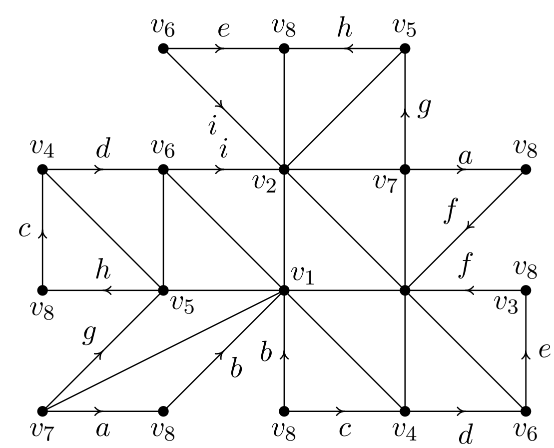


Checking the link condition for v_1 and v_8 we find the following:

These are simple closed polygons. Hence K is a simplicial surface.

[3 marks]

(b) Now identifying edges of the triangles leads to the following polygon with edges to be identified in pairs as indicated.



This is represented by the symbol $abb^{-1}cdeff^{-1}a^{-1}ghe^{-1}ii^{-1}d^{-1}c^{-1}h^{-1}g^{-1}$.

[5 marks]

(c) Reducing this symbol to canonical form using the standard algorithm gives the following.

$$\begin{aligned}
& abb^{-1}cdeff^{-1}a^{-1}ghe^{-1}ii^{-1}d^{-1}c^{-1}h^{-1}g^{-1} \\
\sim\ & \dot{a}cd\dot{e}(\dot{a}^{-1})(gh)\dot{e}^{-1}d^{-1}c^{-1}h^{-1}g^{-1} \quad \text{(cancelling } xx^{-1}\text{)} \\
\sim\ & \dot{a}(cd\dot{e})(gh)\dot{a}^{-1}\dot{e}^{-1}d^{-1}c^{-1}h^{-1}g^{-1} \quad \text{(since } \ldots xUVx^{-1}\ldots \sim \ldots xVUx^{-1}\ldots\text{)} \\
\sim\ & \dot{a}ghcd(\dot{e}\dot{a}^{-1}\dot{e}^{-1})d^{-1}c^{-1}h^{-1}g^{-1} \quad \text{(since } \ldots xUVx^{-1}\ldots \sim \ldots VUx^{-1}\ldots\text{)} \\
\sim\ & (aea^{-1}e^{-1})ghcdd^{-1}c^{-1}h^{-1}g^{-1} \quad \text{(since } xUx^{-1} \text{ commutes with other terms)} \\
\sim\ & xyx^{-1}y^{-1} \quad \text{(cancelling } xx^{-1} \text{ and relabelling).}
\end{aligned}$$

Hence the surface is orientable of genus 1 (the torus).

[5 Marks]

[Total: 15 marks, similar to questions set]

B6. (a) A topological surface is a non-empty Hausdorff second countable topological space S which is locally planar, i.e. each point $x \in X$ lies in an open subset $U \subset X$ which is homeomorphic to an open subset of the plane $\mathbb{R}^2$ with the usual topology.

Suppose that S_1 and S_2 are non-empty path-connected topological surfaces. Choose subspaces $V_1 \subset S_1$ and $V_2 \subset S_2$ which are homeomorphic to the open disc $B_1(\mathbf{0}) \subset \mathbb{R}^2$ by homeomorphisms

$$\phi_i \colon B_1(\mathbf{0}) \to V_i \quad \text{for } i = 1 \text{ and } i = 2$$

We form the connected sum $S_1 \# S_2$ by removing the interiors of smaller discs, i.e. $\phi_i\big(B^2_{1/2}(\mathbf{0})\big)$ and gluing along the boundary circles. More precisely, it is the quotient space of the disjoint union

$$S = \Big[\Big(S_1 - \phi_1\big(B^2_{1/2}(\mathbf{0})\big)\Big) \sqcup \Big(S_2 - \phi_2\big(B^2_{1/2}(\mathbf{0})\big)\Big)\Big] \Big/\!\sim$$

where $\phi_1(\boldsymbol{u}) \sim \phi_2(\boldsymbol{u})$ for $\boldsymbol{u} \in B^2_1(\mathbf{0})$ with $|\boldsymbol{u}| = 1/2$.

[5 marks]

(b) A triangulation of a path-connected compact surface S is a homeomorphism $h \colon |K| \to S$ where $|K|$ is the underlying space of a simplicial surface K.

Given such a triangulation of a surface S, then the Euler characteristic of S, $\chi(S)$, is defined by $\chi(S) = v - e + f$ where v is the number of vertices in K, e is the number of edges in K and f is the number of triangles in K. This can be shown to be a topological invariant.

[3 marks]

(c) Suppose that S_1 and S_2 are two such surfaces with $|K_1| \cong S_1$ and $|K_2| \cong S_2$ then we can form K such that $|K| \cong S_1 \# S_2$ by removing a triangle from each of K_1 and K_2 and identifying the corresponding edges and vertices of these two triangles. Then $f = f_1 + f_2 - 2$ (two triangles removed), $e = e_1 + e_2 - 3$ (three pairs identified), $v = v_1 + v_2 - 3$ (three pairs identified). Thus $\chi(K) = (v_1+v_2-3)-(e_1+e_2-3)+(f_1+f_2-2) = (v_1-e_1+f_1)+(v_2-e_2+f_2)-2 = \chi(S_1)+\chi(S_2)-2$.
$\chi(S^2) = 2$.

Hence, by induction on g, $\chi(T_g) = 2 - 2g$ since $\chi(T_1) = 0$ and, for $k \geqslant 1$, if the result holds for $g = k$, $\chi(T_k) = 2 - 2k$ and so $\chi(T_{k+1}) = \chi(T_k \# T_1) = (2-2k) + 0 - 2 = 2 - 2(k+1)$ and so the result holds for $g = k + 1$.

Similarly, $\chi(P_g) = 2 - g$ since $\chi(P_1) = 1$ and, for $k \geqslant 1$, if the result holds for $g = k$, $\chi(P_k) = 2 - k$ and so $\chi(P_{k+1}) = \chi(P_k \# P_1) = (2-k) + 1 - 2 = 2 - (k+1)$ and so the result holds for $g = k+1$.

[5 marks]

(d) The Euler characteristic is used in the proof of the classification theorem to help distinguish the spaces in the list.

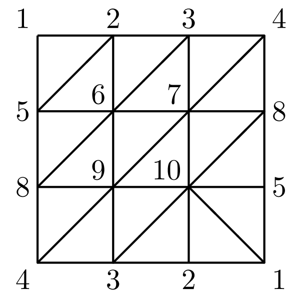
The Euler characteristic shows that the surfaces T_g for $g \geqslant 1$ are all topologically distinct from each other and from S^2, and the surfaces P_g for $g \geqslant 1$ are all topologically distinct from each other and from S^2. However, for even numbers $2 - 2k$ ($k > 0$) there are two surfaces in the list, T_k and P_{2k}, with this Euler characteristic.

[2 marks]

[**Total: 15 marks**]

[This is a summary of coursework but requires the student to have a good overview of the first three sections of the course. The inductive proof in (c) was left as an exercise.]

B7. Write v_i for the ith standard basis vector in $\mathbb{R}^{10}$, $1 \leqslant i \leqslant 9$. Let K be the set of 2-simplices $\langle v_i, v_j, v_k\rangle$ where (i, j, k) are the vertices of a triangle in the triangulation of the unit square I^2 shown below together with their faces. Then K is a simplicial complex with underlying space $|K|$ homeomorphic to the projective plane.



The intersection condition is automatic since the vertices are linearly independent vectors and the face condition is automatic by definition.

Now we can define a continuous function $f\colon I^2 \to |K|$ by mapping the point i in the unit square (in the above picture) by $i \mapsto v_i$ and extending linearly over each triangle. This is continuous by the Gluing Lemma (since the triangles are all closed subsets of I^2) and induces a continuous bijection $F\colon I^2/\sim \;\to |K|$ which is therefore a homeomorphism where $\sim$ is the equivalence relation given by $(s, 0) \sim (s-1, 1)$ and $(0, t) \sim (1, 1-t)$ which is known to give the projective plane.

[6 marks, similar to bookwork]

Since K is clearly connected $H_0(K) \cong \mathbb{Z}$ and since K is 2-dimensional $H_i(K) = 0$ for $i > 2$ and $i < 0$.

To find $Z_1(K)$ notice that if $x \in Z_1(K)$ then $x \sim x'$ where x' only involves edges corresponding to the edges of the template together with three 'internal' edges, say $\langle v_5, v_6\rangle$, $\langle v_7, v_8\rangle$ and $\langle v_2, v_{10}\rangle$. Since other edges can be eliminated. For example $\langle v_2, v_5\rangle \sim \langle v_1, v_5\rangle - \langle v_1, v_2\rangle$ since $d_2\langle v_1, v_2, v_5\rangle = \langle v_2, v_5\rangle - \langle v_1, v_5\rangle + \langle v_1, v_2\rangle \sim 0$. However, since $x \in Z_1(K)$, $x' \in Z_1(K)$ and so x' cannot involve these internal edges since they have vertices which would cancel out on taking the boundary.

Considering the edges corresponding the boundary of the template we see that the cycles containing these edge are generated by

$$x = \langle v_1, v_2\rangle + \langle v_2, v_3\rangle + \langle v_3, v_4\rangle + \langle v_4, v_8\rangle - \langle v_5, v_8\rangle - \langle v_1, v_5\rangle.$$

Let V be the subgroup of $C_1(K)$ generated by x. Then $Z_1(K) = V + B_1(K)$.

Hence $H_1(K) = Z_1(K)/B_1(K) = (B_1(K) + V)/B_1(K) = V/\big(V \cap B_1(K)\big)$ by the Second Isomorphism Theorem.

If $d_2(z) \in V$ then z must be a multiple of $y = \langle v_1, v_2, v_5\rangle + \dots$ (all the 2-simplices oriented clockwise). But $d_2(y) = 2x$. Hence $V \cap B_1(K) \cong \mathbb{Z}$ generated by $2x$. Hence $H_1(K) \cong \mathbb{Z}_2$ generated by $[x]$.

For $z \in Z_2(K)$, z must be a multiple of y but since $d_2(y) \neq 0$ if follows that $Z_2(K) = 0$ and $H_2(K) = 0$.

Conclusion; $H_i(K) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}_2 & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$ [9 marks, example set]

[**Total: 15 marks**]

B8. (a) Two continuous functions of topological spaces $f_0\colon X \to Y$ and $f_1\colon X \to Y$ are *homotopic*, written $f_0 \simeq f_1$, if there is a continuous map $H\colon X \times I \to Y$ such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$. We call H a *homotopy* between f_0 and f_1 and write $H\colon f_0 \simeq f_1\colon X \to Y$. [2 marks, bookwork]

There are three conditions for an equivalence relation.

reflexivity: Given a continuous function $f\colon X \to Y$ then $f \simeq f$. A homotopy is given by $H(x,t) = f(x)$.

symmetry: Given homotopic functions $f_0 \simeq f_1\colon X \to Y$ then $f_1 \simeq f_0$. Given a homotopy $H\colon f_0 \simeq f_1$ then a homotopy $K\colon f_1 \simeq f_0$ is given by $K(x,t) = H(x,1-t)$.

transitivity: Given homotopic functions $f_0 \simeq f_1\colon X \to Y$ and $f_1 \simeq f_2\colon X \to Y$ then $f_0 \simeq f_2\colon X \to Y$. Given homotopies $H\colon f_0 \simeq f_1$ and $K\colon f_1 \simeq f_2$ then a homotopy $L\colon f_0 \simeq f_2$ is given by

$$L(x,t) = \begin{cases} H(x,2t) & \text{for } 0 \leqslant t \leqslant 1/2, \\ K(x,2t-1) & \text{for } 1/2 \leqslant t \leqslant 1. \end{cases}$$

This is well-defined since $H(x,1) = f_1(x) = K(x,0)$ and is continuous by the Gluing Lemma.

Hence homotopy is an equivalence relation. [5 marks, exercise set]

(b) A continuous function $f\colon X \to Y$ is a *homotopy equivalence* when there it has a *homotopy inverse* $g\colon Y \to X$ which means that $g \circ f \simeq I_X\colon X \to X$, the identity map, and $f \circ g \simeq I_Y\colon Y \to Y$. In this case we say that X and Y are *homotopy equivalent* spaces and denote this by $X \equiv Y$ (or sometimes $X \simeq Y$). [3 marks, bookwork]

Suppose that X and Y are homotopy equivalent spaces with maps as above. Suppose that X is path-connected. To see that Y is path-connected, let y_0, $y_1 \in Y$. Then since X is path-connected there is a path $\sigma\colon [0,1] \to X$ from $g(y_0)$ to $g(y_1)$. Hence $f \circ \sigma\colon [0,1] \to Y$ is a path in Y from $f\bigl(g(y_0)\bigr)$ to $f\bigl(g(y_1)\bigr)$.

Let $H\colon f \circ g \simeq I_Y$. Then $\sigma_0(t) = H(y_0,t)$ gives a path in Y from $f\bigl(g(y_0)\bigr)$ to y_0 and $\sigma_1(t) = H(y_1,t)$ gives a path in Y from $f\bigl(g(y_1)\bigr)$ to y_1. The product of the three paths $\overline{\sigma_0}$ (reverse path), σ and σ_1 gives a path in Y from y_0 to y_1. Hence Y is path-connected.

In just the same way, reversing the roles of f and g, if Y is path-connected then so is X

[5 marks, similar to example set]

[**Total: 15 marks**]

C9. (a) A *p-symmetry* of a topological surface S is a homeomorphism $f\colon S \to S$ such that $f^p = f \circ \cdots \circ f = 1$, the identity, and $f \neq 1$.

A *fixed point* of a p-symmetry $f\colon S \to S$ is a point $x \in S$ such that $f(x) = x$.

Let $f\colon S^2 \to S^2$ be a rotation about a diameter through an angle $2\pi/p$. This is a p-symmetry with two fixed points (at the ends of the diameter). The map f induces $F\colon P^2 = S^2/(x \sim \pm x) \to P^2$ with one fixed point.

[5 marks, bookwork]

(b) Let U be an open set as in the question. Since S is a surface there is a closed set $A_1 \subset U$ such that $A_1 \cong D^2$. Choose a closed set $A_2 \subset P^2$ such that $A_2 \cong D^2$. Then we can form $S' = S\#P_p$ as the connected sum of S with p copies of P^2 by removing the interiors of the sets $f^i(A_1)$, $0 \leqslant i \leqslant p-1$, from S, taking p copies of P^2 with the interior of A_2 removed and identifying the boundary circles. Then the p-symmetry $f\colon S \to S$ extends to a p-symmetry $f'\colon S' \to S'$ which cyclically permutes the p projective planes. Since f is free so is f'.

[5 marks, problem set, similar to bookwork]

(c) We have shown that $P^2 = P_1$ has a p-symmetry with one fixed point. So, if p divides $g-1$, then $g - 1 = pr$, for some r and so $g = 1 + pr$. So applying the above result r times gives a p-symmetry on P_g with a single fixed point.

[3 marks, problem set, similar to bookwork]

(d) Suppose that $f\colon S \to S$ is a p-symmetry on a closed surface S with a single fixed point a. The we can define an equivalence relation on S by $x \sim f^i(x)$ for all $x \in S$, $i \geqslant 0$. The quotient space $S' = S/{\sim}$ is also a closed surface. In this case, under the quotient map $q\colon S \to S'$, each point of S' has precisely p preimages apart from the point $[a] = \{a\} \in S'$ which has only one preimage. Choose a triangulation $|K'| \cong S'$ so that the the point $[a] = \{a\} \in S'$ corresponds to a vertex of K'. Then using the quotient map q we can construct a simplicial surface K such that $|K| \cong S$ in such a way that the map $|K| \to |K'|$ corresponding to q maps vertices to vertices, edges to edges and triangles to triangles. Hence $v(K) = pv(k') - (p-1)$ (because of each vertex of K' corresponds to p vertices of K apart from $[a]$ which corresponds to a single vertex of K), $e(K) = pe(K')$ and $f(K) = pf(K')$. Hence $\chi(K) = p\chi(K') - (p-1)$.

Now, if $S = P_g$, $\chi(K) = 2 - g$ and so $2 - g = p(\chi(K') - 1) + 1$ which gives $g - 1 = p\big(1 - \chi(K')\big)$ so that p divides $g - 1$ as required.

[7 marks, problem set, similar to bookwork]

[**Total: 20 marks**]

C10. (a) Suppose that the triangulation has e edges and f triangles. Then we know the following.

(i) $v - e + f = \chi$ (from the definition of the Euler characteristic).

(ii) $e \leqslant v(v-1)/2$ (since the maximum number of edges has every pair of vertices joined by an edge).

(iii) $2e = 3f$ (since each triangle has three edges and each edge is an edge of two triangles).

Then $\chi = v - e + f$ (by (i)) $= v - e/3$ (by (iii)) $\geqslant v - v(v-1)/6$ (by (ii)) $= (7v - v^2)/6$. Hence $v^2 - 7v + 6\chi \geqslant 0$.
Let the roots of the equation $v^2 - 7v + 6\chi = 0$ be $v_1 < v_2$. Then $v^2 - 7v + 6\chi = (v - v_1)(v - v_2) \geqslant 0 \Leftrightarrow v \leqslant v_1$ or $v \geqslant v_2$. From the usual formula the roots are given by $\big(7 \pm \sqrt{49 - 24\chi}\big)/2$ and so $v \geqslant (7 + \sqrt{49 - 24\chi})/2$ or $v \leqslant (7 - \sqrt{49 - 24\chi})/2$.
Since $v \geqslant 3$ (a triangulation includes at least one triangle), if $v \leqslant \big(7 - \sqrt{49 - 24\chi}\big)/2$, $3 \leqslant \big(7 - \sqrt{49 - 24\chi}\big)/2$ which gives $\chi \geqslant 2$ and so $\chi = 2$ (since the Euler characteristic of a surface is at most 2. This gives $v \leqslant 3$ and so $v = 3$ which means that $e = 3$ and $f = 2$. This corresponds to two triangles with the same edges and vertices which would violate the intersection condition. So this case does not arise and we must have $v \geqslant (7 + \sqrt{49 - 24\chi})/2$, as required.

[12 marks]

(b) If $v = \big(7 - \sqrt{49 + 24\chi}\big)/2$ then $v^2 - 7v + 6\chi = 0$ and so (using the equation $\chi = v - e/3$ obtained above) $e = v(v-1)/2$ which means that there is an edge between each pair of vertices and so the 1-skeleton of the triangulation must be the complete graph on v vertices.

[3 marks]

[**Total; 15 marks**]

[**These proofs were outlined in the notes with details left as exercises.**]

C11. (a) A *triangulable pair* of spaces (X, A) is a topological space X with a subspace A such that there is a homeomorphism $h\colon X \to |K|$, the underlying space of a simplicial complex K, with $h(A) = |L|$ the underlying space of a subcomplex L of K. [1 mark, bookwork]

A reduced homology theory assigns to each non-empty triangulable space X a sequence of groups $\tilde{H}_n(X)$ (for $n \in \mathbb{Z}$) and for each continuous map of triangulable spaces $f\colon X \to Y$ a sequence of homomorphisms $f_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$ such that

(i) for continuous functions $f\colon X \to Y$ and $g\colon Y \to Z$, $g_* \circ f_* = (g \circ f)_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Z)$ for all n;

(ii) for the identity map $I\colon X \to X$, $I_* = I\colon \tilde{H}_n(X) \to \tilde{H}_n(X)$ the identity map for all n;

(iii) [homotopy axiom] for homotopic maps $f \simeq g\colon X \to Y$, $f_* = g_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$ for all n;

(v) [exactness axiom] for any triangulable pair (X, A) there are boundary homomorphisms $\partial\colon \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A)$ for all n which fit into a long exact sequence

$$\ldots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \ldots$$

and such that for any continuous function of triangulable pairs $f\colon (X, A) \to (Y, B)$ inducing a map of quotient spaces $\bar{f}\colon X/A \to Y/B$ the following diagram commutes for all n;

$$\begin{array}{ccc} \tilde{H}_n(X/A) & \xrightarrow{\bar{f}_*} & \tilde{H}_n(Y/B) \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ \tilde{H}_{n-1}(A) & \xrightarrow{f_*} & \tilde{H}_{n-1}(B) \end{array}$$

(vi) [dimension axiom] $\tilde{H}_0(S^0) \cong \mathbb{Z}$ and $\tilde{H}_n(S^0) = 0$ for all $n \neq 0$.

[7 marks, bookwork]

(c) Suppose that $f\colon X \to Y$ is a homotopy equivalence with homotopy inverse $g\colon Y \to X$. Then

$$g_* \circ f_* = (g \circ f)_* \text{ (using (i))} \ = I_* \text{ (using (iii))} \ = I\colon \tilde{H}_n(X) \to \tilde{H}_n(X) \text{ (by (ii))}$$

and similarly $f_* \circ g_*\colon I\colon \tilde{H}_n(Y) \to \tilde{H}_n(X)$ so that $f_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$ is an isomorphism.
[2 marks, exercise set]

(d) Now consider the pair (D^n, S^{n-1}) for which $D^n/S^{n-1} \cong S^n$. Then the exactness axiom gives the long exact sequence

$$\ldots \to \tilde{H}_i(D^n) \xrightarrow{q_*} \tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1}) \xrightarrow{i_*} \tilde{H}_{i-1}(D^n) \ldots$$

The space D^n is contractible (homotopy equivalent to a point) and so by the above all of its homology groups are trivial. Hence from this exact sequence we see that the boundary homomorphisms

$$\partial\colon \tilde{H}_i(S^n) \to \tilde{H}_{i-1}(S^{n-1})$$

are all isomorphisms. Hence, iterating these maps and using the dimension axiom we see that

$$\tilde{H}_i(S^n) \cong \tilde{H}_{i-n}(S^0) \cong \mathbb{Z} \text{ for } i = n, 0 \text{ for } i \neq n.$$

[5 marks, exercise set]
[Total: 15 marks]

pje\teaching\MATH41072\jun15sol.tex 6 March 2015

Solutions

1

(1a) **(3+2+3)**.

Riemannian metric G on n-dimensional manifold M^n defines for every point $\mathbf{p} \in M$ the scalar product of tangent vectors in the tangent space $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$. It means that in every coordinate system $(x^1, \dots, x^n)$ a metric $G = g_{ik}dx^i dx^k$ is defined by a matrix valued function $g_{ik}(x)$ $(i = 1, \dots, n; k = 1, \dots n)$ such that for any two vectors $\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i}, \mathbf{B} = B^i(x)\frac{\partial}{\partial x^i}$, tangent to the manifold M at the point $\mathbf{p}$ with coordinates $x = (x^1, x^2, \dots, x^n)$ $(\mathbf{A}, \mathbf{B} \in T_{\mathbf{p}}M)$ the scalar product is equal to:

$$\langle \mathbf{A}, \mathbf{B} \rangle_G \big|_{\mathbf{p}} = G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}} = A^i(x) g_{ik}(x) B^k(x)\,,$$

where

1. $G(\mathbf{A}, \mathbf{B}) = G(\mathbf{B}, \mathbf{A})$, i.e. $g_{ik}(x) = g_{ki}(x)$ (symmetricity condition)
2. $G(\mathbf{A}, \mathbf{A}) > 0$ if $\mathbf{A} \neq \mathbf{0}$, i.e.
   $g_{ik}(x)u^i u^k \geq 0$, $g_{ik}(x)u^i u^k = 0$ iff $u^1 = \ldots = u^n = 0$ (positive-definiteness)
3. $G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}=x}$,i.e. $g_{ik}(x)$ are smooth functions. $\blacksquare$

For arbitrary x and arbitrary index i, $i = 1, \dots, n$ consider non-zero vector $\mathbf{A} \in T_x M$ such that its i-th component $A^i = 1$ and all other components are equal to zero. The positive-definiteness condition means that $G(\mathbf{A}, \mathbf{A}) = A^i g_{ik} A^k = g_{ii} > 0$.

The length of the vector $\mathbf{X} = \partial_u + t\partial_v \neq 0$ is equal to $G(\mathbf{X}, \mathbf{X}) = c^2 + t + t^2 > 0$ due to positive-definitness. We see that $t^2 + t + c^2 = (t + \frac{1}{2})^2 + \left(c - \frac{1}{4}\right) > 0$ for all t. Hence $c > \frac{1}{4}$.

(1b) **(2+2)** n-dimensional Riemannian manifold (M, G) is locally Euclidean Riemannian manifold, if for every point $\mathbf{p} \in M$ there exists an open neighboorhood D (domain) containing this point, $\mathbf{p} \in D$ such that D is isometric to a domain in Euclidean plane, i.e. in a vicinity of every point $\mathbf{p}$ there exist local coordinates $u^1, \dots, u^n$ such that Riemannian metric G in these coordinates has an appearance

$$G = du^i \delta_{ik} du^k = (du^1)^2 + \ldots + (du^n)^2\,.$$

For cylindrical surface $x^2 + y^2 = 4$ consider parametrisation $x = 2\cos\varphi, y = 2\sin\varphi, z = h$. Then induced Riemannian metric is $G = (dx^2 + dy^2 + dz^2)_{x=2\cos\varphi, y=2\sin\varphi, z=h} = 4d\varphi^2 + dh^2$. In a vicinity of every point one can consider new coordinates $u = h, v = 2\varphi$ then $du^2 + dv^2 = dh^2 + 4d\varphi^2$. We see that in coordinates u, v metric is Euclidean. Hence surface of cylinder as Riemannian manifold with induced Rimeannian metric is locally Euclidean.

(1c) **(2+3+1+2)** A volume form on a Riemannian manifold M^n with metric $G = g_{ik}dx^i dx^k$ is $\sqrt{\det g}dx^1 dx^2 \dots dx^n$.

For Riemannian metric $G = \frac{dx^2 + dy^2}{(1+x^2+y^2)^2}$, $\det G = \det \begin{pmatrix} \frac{1}{(1+x^2+y^2)^2} & 0 \\ 0 & \frac{1}{(1+x^2+y^2)^2} \end{pmatrix} = \frac{1}{(1+x^2+y^2)^4}$,

Hence the area of a domain is equal to

$$\int_{x^2+y^2 \leq a^2} \sqrt{\det G} dx dy = \int_{x^2+y^2 \leq a^2} \frac{1}{(1+x^2+y^2)^2} dx dy =$$

$$\int_{r\le a}\int_0^{2\pi}\frac{1}{(1+r^2)^2}rdrd\varphi = 2\pi\int_{r\le a}\frac{1}{(1+r^2)^2}rdr = \pi\int_{u\le a^2}\frac{1}{(1+u)^2}du = -\pi\frac{1}{1+u}\Big|_0^a = \pi\left(1-\frac{1}{1+a}\right)$$

Taking $a \to \infty$ we see that $S_a = \pi\left(1-\frac{1}{1+a}\right) \to \pi$:

Area of all the plane is equal to π. on the other hand the area of Euclidean plane with standard Euclidean metric is equal to infinity. Hence they are not isometric.

2

(**2a**) (**2+1+3**).

Affine connection on M is the *operation* ∇ which assigns to every vector field $\mathbf{X}$ a linear map $\nabla_{\mathbf{X}}$ on the space of vector fields: $\nabla_{\mathbf{X}}(\lambda\mathbf{Y}+\mu\mathbf{Z}) = \lambda\nabla_{\mathbf{X}}\mathbf{Y}+\mu\nabla_{\mathbf{X}}\mathbf{Z}$ $(\lambda,\mu\in\mathbf{R})$, which satisfies the following additional conditions:

1. For arbitrary (smooth) functions f, g on M

$$\nabla_{f\mathbf{X}+\mathbf{g}\mathbf{Y}}(\mathbf{Z}) = f\nabla_{\mathbf{X}}(\mathbf{Z}) + g\nabla_{\mathbf{Y}}(\mathbf{Z}) \qquad (C(M)\text{-linearity})$$

2 For arbitrary function f

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = (\nabla_{\mathbf{X}}f)\,\mathbf{Y} + f\nabla_{\mathbf{X}}(\mathbf{Y}) \qquad (\text{Leibnitz rule})$$

($\nabla_{\mathbf{X}}f$ is just usual derivative of a function f along vector field: $\nabla_{\mathbf{X}}f = \partial_{\mathbf{X}}f$.) ■

Canonical flat connection ∇^{can} is a connection which Christoffel symbols vanish in Cartesian coordinates.

$$\Gamma^r_{rr} = \frac{\partial^2 x}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r^2}\frac{\partial r}{\partial y} = 0$$

since $x_{rr} = y_{rr} = 0$ and

$$\Gamma^r_{\varphi\varphi} = \frac{\partial^2 x}{\partial\varphi^2}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial\varphi^2}\frac{\partial r}{\partial y} = -r\cos\varphi\cdot\frac{x}{\sqrt{x^2+y^2}} - r\cos\varphi\cdot\frac{x}{\sqrt{x^2+y^2}} = -r,$$

(**2b**) (**3+3**). Let M be a surface embedded in $\mathbf{E}^3$. Let $\nabla^{\text{can.flat}}$ be canonical flat connection in $\mathbf{E}^3$ (It is defined by the condition that its Christoffel symbols vanish in Cartesian coordinates on $\mathbf{E}^3$: $\nabla^{\text{can.flat}}_{\mathbf{X}}Y = X^i\frac{\partial Y^m}{\partial x^i}\frac{\partial}{\partial x^m}$.) The induced connection $\nabla^{(M)}$ is defined in the following way: for arbitrary vector fields $\mathbf{X}, \mathbf{Y}$ tangent to the surface M, $\nabla^M_{\mathbf{X}}\mathbf{Y}$ equals to the projection on the tangent space of the vector field $\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}$:

$$\nabla^M_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}},$$

where $\mathbf{A}_{\text{tangent}}$ is a projection of the vector A attached at the point of the surface on the tangent space: $\mathbf{A}_{\text{tangent}} = \mathbf{A} - \mathbf{A}_\perp$, where $\mathbf{A}_\perp = \mathbf{n}(\mathbf{A},\mathbf{n})$. ($\mathbf{n}$ is normal unit vector field to the surface.)

For saddle $\partial_u = \mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix}$, $\partial_v = \mathbf{r}_u = \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix}$,

Calculate $\nabla^M_u \partial_u, \nabla^M_u \partial_v, \nabla^M_v \partial_u, \nabla^M_v \partial_v$ at the point $u = v = 0$.

$\nabla^{\text{can.flat}}_{\partial_u} \partial_u = \mathbf{r}_{uu} = \nabla^{\text{can.flat}}_{\partial_v} \partial_u = \mathbf{r}_{vv} = 0$ hence its projection on the surface also vanishes. Thus we see that $\nabla^M_u \partial_u = \nabla^M_v \partial_v = 0$ (at all points of saddle)

$\nabla^{\text{can.flat}}_{\partial_u} \partial_v = \nabla^{\text{can.flat}}_{\partial_v} \partial_u = \mathbf{r}_{uv} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. This vector is orthogonal to vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ at the point $u = v = 0$: $\partial_u |_{u=v=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\partial_v |_{u=v=0} = \mathbf{r}_u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Hence its projection on M vanishes: $\nabla^M_u \partial_v, \nabla^M_v \partial_u = 0$ too.

(**2c**) (**2+3+3**). Let M be a Riemannian manifold with metric $G = g_{ik} dx^i dx^k$.

Christoffel symbols of Levi-Civita connection have the following appearance:

$$\Gamma^m_{ik}(x) = \frac{1}{2} g^{mn}(x) \left(\frac{\partial g_{in}(x)}{\partial x^k} + \frac{\partial g_{kn}(x)}{\partial x^i} - \frac{\partial g_{ik}(x)}{\partial x^n} \right) . \blacksquare \tag{1}$$

We have that $G = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$, $G^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{u} \end{pmatrix}$;

$$\Gamma^u_{vv} = \frac{1}{2} g^{uu} \left(-\partial g_{\frac{vv}{\partial u}} \right) = -\frac{1}{2} 2u = -u \,.$$

These coordinates look like 'polar coordinates' one can consider new coordinates $u' = u \cos v, v' = u \sin v$ and $du'^2 + dv'^2 = du^2 + dv^2$. In these new coordinates due to Levi-Civita formula Chrsitoffel symbols vanish.

3

(**3a**) (**2+2+3**).

A geodesic on Riemannian manifold M is a parameterised curve $x^i = x^i(t)$ such that velocity vector is covariantly constant with respect to parallel transport along the curve.

We say that vector is covariantly constant on the curve if it remains parallel at all the points of the curve. $\blacksquare$

Parallel transport is defined with respect to the Levi-Civita connection of the Riemannian manifold. This means that

$$\nabla_{\mathbf{v}} \mathbf{v} = \frac{\nabla \mathbf{v}}{dt} = \frac{dv^i(t)}{dt} + v^k(t) \Gamma^i_{km}(x^i(t)) v^m(t) = 0 \,, \text{where } v^i(t) = \tfrac{dx^i(t)}{dt} , \tag{2}$$

where Γ^i_{km} is Levi-Civita connection.

Consider cylindrical surface $\mathbf{r}(\varphi, h)$: $x = a \cos \varphi, y = a \sin \varphi, z = h$. Induced Riemannian metric is $G = dh^2 + a^2 d\varphi^2$. The Christoffel symbols of Levi-Civita connection in coordinates (h, φ) obviously vanish. The differential equation for geodesics becomes: $\frac{d^2\varphi(t)}{dt^2} = 0$, $\frac{d^2h(t)}{dt^2} = 0$ i.e. $\varphi(t) = \varphi_0 + \Omega t$ and $h = h_0 + v^i t$. This is equations of the helix. In the case $v = 0$ helix becomes the circle. In the case if $\Omega = 0$, helix becomes vertical line.

(3b) (1+1+3+3)) A Lagrangian L of the "free" particle on Riemannian manifold with metric $G = g_{ik}dx^i dx^k$ is a function on tangent vectors which is expressed via metric in the following way: $L(x, \dot{x}) = \frac{1}{2}g_{ik}(x)\dot{x}^i\dot{x}^k$. $\blacksquare$

Euler-Lagrange second order differential equations $\frac{\partial L}{\partial x^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)$ for the Lagrangian $L(x, \dot{x})$ of the "free" particle on the Riemannian manifold are equivalent to the second order differential equations (2) for parameterised geodesics for this Riemannian manifold.

$\blacksquare$

Euler-Lagrange equations for Lagrangian of free particle on the sphere $L = R^2\frac{\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2}{2}$ are:

$$\frac{\partial L}{\partial \varphi} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{d}{dt}\left(R^2\sin^2\theta\dot{\varphi}\right) = R^2\sin^2\theta\ddot{\varphi} - 2R^2\sin\theta\cos\theta\dot{\theta}\dot{\varphi} \text{ i.e. } \quad \ddot{\varphi} + 2\mathrm{cotan}\theta\ \dot{\theta}\dot{\varphi} = 0\,,$$

$$\frac{\partial L}{\partial \theta} = R^2\sin\theta\cos\theta\dot{\varphi}^2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = R^2\ddot{\theta}, \text{ i.e. } \quad \ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 = 0\,.$$

Comparing these equations with equations for geodesics: $\ddot{x}^i - \dot{x}^k\Gamma^i_{km}\dot{x}^m = 0$ ($i = 1, 2$, $x^1 = \theta, x^2 = \varphi$) we come to

$$\Gamma^\varphi_{\varphi\varphi} = \Gamma^\varphi_{\theta\theta} = 0, \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \mathrm{cotan}\theta, \Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0, \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta. \quad \blacksquare$$

The first Euler-Lagrange equation for geodesic, $\frac{\partial L}{\partial \varphi} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{d}{dt}\left(R^2\sin^2\theta\dot{\varphi}\right)$ implies that $\sin^2\theta\dot{\varphi}$ is integral of motion, i.e. it is preserved on geodesics.

(**3c**) (**5**).

Since semicircle C is geodesic and vector $\mathbf{X}_0 = \partial_x$ is tangent to C, i.e. it is proportional to velocity vector at the point A, then during parallel transport vector $\mathbf{X}(t)$ remains proportional to velocity vector. Velocity vector at the point B. is orthogonal to vector $(1, \sqrt{3})$. Hence it is proportional to radius-vector $\sqrt{3}\partial_x - \partial_y$ (Lobachevsky metric is conformally Euclidean hence orthogonality is the same). We see that vector X_1 attached at the point B is proportional to vector $\sqrt{3}\partial_x - \partial_y$, $\mathbf{X}_1 = k(\sqrt{3}, -k)$. On the other hand during parallel transport its length is not changed, since the connection is Levi-Civita connection. We have

$$\langle \mathbf{X}_0, \mathbf{X}_0 \rangle_A = \langle \partial_x, \partial_r x \rangle_A = \frac{1}{4}$$

$$\langle \mathbf{X}_1, \mathbf{X}_1 \rangle_B = \langle k\sqrt{3}\partial_x - k\partial_y, k\sqrt{3}\partial_x - \partial_y \rangle_B = \frac{3k^2 + k^2}{3} = \frac{4k^2}{3}$$

We have $\frac{3k^2}{4} = \frac{1}{4}$, hence $k = \frac{\sqrt{3}}{16}$ and $\mathbf{X}_1 = \frac{\sqrt{3}}{16}(\sqrt{\partial}_x - \partial_y)$.

4

4a (2+3+3). Perform calculations for cone $\mathbf{r}_h = \begin{pmatrix} 2\cos\varphi \\ 2\sin\varphi \\ 1 \end{pmatrix}$, $\mathbf{r}_\varphi = \begin{pmatrix} -2h\sin\varphi \\ 2h\cos\varphi \\ 0 \end{pmatrix}$.

We take $\mathbf{e} = \frac{\mathbf{r}_h}{|\mathbf{r}_h|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2\cos\varphi \\ 2\sin\varphi \\ 1 \end{pmatrix}$ and $\mathbf{f} = \frac{\mathbf{r}_\varphi}{|\mathbf{r}_\varphi|} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$. Vectors $\mathbf{e}, \mathbf{f}$ are unit tangent vectors and they are orthogonal to each other.

The vector $\mathbf{n} = \mathbf{e} \times \mathbf{f} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2\cos\varphi \\ 2\sin\varphi \\ 1 \end{pmatrix} \times \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \\ 1 \end{pmatrix}$ is a unit vector which is orthogonal to the cone.
Calculate in derivation formulae $d\mathbf{e}$ and $d\mathbf{n}$ and expand this vector-valued 1-form over $\mathbf{e}, \mathbf{f}$:

$$d\mathbf{e} = \frac{1}{\sqrt{5}} d \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} d\varphi = \frac{d\varphi}{\sqrt{5}} \mathbf{f}\,,$$

$$d\mathbf{n} = \frac{1}{\sqrt{2}} d \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 1 \end{pmatrix} d\varphi = -\frac{d\varphi}{\sqrt{5}} \mathbf{f}\,.$$

We see that 1-form $a = \frac{d\varphi}{\sqrt{5}}$, $b = 0$ and 1-form $-c = -\frac{d\varphi}{\sqrt{5}}$. Hence $a = c = \frac{d\varphi}{\sqrt{5}}$, $b = 0$.

Let S be the shape (Weingarten) operator: $S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n}$ for an arbitrary tangent vector $\mathbf{X}$. From the derivation equation $d\mathbf{n} = -\frac{d\varphi}{\sqrt{5}}\mathbf{f}$ it follows that $S\mathbf{X} = -d\mathbf{n}(\mathbf{X}) = \frac{d\varphi(\mathbf{X})}{\sqrt{5}}\mathbf{f}$. In particularly it means that for basic vectors $\mathbf{e}, \mathbf{f}$ we have

$$S\mathbf{e} = \frac{d\varphi(\mathbf{e})}{\sqrt{2}}\mathbf{f} = \frac{1}{\sqrt{5}} d\varphi \left(\frac{\mathbf{r}_h}{\sqrt{5}}\right) = 0, \quad S\mathbf{f} = \frac{d\varphi(\mathbf{f})}{\sqrt{5}}\mathbf{f} = \frac{1}{\sqrt{5}} d\varphi \left(\frac{\mathbf{r}_\varphi}{h}\right) = \frac{1}{h\sqrt{5}}\,.$$

A matrix of the shape operator in the basis $\{\mathbf{e}, \mathbf{f}\}$ is $\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h\sqrt{5}} \end{pmatrix}$. Hence Gaussian curvature equals to zero and mean curvature $H = \mathrm{Tr} S = \frac{1}{h\sqrt{5}}$.

4b (3+3) Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be an arbitrary vector fields on the manifold equipped with affine connection ∇. Consider the operation which assigns to the vector fields $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ the new vector field: $\mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \left(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}} - \nabla_{[\mathbf{X},\mathbf{Y}]}\right)\mathbf{Z}$. One can show that it is $C^\infty(M)$-linear operation with respect to vector fields $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$, i.e. for an arbitrary functions f, g, h, $\mathcal{R}(f\mathbf{X}, g\mathbf{Y})(h\mathbf{Z}) = fgh\mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z}$. Thus it it defines the tensor field of the type $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$: If $\mathbf{X} = X^i\partial_i, \mathbf{X} = X^i\partial_i, \mathbf{X} = X^i\partial_i$ then

$$\mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \mathcal{R}(X^m\partial_m, Y^n\partial_n)(Z^r\partial_r) = Z^r R^i_{rmn} X^m Y^n$$

where we denote by R^i_{rmn} the components of the tensor $\mathcal{R}$ in the coordinate basis ∂_i $R^i{}_{rmn}\partial_i = \mathcal{R}(\partial_m, \partial_n)\partial_r$. This $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ tensor field is called curvature tensor of the connection ∇.

The surface of cylindre in $\mathbf{E}^2$ is locally Euclidean, induced Riemannian metric is $dh^2 + a^2 d\varphi^2$, hence the Levi-Civita connection of the Riemannian metric has vanishing Christoffel symbols in coordinates $(h, varphi)$. This implies that Riemann curvature tensor is equal to zero.

4c (3+3) Let M be a surface in Euclidean space $\mathbf{E}^3$. Let C be a closed curve C on M such that C is a boundary of a compact oriented domain $D \subset M$. Consider the parallel transport of an arbitrary tangent vector along the closed curve C. As a result of parallel transport along this closed curve any tangent vector rotates through the angle

$$\angle\phi = \angle\left(\mathbf{X}, \mathbf{R}_C\mathbf{X}\right) = \int_D K d\sigma \, ,$$

where K is the Gaussian curvature and $d\sigma = \sqrt{\det g} dudv$ is the area element induced by the Riemannian metric on the surface M, i.e. $d\sigma = \sqrt{\det g} dudv$.

The circle C is a boundary of the sphere segment of the height H. The area of this domain is equal to $2\pi Rh$. The Gaussian curvature of sphere iis equal to $\frac{K=1}{R^2}$. Hence due to Theorem we see that vector $\mathbf{X}$ through parallel transport rotates on the angle $KS = \frac{2\pi R}{h}$.

5 (for students who earn 15 credits)

5a (3+7).

A vector field $\mathbf{K}$ on Riemannian manifold M. induces infinitesimal diffeomorphism $F_{\mathbf{K}}\colon x^{i'} = x^i + \varepsilon X^i(x)$, $(\varepsilon^2 = 0)$.

We say that K is infinitesimal isometry if this diffeomorphism is an isometry, i.e. $F^*_{\mathbf{K}} G = G$. In local coordinates the condition that K is Killing vector fields reads as:

$$\mathcal{L}_{\mathbf{K}} G = 0, \quad \text{i.e.} g_{ik}(x) = K^r(x)\frac{\partial g_{ik}(x)}{\partial x^r} + \frac{\partial K^r(x)}{\partial x^i} g_{rk}(x) + \frac{\partial K^r(x)}{\partial x^k} g_{ri}(x) \, .$$

for $\mathbf{K} = K^i(x)\frac{\partial}{\partial x^i} G = g_{ik}(x) dx^i dx^k$

Let $\mathbf{K}$ be Killing vector field, and ∇ be Levi-Civita connection. Killing vector field does not change metric and respectively the corresponding Lev–Civita connection:

$$\mathcal{L}_{\mathbf{K}} G = 0 \qquad \text{i.e. } \forall \mathbf{X}, \mathbf{Y}, \quad \partial_{\mathbf{K}} \langle \mathbf{X}, \mathbf{Y}\rangle = \langle \mathcal{L}_{\mathbf{K}} \mathbf{X}, \mathbf{Y}\rangle + \langle \mathbf{X}, \mathcal{L}_{\mathbf{K}} \mathbf{Y}\rangle \tag{5.1a}$$

(invariance of metric with respect to infinitesimal isometry),and

$$\forall \mathbf{X}, \mathbf{Y}, \quad \partial_{\mathbf{K}} \langle \mathbf{X}, \mathbf{Y}\rangle = \langle \nabla_{\mathbf{K}} \mathbf{X}, \mathbf{Y}\rangle + \langle \mathbf{X}, \nabla_{\mathbf{K}} \mathbf{Y}\rangle \tag{5.1b}$$

(invariance of Levi-Civita connection with respect to metric)

Substracting the first relation from the second one we will come to the equation:

$$\forall \mathbf{X}, \mathbf{Y}, \quad \langle (\nabla_{\mathbf{K}} - \mathcal{L}_K)\mathbf{X}, \mathbf{Y}\rangle + \langle \mathbf{X}, (\nabla_{\mathbf{K}} - \mathcal{L}_K)\mathbf{Y}\rangle = 0 \tag{5.1c}$$

Notice that the condition (5.1c) is equivalent to the condition (5.1a) provided that ∇ is the Levi-Civita condition.

The operator $A(\mathbf{X}) = \nabla_{\mathbf{K}}\mathbf{X} - \mathcal{L}_{\mathbf{K}}\mathbf{X}$ is linear operator on tangent vectors:

$$A(f\mathbf{X}) = f\nabla_K\mathbf{X} + (\partial_{\mathbf{K}}f)\mathbf{X} - (\partial_K f)\mathbf{X} - f\mathcal{L}_{\mathbf{K}}\mathbf{X} = fA_{\mathbf{K}}(\mathbf{X}).$$

The condition (5.1c) means that this lienar operator is antisymmetric (with respect to metric G):

$$\forall \mathbf{X}, \mathbf{Y}, \quad \langle A_{\mathbf{K}}(\mathbf{X}), \mathbf{Y}\rangle + \langle \mathbf{X}, A_{\mathbf{K}}(\mathbf{Y})\rangle = 0\,. \tag{5.1d}$$

We have that

$$A(\mathbf{X}) = \nabla_K\mathbf{X} - \mathcal{L}_{\mathbf{K}}\mathbf{X} = \underbrace{(\nabla_K\mathbf{X} - \nabla_{\mathbf{X}}\mathbf{K})}_{[\mathbf{K},\mathbf{X}]+S(\mathbf{K},\mathbf{X})} + \nabla_{\mathbf{X}}\mathbf{K} - [\mathbf{K},\mathbf{X}]$$

Since ∇ is the Levi-Civita connection, it is symmetric, i.e. torsion tensor S identically vanishes. We see that if ∇ is Levi-Civita condition, then $\mathbf{K}$ is Killing if and only if the operator $A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{X}}\mathbf{K}$ is antisymmetric.

Rewrite the condition (5.1d) in local coordinates $\{x^i\}$: for any basic vectors $\frac{\partial}{\partial x^m}$, $\frac{\partial}{\partial x^m}$ we have

$$0 = \langle A_{\mathbf{K}}(\partial_m), \partial_n\rangle + \langle \partial_m, A_{\mathbf{K}}(\partial_n)\rangle = \langle \nabla_m(K^i\partial_i), \partial_n\rangle + \langle \partial_m, \nabla_n(K^i\partial_i)\rangle\,,$$

i.e.

$$(\partial_m K^i + K^r\Gamma^i_{rm})g_{in} + (m \leftrightarrow n) = 0\,, \tag{5.1e}$$

where Γ^i_{rm} are Christoffel symbols of Levi-Civita connection.

5b (4+3+3) Let x^i are standard coordinates in $\mathbf{E}^n$. Metric in these coordinates is $G = dx^i\delta_{ik}dx^k$. Chrstophel symbols vanish and equation (5.1e) becomes:

$$\frac{\partial K^i(x)}{\partial x^m}\delta_{in} + \frac{\partial K^p(x)}{\partial x^n}\delta_{im} = 0\,,$$

i.e.

$$\frac{\partial K^i(x)}{\partial x^k} + \frac{\partial K^k(x)}{\partial x^i} = 0\,.$$

Solve this equation. Differentiating by x we come to

$$\frac{\partial^2 K^i(x)}{\partial x^m \partial x^k} + \frac{\partial K^k(x)}{\partial x^m \partial x^i} = 0\,.$$

Consider tensor field

$$T^i_{mk} = \frac{\partial^2 K^i}{\partial x^m \partial x^k}\,.$$

We see that

$$T^i_{mk} = T^i_{km} = -T^m_{ik}\,.$$

It is easy to see that this implies that $T^i_{mk} \equiv 0$!!!:

$$T^i_{mk} = -T^m_{ik} = -T^m_{ki} = T^k_{mi} = T^k_{im} = -T^i_{km} = -T^i_{mk} \Rightarrow T^i_{mk} = \ .$$

We see that $T^i_{mk} = \frac{\partial^2 K^i(x)}{\partial x^m \partial x^k} = 0$, i.e.

$$K^i(x) = C^i + B^i_k x^k$$

where C^i are arbitrary constants and B^i_k is an arbitrary antisymmetrical matrix.

Calculate the dimension $\kappa(\mathbf{E}^n)$ of the space of Killing vector fields for $\mathbf{E}^n$. The space of constant vectors C^i has dimension n and a space of $n \times n$ antisymmetrical matrices has dimension $\frac{n\times n-n}{2}$. Hence

$$\kappa(\mathbf{E}^n) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

We know that for arbitary 2-dimensional manifold, $\kappa(M) \leq 3$.

Consider M_1-sphere in $\mathbf{E}^3$ and M_2 cylindrical surface. Rotations of $\mathbf{E}^3$ define three independent KIlling vector fields on sphere.; $\kappa(S^2) = 3$, its Gaussian curvature $K = \frac{1}{R^2}$. Cylindrical surface is locally Euclieanean: $G = dh^2 = a^2 d\varphi^2 dh^2 + du^2$ $(u = a\varphi)$, hence in a vicinity of every points there are three independent vector fields which preserve metric locally:

$$\partial_h, \partial_u = a\partial_\varphi, h\partial_u - u\partial_h$$

We see that the third vector field is not defined globally: since the angle φ is not one-valued function. There two lienar independent vector fields on cylindre: ∂_h and ∂_φ, $\kappa = 2$.

Every question is worth 20 marks

The marks for every subquestions are indicated above in the text of solutions.

Bookwork

First question :	$(a1) - 3$	$(b1) - 2 + 2$	$(c1) - 2$	$3 + 4 + 2 = 9$
Second question :	$(a1, a2) - 2 + 1$	$(b1) - 3$	$(c1) - 2$	$3 + 3 + 2 = 8$
Third question :	$(a1, a2) 2 + 2$	$(b1, b2, b3) 1 + 1 + 3$		$4 + 5 = 9$
Fourth question :	$(a1, a3) - 2 + 3$	$(b1) - 3$	$c(1) - 3$	$5 + 3 + 3 = 12$
Fifth question :	$a - 10$	$b1 - 4$		$10 + 4 = 14$

Easy questions

First question :	$(a2) - 1$		$c(1)2$	$2 + 2 = 4$
Second question	$(a1, a2) - 3 + 2$			$3 + 2 = 5$
Third question	$(a1, 2) - 2 + 2$	$b(1, 2)1 + 1$		$2 + 2 + 1 + 1 = 6$
Fourth question	$(a1) - 2$			$1 + 3 + 2 = 6$

Difficult or unseen questions

First question	$a(3)-2$difficult and partly unseen
Second question	$c(3)-3$ (not very difficult but unusual)
Third question	$c-5$ (this is little bit difficult)
Fourth question	$b(2)-4$unseen but not difficult
Fifth question	$b2, b3-3+3$ unseen in this framework (b3 is difficult and unseen in this variation)

SECTION A

Answer **ALL** questions in this section (40 marks in total)

A1. The polynomials $f = X^4 - Y + 1$, $g = Y + Z^2 + 1$, $h = YZ + Z$ generate the ideal I of $\mathbb{Q}[X, Y, Z]$.

(a) Find a Gröbner basis of I with respect to the lexicographic order Lex with $X \succ Y \succ Z$.

Answer. [*routine,* 10 *marks*]

Buchberger's algorithm. Start with $f = \underline{\underline{X^4}} - Y + 1$, $g = \underline{\underline{Y}} + Z^2 + 1$, $h = \underline{\underline{YZ}} + Z$. We double-underline leading monomials with respect to Lex.

The leading monomials of f and g are relatively prime so $S(f, g) \to 0$. Same applies to f and h, so $S(f, h) \to 0$. One has $S(g, h) = Z(Y + Z^2 + 1) - (YZ + Z) = Z^3$, reduced mod $\{f, g, h\}$.

The leading monomials of f, g and $\underline{\underline{Z^3}}$ are pairwise relatively prime, so $S(f, Z^3)$, $S(g, Z^3) \to 0$. $S(h, Z^3) = Z^2(YZ + Z) - Y(Z^3) = Z^3 \xrightarrow{Z^3} 0$. There are no more S-polynomials to compute. A Gröbner basis is $\{f, g, h, Z^3\}$.

(b) Find the reduced Gröbner basis of I.

Answer. [*routine,* 3 *marks*]

The polynomial $h = YZ + Z$ is redundant as $\operatorname{lm} h = YZ$ is divisible by $\operatorname{lm} g = Y$. Delete h.

The polynomial $f = X^4 - Y + 1$ is not reduced mod $g = Y + Z^2 + 1$: $f \xrightarrow{g} (X^4 - Y + 1) + Y + Z^2 + 1 = X^4 + Z^2 + 2$.

The reduced Gröbner basis is $\{X^4 + Z^2 + 2,\ Y + Z^2 + 1,\ Z^3\}$.

(c) Is the variety $\mathcal{V}(I) \subset \mathbb{Q}^3$ non-empty? Justify your answer.

Answer. [*unseen,* 2 *marks*]

The ideal I contains the polynomial $X^4 + Z^2 + 2$ which is strictly positive on $\mathbb{Q}^3$, so $\mathcal{V}(I) = \varnothing$. (*The same can be easily arrived at by looking at the original polynomials f, g, h.*)

[15 marks]

A2.

(a) Give a definition of a noetherian ring.

Answer. [*bookwork,* 2 *marks*]

A noetherian ring is a ring where every ideal is finitely generated.

(b) Is it true that every subring of every noetherian ring is noetherian? Justify your answer briefly.

Answer. [*seen in class,* 2 *marks*]

No: there is a non-noetherian domain R, e.g., $R = \mathbb{Q}[X_1, X_2, \dots]$, the "polynomial ring" in infinitely many variables. Then R is a subring of its field of fractions, $\mathcal{F}r(R)$, a noetherian ring.

(c) Give a definition of a euclidean norm and a euclidean ring. Briefly state a reason why a euclidean ring is noetherian.

Answer. [*bookwork/straightforward, 3 marks*]

A euclidean norm on a ring R is a function $N\colon R \to \mathbb{N}$ such that for all $a, b \in R \setminus \{0\}$, there exists $q \in R$ such that $a = qb$ or $N(a - qb) < N(b)$. A ring with a euclidean norm is called a euclidean ring. It is noetherian because, by a theorem in the course, it is a principal ideal ring (every ideal has a generating set of cardinality 1).

(d) Write down an example of a ring which is noetherian but not euclidean.

Answer. *[seen, 1 mark]*

For example, $\mathbb{Q}[X, Y]$.

(e) State without proof Hilbert's Basis Theorem for polynomial rings.

Answer. *[bookwork, 2 marks]*

If K is a field, $K[X_1, \ldots, X_n]$ is a noetherian ring.

[10 marks]

A3. Let R be a commutative domain and let $a, b \in R$.

(a) What is meant by saying that a is irreducible in R?

Answer. *[bookwork, 2 marks]*

a is not a unit and not a product of two non-units.

(b) What is meant by saying that b is an associate of a?

Answer. *[bookwork, 1 mark]*

$b = xa$ where x is a unit of R.

(c) Prove: if a is irreducible and b is an associate of a, then b is irreducible.

Answer. *[bookwork, 3 marks]*

Let $b = rs$ where s is not a unit. We need to show that r is a unit. Note that $a = (x^{-1}r)s$, so by irreducibility of a, $x^{-1}r$ is a unit, hence $r = x(x^{-1}r)$ is a unit.

Answer the following questions, giving reasons for your answer.

(d) Is $2X^3 + X^2 + X - 1$ irreducible in $\mathbb{Z}[X]$?

Answer. *[similar to examples done in class, 3 marks]*

Not irreducible: equals $(2X-1)(X^2+X+1)$. (*Can be seen easily by finding the rational roots.*)

(e) Is $\frac{1}{20}X^5 + \frac{2}{15}X^3 + \frac{1}{5}X - \frac{3}{10}$ irreducible in $\mathbb{Q}[X]$?

Answer. *[similar to examples done in class, 3 marks]*

Irreducible: multiply by 60 to get $3X^5 + 8X^3 + 12X - 18$ which is Eisenstein with $p = 2$.

(f) Is $X^8 + X + 1$ irreducible in $\mathbb{Z}_2[X]$?

Answer. *[unseen, 3 marks]*

No: divisible by X^2+X+1. Can be checked by long division, or else $X^8+X+1-(X^2+X+1) = X^2((X^3)^2 - 1)$ is divisible by $X^3 + 1 = (X + 1)(X^2 + X + 1)$. So $X^2 + X + 1$ is a factor.

[15 marks]

SECTION B

Answer **TWO** of the three questions in this section (40 marks in total).

If more than TWO questions from this section are attempted, then credit will be given for the best TWO answers.

B4. Let $M(X_1, \ldots, X_n)$ denote the set of all monomials in $X_1, \ldots, X_n$.

(a) Let S be a subset of $M(X_1, \ldots, X_n)$. State Dickson's Lemma about minimal monomials of S.

Answer. [*bookwork,* 3 *marks*]

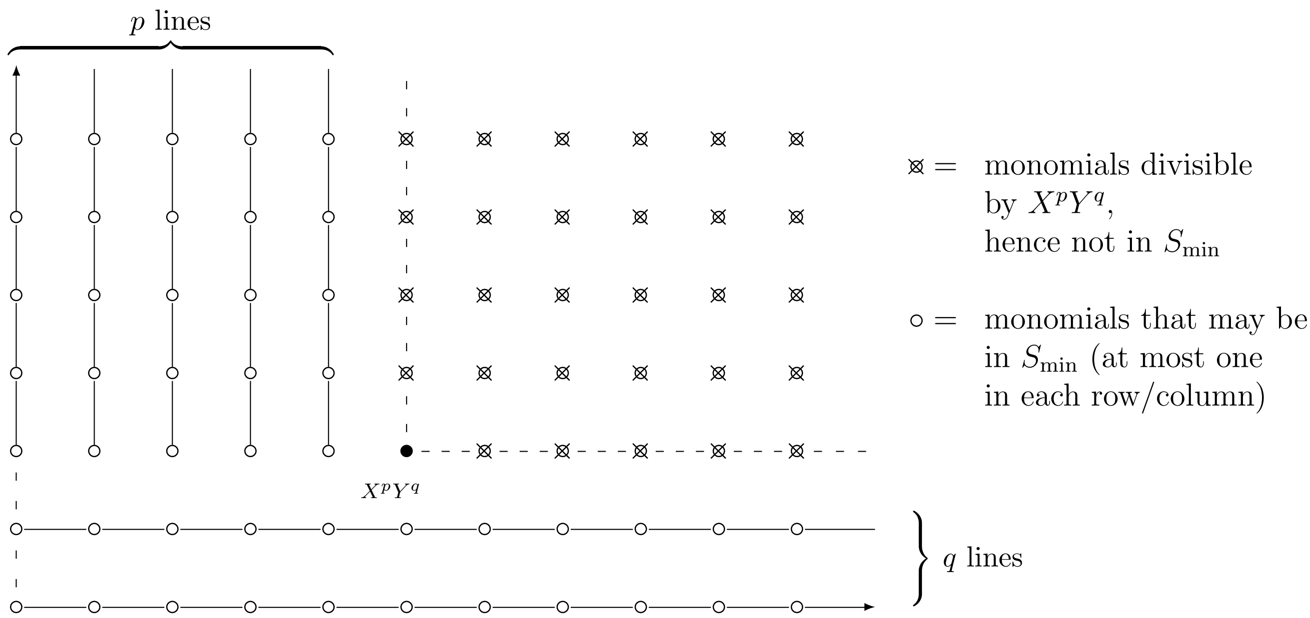
Let $S_{\min}$ be the set of monomials in S which are minimal in S with respect to "|" ("divides"). Then $S_{\min}$ is finite, and every element of S is divisible by at least one element of $S_{\min}$.

(b) Prove Dickson's Lemma for $n = 2$.

Answer. [*bookwork,* 7 *marks*]

Proof of finiteness of $S_{\min}$: let $S \subseteq M(X, Y)$. If $S = \varnothing$, $S_{\min} = \varnothing$ which is finite. Otherwise, pick $X^pY^q \in S$. Then no elements of $S_{\min}$ are strictly divisible by X^pY^q and lie in the infinite quadrant to the top-right of $(p, q) \in \mathbb{N} \times \mathbb{N}$ (the lattice which represents $M(X, Y)$).

The remaining part of the lattice is covered by p vertical and q horizontal lines, see the illustration below. If two monomials from S are on the same line (horizontal or vertical), one monomial divides the other. Minimal monomials must not divide each other, hence a line cannot contain more than one *minimal* monomial. Thus, $|S_{\min}| \leq p + q + 1$ (at most one monomial on each of the $p + q$ lines plus possibly X^pY^q).



Proof of the rest of the lemma: let $m \in S$. Among all the elements of S that divide m, choose one which has the lowest total degree, and denote it $m_{\min}$. Note that $m_{\min}$ cannot be strictly divisible by another element $y \in S$, for y would divide m and have a lower total degree than $m_{\min}$. Hence $m_{\min} \in S_{\min}$.

(c) What is meant by saying that $\preccurlyeq$ is a monomial ordering on $M(X_1, \ldots, X_n)$?

Answer. [*bookwork, 3 marks*]

(1) $\preccurlyeq$ is a total order on $M(X_1, \ldots, X_n)$;
(2) $m \in M(X_1, \ldots, X_n) \implies 1 \preccurlyeq m$;
(3) if $m' \preccurlyeq m$ then, for every $m_1 \in M(X_1, \ldots, X_n)$, $m_1 m' \preccurlyeq m_1 m$.

(d) Show: if $m, m' \in M(X_1, \ldots, X_n)$, m divides m', and $\preccurlyeq$ is a monomial ordering, then $m \preccurlyeq m'$.

Answer. [*bookwork, 2 marks*]

$m' = m_1 m$ for some monomial m_1; by (1) above, $1 \preccurlyeq m_1$, and by (3) above, one can multply both sides of the inequality by m, obtaining $m = 1 \cdot m \preccurlyeq m_1 m = m'$.

(e) Let $I \neq \{0\}$ be a monomial ideal of $\mathbb{Q}[X_1, \ldots, X_n]$. Show that I contains a monomial, m, such that m is divisible by exactly 2015 other monomials contained in I.

Answer. [*unseen, 5 marks*]

$I \neq \{0\}$ means that I is generated by a non-empty set of monomials. Let m_0 be the least monomial in I with respect to the lexicographic order with $X_1 \succ \ldots \succ X_n$. Put $m = m_0 X_n^{2015}$; then $m \in I$. Let $m' \in I$ be such that $m' \mid m$, $m' \neq m$. By the choice of m_0 and part (d) one has $m_0 \preccurlyeq_{\text{Lex}} m' \prec_{\text{Lex}} m_0 X_n^{2015}$. There are exactly 2015 monomials m' in $M(X_1, \ldots, X_n)$ satisfying this inequality, namely m_0, $m_0 X_n, \ldots, m_0 X_n^{2014}$; all of them are multiples of m_0 hence are in I.

[20 marks]

B5.

(a) What is meant by a prime in a commutative domain R?

Answer. [*bookwork, 2 marks*]

$p \in R$ is a prime if p is not a unit and $\forall a, b \in R$, $p \nmid a$, $p \nmid b \implies p \nmid ab$.

(b) Show that a non-zero prime is irreducible.

Answer. [*bookwork, 3 marks*]

Take a prime $p \neq 0$ and let $p = ab$. Then $p \mid ab$ so $p \mid a$ or $p \mid b$. If $p \mid a$ then $a = px$ and $p = pxb$, so by cancellation law $xb = 1$ and b is a unit. If $p \mid b$ then a is a unit. Thus, one of a, b is a unit.

(c) What is a unique factorisation domain (UFD)?

Answer. [*bookwork, 2 marks*]

A domain where every non-unit is a product of primes.

(d) Describe without proof all primes in the domain R, and state whether R is a UFD, if

i. $R = \mathbb{R}$, the field of real numbers;

ii. $R = \mathbb{R}[X]$, the ring of polynomials in X with real coefficients.

Answer. *[routine, $5 = 2 + 3$ marks]*

i. In $\mathbb{R}$, the only prime is 0. It is a UFD.
ii. In $\mathbb{R}[X]$, the primes are 0, $aX + b$ where $a, b \in \mathbb{R}$, $a \neq 0$, and $aX^2 + bX + c$ where $a, b, c \in \mathbb{R}$, $b^2 < 4ac$. It is a UFD.

(e) Is $2 + 15i$ a prime in the UFD $\mathbb{Z}[i]$, the ring of Gaussian integers? Give reasons for your answer.

Answer. *[similar to examples done in class, 3 marks]*

Yes: $|2 + 15i|^2 = 2^2 + 15^2 = 229$ is a prime number; easy to see (*this was shown in class*) that $2 + 15i$ cannot be factorised into non-units, so is irreducible hence a prime in the UFD $\mathbb{Z}[i]$.

(f) Let a, b, c, d, e be non-zero elements of a commutative domain R (not necessarily a UFD) such that $ab = cde$ and $ac = bd$. Show that if c is a prime, then e is not a prime.

Answer. *[unseen, 5 marks]*

$c^2de = abc = b^2d$ so by the cancellation law $c^2e = b^2$. Therefore, $c \mid b^2$, hence $c \mid b$ as c is a prime. Write $b = cx$. Substitute into $c^2e = b^2$ to get $e = x^2$. A square cannot be irreducible, hence by part (b), e is not a prime.

[20 marks]

B6.

(a) What is meant by saying that an ideal I of a commutative ring R is a maximal ideal?

Answer. *[bookwork, 2 marks]*

$I \neq R$, and there is no ideal J such that $I \subsetneq J \subsetneq R$.

(b) What is meant by the radical, $\sqrt{I}$, of an ideal I? What is a radical ideal?

Answer. *[bookwork, 2 marks]*

$\sqrt{I} = \{a \in R \mid \exists n \in \mathbb{N} : \ a^n \in I\}$; I is a radical ideal if $\sqrt{I} = I$.

(c) Show that a maximal ideal is a radical ideal. You may assume basic properties of the radical without particular comment.

Answer. *[seen, 2 marks]*

Let I be a maximal ideal. Then $I \neq R$ so $1 \notin I$ hence $1^n \notin I$ for all $n \in \mathbb{N}$ and $1 \notin \sqrt{I}$. Therefore, $I \subseteq \sqrt{I} \subsetneq R$. By maximality of I, $I = \sqrt{I}$.

(d) Describe all maximal ideals of the ring $\mathbb{C}[X, Y]$. Prove that the ideals in your list are maximal. (You do not have to prove that there are no other maximal ideals.)

Answer. *[seen, 5 marks]*

The maximal ideals are $<X - a, Y - b>$ for all $(a, b) \in \mathbb{C}^2$. Proof that the ideal $I = <X - a, Y - b>$ is maximal: $G = \{\underline{\underline{X}} - a, \underline{\underline{Y}} - b\}$ is a Gröbner basis for any monomial ordering (the leading monomials are relatively prime) which is reduced and does not contain 1. Hence $I \neq \mathbb{C}[X, Y]$. If $f \notin I$, then remainder(f, G) must be a non-zero constant, hence $<\{f\} \cup G> = \mathbb{C}[X, Y]$ — this proves maximality of I. (*There are other ways to prove that I is maximal.*)

(e) Give an example of an ideal $J \neq \{0\}$ of $\mathbb{C}[X, Y]$ such that J cannot be generated by two polynomials. Justify your example.

Answer. [*similar to an example on example sheets,* 5 *marks*]

For example, let $J = \langle X^2, XY, Y^2 \rangle$. Assume for contradiction that J is generated by $f, g \in J$. As J is a monomial ideal, every monomial in f and in g is divisible by X^2, XY or Y^2, hence is of total degree ≥ 2. So, writing $X^2 = h_1 f + h_2 g$ where $h_1, h_2 \in \mathbb{C}[X, Y]$, we conclude that X^2 is a linear combination — with scalar coefficients — of $\overline{f}$ and $\overline{g}$ (where $\overline{}$ denotes the terms of total degree 2). But so are XY and Y^2 — a contradiction, as the span of $\overline{f}$ and $\overline{g}$ cannot contain three linearly independent elements.

(f) For the ideal J from your example in part (e), find $\mathcal{V}(J)$ and $\sqrt{J}$.

Answer. [*varies depending on J,* 4 *marks*]

In our particular example, $\mathcal{V}(J) = \{(0, 0)\}$ (obvious) hence $\sqrt{J} = \mathcal{I}(\{(0, 0)\}) = \langle X, Y \rangle$.

[20 marks]

END OF EXAMINATION PAPER

(1)

MATH32062 Examination Solutions 2014/15

1.(a)(i) $V(J) = \{(a_1, a_2, \ldots, a_n) \in K^n \mid f(a_1, a_2, \ldots a_n) = 0 \;\; \forall f \in J\}$ (1*)

(ii) $I(X) = \{f \in K[x_1 \ldots x_n] \mid f(a_1, a_2, \ldots a_n) = 0 \;\; \forall (a_1 \ldots a_n) \in X\}$ (1*)

(iii) Let $P \in V(J)$ and let $f \in \sqrt{J}$. By the definition of $\sqrt{J}$, there exists $n \in \mathbb{N}$ such that $f^n \in J$.
Then $0 = (f^n)(P) = (f(P))^n$, which implies $f(P) = 0$. (2*)
As this holds for every $P \in V(J)$, we have $f \in I(V(J))$, so $\sqrt{J} \subseteq I(V(J))$ as required. Bookwork

(iv) K has to be algebraically closed (1*)

(b) Let $f_1 = (x - y^2 + z^3)^2$, $f_2 = yz^2 - z^3 + 2x^2 - y^2 + y$.
Any $h \in J$ can be written as $h = f_1 g_1 + f_2 g_2$ for some $g_1, g_2 \in \mathbb{C}[x,y,z]$, therefore

$$\frac{\partial h}{\partial z} = \frac{\partial f_1}{\partial z} g_1 + f_1 \frac{\partial g_1}{\partial z} + \frac{\partial f_2}{\partial z} g_2 + f_2 \frac{\partial g_2}{\partial z}$$ (4*)

$\frac{\partial f_1}{\partial z} = 3z^2(x - y^2 + z^3)$, $\frac{\partial f_2}{\partial z} = 2yz - 3z^2$, so at (Similar to a question on the problem sheets)
$P = (1,3,2)$, $f_1(P) = f_2(P) = \frac{\partial f_1}{\partial z}\Big|_P = \frac{\partial f_2}{\partial z}\Big|_P = 0$,
therefore $\frac{\partial h}{\partial z}\Big|_P = 0$ for every $h \in J$.

$\frac{\partial (x - y^2 + z^3)}{\partial z} = 3z^2$, $\frac{\partial (x - y^2 + z^3)}{\partial z}\Big|_{(1,3,2)} = 12 \neq 0$, so (2*)
$x - y^2 + z^3 \notin J$, but $(x - y^2 + z^3)^2 \in J$, so
$x - y^2 + z^3 \in \sqrt{J} \subseteq I(V(J))$.

(c)(i) W is irreducible iff it cannot be written as $W = W_1 \cup W_2$, where W_1, W_2 are also affine algebraic varieties and $W_1 \neq W \neq W_2$ (1*)

(2) (c)(ii) We shall prove the contrapositive, i.e. if $I(W)$ is not a prime ideal, then W is reducible. (3) Bookwork.

Assume that $I(W)$ is not prime, then there exist polynomials f_1, f_2 such that $f_1, f_2 \notin I(W)$ but $f_1 \cdot f_2 \in I(W)$. Let $W_i = V(\langle I(W), f_i \rangle) = \{P \in W \mid f_i(P) = 0\}$ $(i = 1, 2)$.

Let $P \in W$. $0 = \underbrace{(f_1 f_2)}_{I(W)}(P) = f_1(P) f_2(P)$, so $f_1(P) = 0$ or $f_2(P) = 0$. If $f_1(P) = 0$, then $P \in W_1$, if $f_2(P) = 0$, $P \in W_2$. Hence $W \subseteq W_1 \cup W_2$. W_1, W_2 are subsets of W, so $W = W_1 \cup W_2$. W_1, W_2 are also affine algebraic varieties and they are not equal to W as $f_1, f_2 \notin I(W)$, so W is reducible.

(d) $yz(y-1) = 0$ implies that $y = 0$, $y = 1$ or $z = 0$

If $y = 0$, then from the 1st factor we get $x^2 - 2xz - 2x = 0$
$x(x - 2z - 2) = 0$, so $x = 0$ or $x - 2z - 2 = 0$
We get the lines $V(\langle x, y \rangle)$ and $V(\langle x - 2z - 2, y \rangle)$ which are isomorphic to $\mathbb{A}^1$, so they are irreducible. (5)

If $y = 1$, then we get $x^2 - 2x + z + 1 = 0$, which is the equation of a parabola in the $y = 1$ plane, which is also irreducible, because it is also isomorphic to $\mathbb{A}^1$ or because it is a non-degenerate conic.

If $z = 0$ then we get $x^2 - 2x + y^2 = 0$,
$(x-1)^2 + y^2 - 1 = 0$, which is the equation of a circle in the $z = 0$ plane, which is also irreducible as a non-degenerate conic.

Therefore the irreducible components are
$V(\langle x, y \rangle)$, $V(\langle x - 2z - 2, y \rangle)$, $V(\langle y - 1, x^2 - 2x + z + 1 \rangle)$
and $V(\langle z, x^2 - 2x + y^2 \rangle)$.

(3) 2. (a)(i) ℓ is tangent to V at P, iff $f(P+t\underline{v})$ (a polynomial in t) has a zero of multiplicity at least 2 at $t=0$ or is identically 0 for every $f \in \mathcal{I}(V)$. (2*)

$T_P V$ is the union of P and the tangent lines to V at P.

(ii) There exist a non-empty Zariski open subset $U \subseteq V$ and a non-negative integer d such that $\dim T_P V = d$ for every $P \in U$. This d is the dimension of V. The set $\{P \in V \mid \dim T_P V > d\}$ is the singular locus of V. (2*)

(b)(i) Let $f(x,y,z) = (x-1)^3 - y^2$, $g(x,y,z) = x^2+y^2-2x-z$. (6*)

The Jacobian matrix is (Standard problem type)

$$J = \begin{pmatrix} 3(x-1)^2 & -2y & 0 \\ 2x-2 & 2y & -1 \end{pmatrix}$$

$\mathrm{rk}\, J \neq 0$ because of the -1 in the bottom right hand corner. The point where $\mathrm{rk}\, J = 1$ are those where all the 2×2 minors vanish, these are $3(x-1)^2 \cdot 2y + 2y(2x-2)$, $-3(x-1)^2$ and $2y$. $-3(x-1)^2 = 2y = 0$ implies $x=1, y=0$. Then the first minor is also 0. By substituting $x=1, y=0$ into g, we get $z=-1$. $f(1,0,-1)=0$, too, so $(1,0,-1) \in W$, and this is the only point of W where $\mathrm{rk}\, J = 1$.

At all other points $P \in W$ $\mathrm{rk}\, J_P = 2$ as there is no other possibility (W does have other points, e.g. $x=2, y=1, z=1$.)

Hence $\underline{\dim W = 3-2 = 1}$ and $\underline{\mathrm{Sing}\, W = \{(1,0,-1)\}}$

(ii) $t^2+1, t^3, t^6+t^4-1 \in K[t] = K[\mathbb{A}^1]$, therefore φ is a morphism $\mathbb{A}^1 \to \mathbb{A}^3$.

If we substitute $x = t^2+1$, $y=t^3$, $z = t^6+t^4-1$ into f and g, we get

$((t^2+1)-1)^3 - (t^3)^2 = t^6 - t^6 = 0$ and

$(t^2+1)^2 + (t^3)^2 - 2(t^2+1) - t^6 - t^4 + 1 = t^4 + 2t^2 + 1 + t^6 - 2t^2 - 2 - t^6 - t^4 + 1 = 0.$

(4) (b)(ii) cont'd.

Therefore $\varphi(t) \in W$ for every $t \in \mathbb{C}$, so (3*)
φ is indeed a morphism $\mathbb{A}^1 \to W$.

(iii) $(\psi \circ \varphi)(t) = \psi(t^2+1, t^3, t^6+t^4-1) = \frac{t^3}{t^2+1-1} = \frac{t^3}{t^2} = t \ (t \neq 0)$,

so $\psi \circ \varphi = \mathrm{id}_{\mathbb{A}^1}$.

$(\varphi \circ \psi)(x,y,z) = \left(\frac{y^2}{(x-1)^2}+1, \frac{y^3}{(x-1)^3}, \frac{y^6}{(x-1)^6} + \frac{y^4}{(x-1)^4} - 1 \right)$.

$(x-1)^3 - y^2 \in I$, therefore

$\frac{y^2}{(x-1)^2} + 1 = \frac{(x-1)^3}{(x-1)^2} + 1 = x-1+1 = x$ in $K(W)$. (5)

Similarly, $\frac{y^3}{(x-1)^3} = \frac{y^3}{y^2} = y \in K(W)$

From these, it follows that

$\frac{y^6}{(x-1)^6} + \frac{y^4}{(x-1)^4} - 1 = y^2 + (x-1)^2 - 1 = y^2 + x^2 - 2x$ in $K(W)$

and $x^2+y^2-2x-z \in I(W)$ implies that $y^2+x^2-2x = z$

in $K(W)$. Hence $(\varphi \circ \psi)(x,y,z) = (x,y,z)$ $(x \neq 1)$.

Therefore φ and ψ are inverses of each other as required.

(iv) We have just shown that W is birationally equivalent to $\mathbb{A}^1$, therefore it is rational. (2)
W has a singular point, while $\mathbb{A}^1$ has none, and singularities are preserved under isomorphism, therefore W, $\mathbb{A}^1$ are not isomorphic.

(5) 3(a) A rational map $\varphi : V \dashrightarrow W$ is a function defined on a non-empty subset of V given by an equivalence class of $(n+1)$ tuples of homogeneous elements of $K[V]$ of the same degree. $(\varphi_0 : \varphi_1 : \dots : \varphi_n) \sim (\psi_0 : \psi_1 : \dots : \psi_n)$ iff $\varphi_i \psi_j = \varphi_j \psi_i \quad \forall i,j \ 0 \le i \le n, 0 \le j \le n$. φ is defined at $P \in V$ iff there exists an $(n+1)$ tuple $(\varphi_0 : \dots : \varphi_n)$ representing φ such that $\varphi_i(P) \neq 0$ for some i, $0 \le i \le n$. Then $\varphi(P) = (\varphi_0(P) : \varphi_1(P) : \dots : \varphi_n(P))$ and we require $\varphi(P) \in W$.

4x (Basic definitions)

φ is a morphism iff it is defined at every point of V.

(b) (i) φ can be written as $\varphi(z) = \frac{az+b}{cz+d}$ for some $a,b,c,d \in K$, $ad-bc \neq 0$.

5 (Bookwork)

If $c = 0$, then $\varphi(z) = \frac{a}{d} z + \frac{b}{d}$, if $c \neq 0$, then
$\varphi(z) = \frac{bc-ad}{c} \cdot \frac{1}{cz+d} + \frac{a}{c}$. In either case,
φ can be composed of functions of the following form $z \mapsto z+\alpha \ (\alpha \in K)$, $z \mapsto \lambda z \ (\lambda \in K \setminus \{0\})$ and $z \mapsto \frac{1}{z}$.

The first two clearly preserve the cross ratio, while

$$\left(\frac{1}{z_1}, \frac{1}{z_2}; \frac{1}{z_3}, \frac{1}{z_4}\right) = \frac{\left(\frac{1}{z_1}-\frac{1}{z_3}\right)\left(\frac{1}{z_2}-\frac{1}{z_4}\right)}{\left(\frac{1}{z_1}-\frac{1}{z_4}\right)\left(\frac{1}{z_2}-\frac{1}{z_3}\right)} = \frac{\frac{(z_3-z_1)(z_4-z_2)}{z_1 z_2 z_3 z_4}}{\frac{(z_4-z_1)(z_3-z_2)}{z_1 z_2 z_3 z_4}} =$$

$$= \frac{(z_3-z_1)(z_4-z_2)}{(z_4-z_1)(z_3-z_2)} = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} = (z_1, z_2; z_3, z_4)$$

All three types of functions preserve the cross ratio, therefore so does φ.

(ii) Let $\varphi_1(z) = (z_1, z_2; z_3, z)$. Then $\varphi_1(z_1) = \infty$, $\varphi_1(z_2) = 0$, $\varphi_1(z_3) = 1$. $\varphi_1(z) = \frac{az+b}{cz+d}$ for suitable $a,b,c,d \in K$ and $ad-bc \neq 0$ as φ_1 is not constant. Therefore φ_1 is a projective transformation $\mathbb{P}^1 \to \mathbb{P}^1$

6 (b)(ii) (cont'd)

Let $\varphi_2(z) = (w_1, w_2; w_3, z)$. Then similarly φ_2 is a projective transformation and $\varphi_2(w_1)=\infty$, $\varphi_2(w_2)=0$, $\varphi_2(w_3)=1$.

4

φ_2^{-1} is also a projective transformation, therefore (Bookwork) so is $\varphi = \varphi_2^{-1} \circ \varphi_1$, and φ satisfies $\varphi(z_i) = w_i$ for $i=1,2,3$.

This proves the existence of φ.

Any projective transformation φ with $\varphi(z_i) = w_i$ for $i=1,2,3$ satisfies $(z_1, z_2, z_3, z) = (w_1, w_2, w_3, \varphi(z))$ $\forall z \in \mathbb{C} \cup \{\infty\}$ since φ preserves the cross ratio.

2

The LHS is $\varphi_1(z)$, the RHS is $\varphi_2(\varphi(z))$, by applying φ_2^{-1} to both we obtain $(\varphi_2^{-1} \circ \varphi_1)(z) = \varphi(z)$, so φ we constructed above is the only possibility

(c) φ preserves the cross ratio, so it has to satisfy

$$(3, 2; -1, z) = (4, 3, 6; \varphi(z))$$

for every $z \in \mathbb{C} \cup \{\infty\}$.

$$\frac{(3-(-1))(2-z)}{(3-z)(2-(-1))} = \frac{(4-6)(3-\varphi(z))}{(4-\varphi(z))(3-6)}$$

$$\frac{4(2-z)}{(3-z)\cdot 3} = \frac{(-2)(3-\varphi(z))}{(4-\varphi(z))\cdot(-3)} \qquad \Big| \times (3-z)(4-\varphi(z))\cdot\frac{3}{2}$$

$$2(2-z)(4-\varphi(z)) = (3-z)(3-\varphi(z))$$

$$16 - 8z - 4\varphi(z) + 2z\varphi(z) = 9 - 3z - 3\varphi(z) + z\varphi(z)$$

$$(z-1)\varphi(z) = 5z-7$$

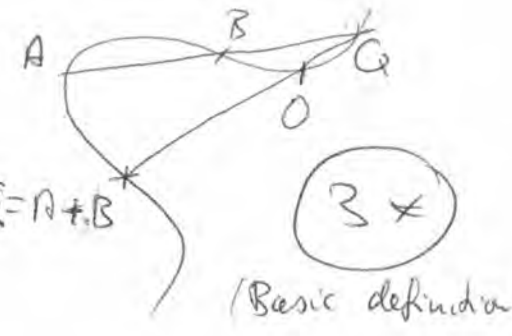
$$\varphi = \frac{5z-7}{z-1}$$

5 ✗

(Standard problem type)

$a=5$, $b=-7$, $c=1$, $d=-1$ is a solution (and all solutions are multiples of this one).

(7) 4(a) Let $A, B \in E$. Take the line AB (the tangent line at A if $A = B$) and let Q be its 3rd point of intersection with E. $A+B$ is the 3rd point of intersection of the line OQ with E.
(If a line is tangent to E, the '3rd' point is defined using intersection multiplicities e.g. if $A \neq B$ and the line AB is tangent to E at A, then $Q = A$.)



3 ×

(Basic definition)

(b) The equation of the line through P and Q is $y = \frac{x+3}{2}$. By substituting this into the equation of F we obtain

$$\left(\frac{x+3}{2}\right)^2 = x^3 - x^2 - 3x$$

$$\frac{x^2}{4} + \frac{3}{2}x + \frac{9}{4} = x^3 - x^2 - 3x$$

$$0 = x^3 - \frac{5}{4}x^2 - \frac{9}{2}x - \frac{9}{4}.$$

4 ×

$x+1$ and $x-3$ are factors of the RHS, therefore it must factor as $(x+1)(x-3)(x+\frac{3}{4})$, so the 3rd point of intersection has x-coordinate $-\frac{3}{4}$ and y-coordinate $\frac{-\frac{3}{4}+3}{2} = \frac{9}{8}$. Hence $\underline{P+Q = \left(-\frac{3}{4}, -\frac{9}{8}\right)}$

To calculate $2P$, we need the equation of the tangent line to E at P. We calculate its slope by implicit differentiation.

(Standard problem type)

Let $f(x,y) = x^3 - x^2 - 3x - y^2$

$$\frac{\partial f}{\partial x} = 3x^2 - 2x - 3 \qquad \left.\frac{\partial f}{\partial x}\right|_P = 2$$

$$\frac{\partial f}{\partial y} = -2y \qquad \left.\frac{\partial f}{\partial y}\right|_P = -2$$

5 ×

The slope is $-\frac{\left.\frac{\partial f}{\partial x}\right|_P}{\left.\frac{\partial f}{\partial y}\right|_P} = -\frac{2}{-2} = 1$, and the equation of the tangent line is $y = x + 2$

8) By substituting $y = x+2$ into the equation of F we get

$$(x+2)^2 = x^3 - x^2 - 3x$$

$$x^2+4x+4 = x^3 - x^2 - 3x$$

$$0 = x^3 - 2x^2 - 7x - 4.$$

We know that $(x+1)^2$ is a factor of the RHS, therefore it must factorize as $(x+1)^2(x-4)$, so the x coordinate of the "3rd" point of intersection with E is 4, and the y coordinate is $4+2=6$. Hence $\underline{2P = (4,-6)}$.

(ii) The points of order 2 are $(\alpha_i, 0)$, $(i=1,2,3)$, where the α_i $(i=1,2,3)$ are the roots of $x^3 - x^2 - 3x = 0$. $x=0$ is a root, the quadratic $x^2-x-3=0$ has roots $x = \frac{1 \pm \sqrt{13}}{2}$, so the points of order 2 are $(0,0)$, $\left(\frac{1+\sqrt{13}}{2}, 0\right)$, $\left(\frac{1-\sqrt{13}}{2}, 0\right)$

3

(Similar to examples done in the lectures)

(c) First we complete the square wrt to y.

Let $y_1 = (y-x-1)$, $x_1 = x$, then $y_1^2 = y^2 - 2xy - 2y + (x+1)^2$, so if we add $(x+1)^2$ to both sides, we can write the equation as

5

$$y_1^2 = x_1^3 + 3x_1^2 + 3x_1.$$

Now we "complete the cube" by introducing $x_2 = x_1 + 1$, $y_2 = y_1$. Then the RHS is simply $x_2^3 - 1$. so the equation becomes $y_2^2 = x^3 - 1$, so $p=0$, $q=-1$ is a solution.

(Not too difficult, but it is only the last 2 steps of of a longer process and it is the last topic in the course. An example will be done in the lectures and there are similar questions on the problem sheet.)

MATH32112/42112/62112/ Exam 2015 Solution

A1. (i) L is a Lie algebra if it is anticommutative, i.e. $[x,x]=0$ for all $x\in L$, and $j(x,y,z)=0$ for all $x,y,z\in L$. If L is anticommutative then $0=[x+y,x+y]=[x,x]+[x,y]+[y,x]+[y,y]=[x,y]+[y,x]$ for all $x,y\in L$ implying $[x,y]=-[y,x]$. Also, $j'(x,y,z)=[[x,y],z]+[[y,z],x]+[[z,x],y]=-[z,[x,y]]-[x,[y,z]]-[y,[z,x]]=-j(x,y,z)$ for all $x,y,z\in L$.

(ii) A subspace I is an ideal of L if $[L,I]\subseteq I$. We say L is simple if it is not abeliean, i.e. $[L,L]\neq\{0\}$, and the only ideals of L are $\{0\}$ and L. Lemma on 2 ideals states that if I,J are two ideals of L then so is $[I,J]=\operatorname{span}\{[x,y]\mid x\in I,\, y\in J\}$. Indeed, if $x\in L$, $u\in I$ and $v\in J$ then $[x,[u,v]]=-[u,[v,x]]-[v,[u,x]]=[u,[x,v]]-[[x,u],v]\in[I,J]$, hence the result.

(iii) Set $L^1:=L$ and define $L^{k+1}:=[L,L^k]$ for $k\in\mathbb{N}$. We say L is nilpotent if $L^N=0$ for some N. Clearly, $L=L^1$ is an ideal of L. Suppose L^k is an ideal of L for some k. Then so is $L^{k+1}=[L,L^k]$ by Lemma on 2 ideals. Hence $L^k\supseteq L^{k+1}$ for all k.

(iv) Set $L^{(0)}:=L$ and define $L^{(k+1)}:=[L^{(k)},L^{(k)}]$ for $k\in\mathbb{N}$. Clearly, $L=L^{(0)}$ is an ideal of L. Suppose $L^{(k)}$ is an ideal of L for some k. Then so is $L^{(k+1)}=[L^{(k)},L^{(k)}]$ by Lemma on 2 ideals. So each $L^{(n)}$ is an ideal of L by induction on n. Hence $L^{(n)}\supseteq[L,L^{(n)}]\supseteq L^{(n+1)}$. We say L is solvable if $L^{(N)}=0$ for some N.

notes

We claim that $L^{(n)}\subseteq L^{2^n}$ for all $n\in\mathbb{Z}_{\geq 0}$. The statement holds for $n=0$. If $L^{(k)}\subseteq L^{2^k}$ for some k then $L^{(k+1)}=[L^{(k)},L^{(k)}]\subseteq[L^{2^k},L^{2^k}]$. So it suffices to show that $[L^m,L^n]\subseteq L^{m+n}$ for all $m,n\in\mathbb{N}$. This is clear when $m=1$. Suppose $[L^k,L^n]\subseteq L^{k+n}$ for some k and all n. Then $[L^{k+1},L^n]=[[L,L^k],L^n]\subseteq[[L,L^n],L^k]+[L,[L^k,L^n]]\subseteq[L^k,L^{n+1}]+[L,L^{k+n}]\subseteq L^{k+1+n}$ (we used Lemma on 2 ideals and our induction assumption).

If $L^n=0$ for some $n\in\mathbb{N}$ then $L^{2^n}\subseteq L^n=0$ (as $2^n\geq n$ for $n\geq 1$) implying $L^{(n)}\subseteq L^{2^n}=0$. So any nilpotent Lie algebra is solvable.

(v) The map $\operatorname{ad}x\colon L\to L$ is defined by setting $(\operatorname{ad}x)(y)=[x,y]$ for all $y\in L$. As the operation in L is bilinear, $\operatorname{ad}x$ is an endomorphism of L and the map $\operatorname{ad}\colon L\to\mathfrak{gl}(L)$ is linear. We call $\operatorname{ad}x$ the *adjoint endomorphism* of x. For all $y,z\in L$ we have that $[\operatorname{ad}x,\operatorname{ad}y](z)=\big((\operatorname{ad}x)\circ(\operatorname{ad}y)-(\operatorname{ad}y)\circ(\operatorname{ad}x)\big)(z)=[x,[y,z]]-[y,[z,x]]=[[x,y],z]=(\operatorname{ad}[x,y])(z)$. So $[\operatorname{ad}x,\operatorname{ad}y]=\operatorname{ad}[x,y]$ for all $x,y\in L$. Hence ad is a representation of L.

unseen

(vi) We have $[2u, v] = 2v$, $[2u, 2w] = -4w = -2(2w)$ and $[v, 2w] = 2u$. So the linear map sending h to $2u$, e to v and f to $2w$ is a homomorphism of Lie algebras (here $\{e, h, f\}$ is the standard basis of $\mathfrak{sl}(2, \Bbbk)$). Since $\operatorname{char}(\Bbbk) \neq 2$, the vectors $2u, v, 2w$ form a basis of A. Hence $A \cong \mathfrak{sl}(2, \Bbbk)$ as Lie algebras.

B2. (i) As a vector space $\mathfrak{gl}(V)$ is the space of all endomorphisms of V with Lie bracket given by $[x, y] = x \circ y - y \circ x$ for all $x, y \in \mathfrak{gl}(V)$. A linear map $\rho\colon L \to \mathfrak{gl}(V)$ is called a representation of L if $\rho([x, y]) = [\rho(x), \rho(y)]$ for all $x, y \in L$. The map $L \times V \to V$ given by $x.v := (\rho(x))(v)$ for all $x \in L$ and $v \in V$ is then bilinear and has the property that $[x, y].v = (\rho([x, y]))(v) = x.y.v - y.x.v$ for all $x, y \in V$. This gives V an L-module structure. We say that ρ is irreducible if $V \neq \{0\}$ and the only L-submodules of V are $\{0\}$ and V.

no keys

(ii) $A \in \mathfrak{gl}(V)$ is nilpotent of $A^N = 0$ for some $N \in \mathbb{N}$. If λ is an eigenvalue of A then $A(v) = \lambda v$ for some nonzero $v \in V$. Then $0 = A^N(v) = \lambda^N v$ implying $\lambda = 0$. Define $L_A, R_A\colon \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ by setting $L_A(X) = A \circ X$ and $R_A(X) = X \circ A$ for all $X \in \mathfrak{gl}(V)$. Then L_A and R_A are endomorphisms of $\mathfrak{gl}(V)$ and $\operatorname{ad} A = L_A - R_A$. Since composition is an associative operation, $(L_A \circ R_A)(X) = L_A(R_A(X)) = A \circ (X \circ A) = (A \circ X) \circ A = R_A(L_A(X)) = (R_A \circ L_A)(X)$ for all $X \in \mathfrak{gl}(V)$, the endomorphism L_A and R_A commute. But then

$$(\operatorname{ad} A)^n = (L_A - R_A)^n = \sum_{i=0}^{n} \binom{n}{k} (-1)^{n-k} L_A^k \circ R_A^{n-k} \qquad (\forall n \in \mathbb{N}).$$

So, if $A^N = 0$ then $(\operatorname{ad} A)^{2N-1} = 0$, hence $\operatorname{ad} A \in \mathfrak{gl}(\mathfrak{gl}(V))$ is nilpotent.

seen example

(iii) A chain of subspaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$ is called a flag in V if $\dim V_i = i$ for all $0 \leq i \leq n$. Lie's theorem states that if $\operatorname{char}(\Bbbk) = 0$ then for any finite dimensional solvable Lie subalgebra L of $\mathfrak{gl}(V)$ there exists a flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ in V such that $x(V_i) \subseteq V_i$ for all $x \in L$ and all $1 \leq i \leq n$.

Suppose $\operatorname{char}(\Bbbk) = p > 0$ and let $V = \Bbbk[X]/(X^p)$ be the truncated polynomial ring over $\Bbbk$. Then V has basis $\{1, t, \ldots, t^{p-1}\}$ where t is the coset of X in V. Let H_1 be the 3-dimensional Heisenberg algebra over $\Bbbk$. It has basis $\{u, v, z\}$ and we have that $[u, v] = z$ and $z \in \mathfrak{z}(H_1)$. Note that H_1 is nilpotent, hence solvable. Let $R_t \in \mathfrak{gl}(V)$ be such that $R_t(t^k) = t^{k+1}$ for all k. The linear map $\rho\colon H_1 \to \mathfrak{gl}(V)$ such that $\rho(u) = \partial/\partial t$, $\rho(v) = R_t$ and $\rho(z) = \mathrm{Id}_V$ has the property that $\rho([u, v]) = \mathrm{Id}_V = [\partial/\partial t, R_t] = [\rho(u), \rho(v)]$, hence defines a representation of H_1 in $\mathfrak{gl}(V)$. If W is a nonzero submodule of the H_1-module V then $W \cap \operatorname{Ker} R_t \neq \{0\}$

implying $t^{p-1} \in W$. But then $(\partial/\partial t)^i(t^{p-1}) \in W$ for all i, yielding $V = W$. So the H_1-module V is irreducible and hence cannot have submodules of dimension $p-1$. This shows that Lie's theorem fails in characteristic $p > 0$.

(iv) The radical of L, denoted $\operatorname{rad} L$, is *the* largest solvable ideal of L. We say that L is semisimple if $\operatorname{rad} L = \{0\}$. Let $R = \operatorname{rad}(L)$, $\bar{R} = \operatorname{rad}(L/R)$ and let $\beta\colon L \to L/R$ be the canonical homomorphism. Let $\tilde{R} = \beta^{-1}(\bar{R})$. Then $\beta([L, \tilde{R}]) = [\beta(L), \bar{R}] \subseteq \bar{R}$, so that $\tilde{R}$ is an ideal of L. Also, $\operatorname{Ker} \beta_{|\tilde{R}} = R$ and $\bar{R} \cong \tilde{R}/R$ by the theorem on isomorphism. Since both R and $\bar{R}$ are solvable, so is $\tilde{R}$. But then $\tilde{R} = R$ by the maximality of R. Hence $\bar{R} = \{0\}$ showing that L/R is semismiple,

(v) Clearly, $[h, w] = [h, [u, v]] = 0$ by the Jacobi identity and $[L, L] = \operatorname{span}\{u, v, w\}$. If $z = \lambda_1 h + \lambda_2 u + \lambda_3 v + \lambda_4 w \in \mathfrak{z}(L)$ then $[h, z] = 0$ forcing $\lambda_i = 0$ for $i \in \{2, 3\}$. Since $[u, z] = 0$ we also get $\lambda_1 = 0$. Therefore, $\mathfrak{z}(L) = \Bbbk w$.

B3. (i) A bilinear symmetric form $\gamma\colon L \times L \to \Bbbk$ is *L-invariant* if $\gamma([x, y], z) + \gamma(y, [x, z]) = 0$ for all $x, y, z \in L$. The *radical* $\operatorname{rad}\gamma$ is the set of all $r \in L$ such that $\gamma(r, x) = 0$ for all $x \in L$. This is a subspace of L. If $x \in L$ and $r \in \operatorname{Rad}\gamma$ then $\gamma([x, r], y) = -\gamma(r, [x, y]) = 0$ for all $y \in L$. Then $[L, \operatorname{Rad}\gamma] \subseteq \operatorname{Rad}\gamma$, i.e. $\operatorname{Rad}\gamma$ is an ideal of L.

For $x, y \in L$ set $\kappa(x, y) := \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y)$. The bilinear form $\kappa\colon L \times L \to \Bbbk$ is called the *Killing form* of L. If L is simple and the Killing form κ of L is nonzero, then $\operatorname{Rad}\kappa \neq L$. But then $\operatorname{Rad}\kappa$ is a proper ideal of L and hence equals $\{0\}$. It follows that κ is non-degenerate.

(ii) Since $L^N = \{0\}$ for some N and $(\operatorname{ad} x \circ \operatorname{ad} y)^k(L) \subseteq L^{2k+1}$ for all k, we see that $\operatorname{ad} x \circ \operatorname{ad} y$ is a nilpotent linear operator for all $x, y \in L$. But then $\kappa(x, y) = \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) = 0$, that is $\kappa = 0$.

(iii) Let $S_0 := \{v_1, \ldots, v_m\}$ be a basis of I. By Linear Algebra, there are $v_{m+1}, \ldots, v_n \in L$ such that $S := \{v_1, \ldots, v_m, v_{m+1} \ldots, v_n\}$ is a basis of L. Let $x, y \in I$, and let X_0 and Y_0 (resp., X and Y) be the matrices of $\operatorname{ad}_I x$ and $\operatorname{ad}_I y$ (resp. $\operatorname{ad} x$ and $\operatorname{ad} y$) relative to S_0 (resp., S). As I is an ideal of L we have that

$$XY = \begin{pmatrix} X_0 Y_0 & * \\ 0 & 0 \end{pmatrix}.$$

But then $\kappa(x, y) = \operatorname{tr}(XY) = \operatorname{tr}(X_0 Y_0) = \kappa_I(x, y)$ for all $x, y \in I$.

(iv) Let $a \in A$ and $x \in L$. Then $((\operatorname{ad} a) \circ (\operatorname{ad} x))^2(L) =$
$= [a, [x, [a, [x, L]]]] \subseteq [a, [x, [a, L]]] \subseteq [a, [x, A]] \subseteq [a, A] \subseteq [A, A] = \{0\}$.
Therefore, $(\operatorname{ad} a) \circ (\operatorname{ad} x))^2 = 0$ showing that $(\operatorname{ad} a) \circ (\operatorname{ad} x)$ is nilpotent. Then $\kappa(a, x) = \operatorname{tr}(\operatorname{ad} a) \circ (\operatorname{ad} x)) = 0$ for all $x \in A$. Hence $A \subseteq \operatorname{Rad} \kappa$.

(v) Note that $[I_i, I_j] \subseteq I_i \cap I_j = \{0\}$ if $i \neq j$. Let $R = \operatorname{rad}(L)$ and suppose $R \neq \{0\}$. Let $0 \neq r \in R$ and write $r = \sum_{i=1}^m r_i$ with $r_i \in I_i$. Then $r_s \neq 0$ for some $1 \leq s \leq m$. The projection $\pi_s \colon L \to I_s$ sends any $x = \sum_{i=1} x_i$ with $x_i \in I_i$ to x_s. The above remark shows that π_s is a surjective homomorphism of Lie algebras. Then $\pi_s(R)$ is a nonzero solvable ideal of I_s. Indeed, $[I_s, \pi_s(R)] = [\pi_s(L), \pi_s(R)] = \pi_s([L, R]) \subseteq \pi_s(R)$. Since I_s is a simple Lie algebra, this is impossible. By contradiction, we deduce that $R = \{0\}$, i.e. L is semisimple.

(vi) Since $[u, [v, [u, [v, L]]]] = \Bbbk[u, [v, [u, v]]] = \{0\}$ and $[v, [v, L]] = \{0\}$ we see that $((\operatorname{ad} u) \circ (\operatorname{ad} v))^2 = (\operatorname{ad} v)^2 = 0$. Hence $\kappa(u, v) = \kappa(v, v) = 0$, showing that $\Bbbk v \subseteq \operatorname{Rad} \kappa$. On the other hand, the matrix of $(\operatorname{ad} u)^2$ with respect to the ordered basis $\{u, v\}$ equals $\operatorname{diag}(0, 1)$. Hence $\kappa(u, u) = 1$ implying $u \notin \operatorname{Rad} \kappa$. So $\operatorname{Rad} \kappa = \Bbbk v$.

notes

B4. (i) An element $x \in \mathfrak{gl}(V)$ is semisimple if V has a basis consisting of eigenvectors for x and it is called nilpotent if $x^N = 0$ for some N. A decomposition $x = x_s + x_n$ with $x_s, x_n \in \mathfrak{gl}(V)$ is called a Jordan decomposition of x if x_s is semisimple, x_n is nilpotent and $[x_s, x_n] = 0$. There exists a Jordan decomposition $x = x_s + x_n$ of x such that both x_s and x_n are polynomials in x. Suppose $x = x'_s + x'_n$ is another Jordan decomposition for x. Then $0 = [x'_s, x] = [x'_s + x'_n] = [x'_s, x'_s] + [x'_s, x'_n] = 0$. So x'_s commutes with x, hence with any polynomial in x. As a result, $[x_s, x'_s] = 0$. Two commuting diagonalisable linear operators can be diagonalised simultaneously. Hence $x_s - x'_s$ is diagonalisable, too. Similarly, $[x'_n, x] = 0$ which implies that x'_n commutes with any polynomial in x. In particular, $[x_n, x'_n] = 0$. But then $(x'_n - x_n)^N = \sum_{i=0}^N \binom{N}{i}(-1)^{N-i}(x'_n)^i \circ x_n^{N-i} = 0$ if $N \gg 0$. So $x'_n - x_n$ is nilpotent. As $x_s + x_n = s'_s + x'_n$ we have that $x_s - x'_s = x'_n - x_n$ is both semisimple and nilpotent. As 0 is the only eigenvalue of a nilpotent endomorphism, we get $x'_s = x_s$ and $x'_n = x_n$.

A Lie subalgebra L of $\mathfrak{gl}(V)$ is called *separating* if $x_s, x_n \in L$ for all $x \in L$.

(ii) We call V an L-module if there is a $\Bbbk$-bilinear mapping $L \times V \to V$ sending $(x, v) \in L \times V$ to $x.v \in V$ such that $[x, y].v = x.y.v - y.x.v$ for all $x, y \in L$ and $v \in V$. We call a nonzero L-module V irreducible notes

if $\{0\}$ and V are the only submodules of V. A subspace W of V is an L-submodule if $x.w \in W$ for all $x \in L$ and $w \in W$. An L-module V is called *completely reducible* if there exist irreducible L-submodules $V_1, \ldots, V_s$ in V such that $V = V_1 \oplus \cdots \oplus V_s$.

Weyl's theorem states the following: Let L be a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0. Then any finite dimensional L-module is completely reducible.

seen example

(iii) Since L is semisimple the adjoint representation ad of L is completely reducible. Then Weyl's theorem implies that there exist irreducible L-submodules $I_1, \ldots, I_m$ of the adjoint L-module L such that $L = I_1 \oplus \cdots \oplus I_m$. Each I_k is $(\operatorname{ad} L)$-invariant, implying $[L, I_k] \subseteq I_k$ for all $k \leq m$. So each I_k is an ideal of L. If $i \neq j$ then $[I_i, I_j] \subseteq I_i \cap I_j = \{0\}$, which shows that the ideals I_j pairwise commute. If J_k is a nonzero ideal of I_k then

$$[L, J_k] = [\textstyle\sum_{i=1}^m I_i, J_k] = \sum_{i=1}^m [I_i, J_k] = [I_k, J_k] \subseteq J_k.$$

So J_k is a nonzero ideal of L. But then J_k is a nonzero $(\operatorname{ad} L)$-submodule of I_k. Therefore, $J_k = I_k$ by the irreducibility of I_k. Since L has no nonzero abelian ideals, this yields that each I_k is a simple Lie algebra.

seen example

(iv) $D \in \mathfrak{gl}(A)$ is a derivation of A if $D(x \cdot y) = D(x) \cdot y + x \cdot D(y)$ for all $x, y \in A$. Let $D, D' \in \operatorname{Der}(A)$ and $\lambda, \lambda' \in \Bbbk$. Then $(\lambda D + \lambda' D')(x \cdot y) = \lambda D(x \cdot y) + \lambda' D'(x \cdot y) = (\lambda D + \lambda' D')(x) \cdot y + x \cdot (\lambda D + \lambda' D')(y)$ which shows that $\operatorname{Der}(A)$ is a subspace of $\mathfrak{gl}(A)$. Next, $[D, D'](x \cdot y) = (D \circ D')(x \cdot y) - (D' \circ D)(x \cdot y) = D(D'(x) \cdot y + x \cdot D'(y)) - D'(D(x) \cdot y + x \cdot D(y)) = (D \circ D' - D' \circ D)(x) \cdot y + x \cdot (D \circ D' - D' \circ D)(x) = [D, D'](x) \cdot y + x \cdot [D, D'](y)$. So $[D, D'] \in \operatorname{Der}(A)$, i.e. $\operatorname{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

unseen

(v) Let $S = \{e, h, f\}$ be the standard basis of $L = \mathfrak{sl}(2, \Bbbk)$, so that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Let E, H, F be the matrices of $\operatorname{ad} e$, $\operatorname{ad} h$, $\operatorname{ad} f$ relative to S, respectively. Then

$$E = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Let $R = \operatorname{Rad} \kappa$ and suppose $R \neq \{0\}$. Then R is a nonzero ideal of L, hence contains an eigenvector for $\operatorname{ad} h$. It follows that $R \cap \{e, h, f\} \neq \emptyset$. But then either $\kappa(e, f) = 0$ or $\kappa(h, h) = 0$. However,

$$EF = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

which gives $\kappa(e,f) = 4$ and $\kappa(h,h) = 8$. Since $\mathrm{char}(\Bbbk) \neq 2$, we reach a contradiction. As a result, κ is non-degenerate.

unseen

(vi) Since $e+2f = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, the characteristic polynomial of $e+2f$ equals t^2-2 and has two distinct roots $\pm\sqrt{2}$. Therefore, $e+2f$ is semisimple. We conclude that $(e+2f)_s = e+2f$ and $(e+2f)_n = 0$.

C5. (i) If $v \in V_\mu$ then $h.(e.v) = [h,e].v + e.(h.v) = 2e.v + e.(\mu v) = (\mu+2)(e.v)$ and $h.(f.v) = [h,f].v + f.(h.v) = -2f.v + f.(\mu v) = (\mu - 2)(f.v)$. So $e.v \in V_{\mu+2}$ and $f.v \in V_{\mu-2}$. We say μ is a *weight* of V if $V_\mu \neq \{0\}$.

(ii) Let $\mathfrak{b} = \Bbbk h \oplus \Bbbk e$. Then $\mathfrak{b}^{(1)} = \Bbbk e$ and $\mathfrak{b}^{(2)} = \{0\}$. So $\mathfrak{b}$ is a solvable Lie subalgebra of L. Then $\mathfrak{b}$ stabilises a flag of subspaces in V (Lie's theorem). Hence there is a nonzero $v \in V$ such that $x.v \in \Bbbk v$ for all $x \in \mathfrak{b}$. In particular, $v, e.v \in V_\lambda$ for some weight λ of V. But then part (i) gives $e.v \in V_\lambda \cap V_{\lambda+2} = \{0\}$. So $e.v = 0$ implying $V_{\mathrm{prim}} \neq \{0\}$. If $w \in V_{\mathrm{prim}}$ then $e.(h.w) = [e,h].w + h.(e.w) = -2e.w + h.0 = 0+0 = 0$. Hence $h.w \in V_{\mathrm{prim}}$.

notes

(iii) We call $v \in V$ a highest weight vector of weight λ if $v \neq 0$, $e.v = 0$ and $h.v = \lambda v$. If v_0 is such a vector and $v_k = \frac{1}{k!}f^k.v_0$ then $f.v_k = \frac{1}{k!}f^{k+1}.v = (k+1)v_{k+1}$. Suppose Clearly $h.v_0 = (\lambda - 2\cdot 0)v_0$. Suppose $h.v_k = (\lambda-2k)v_k$. Then $h.v_{k+1} = h.\frac{1}{k+1}f.v_k = \frac{1}{k+1}([h,f]+f.h).v_k = \frac{1}{k+1}(-2f.v_k + (\lambda-2k)f.v_k) = (\lambda - 2(k+1))v_{k+1}$. By induction on k we get $h.v_k = (\lambda-2k)v_k$ for all $k \in \mathbb{Z}_{\geq 0}$. Since $e.v_0 = 0 = (\lambda - 0 + 1)v_{-1}$ the statement about $e.v_k$ holds for $k=0$. Suppose $e.v_k = (\lambda-k+1)v_{k-1}$ for some k. Then $e.v_{k+1} = \frac{1}{k+1}e.f.v_k = \frac{1}{k+1}(h+f.e).v_k = \frac{1}{k+1}(\lambda - 2k)v_k + \frac{1}{k+1}f.(\lambda-k+1)v_{k-1}) =$
$= \left(\frac{\lambda-2k}{k+1} + \frac{k(\lambda-k+1)}{k+1}\right)v_k = \frac{(k+1)(\lambda-k)}{k+1}v_k = (\lambda-(k+1)+1)v_k$. By induction on k we obtain that $e.v_k = (\lambda-k+1)v_{k-1}$ for all $k \in \mathbb{Z}_{\geq 0}$.

Seen example

(iv) We need to check that E, H, F satisfy the standard relations $[H,E] = 2E$, $[E,F] = H$, $[H,F] = -2F$. Note that P is spanned by the monomials $x^m y^n$ with $m,n \in \mathbb{Z}_{\geq 0}$. We have that $E(x^m y^n) = nx^{m+1}y^{n-1}$, $H(x^m y^n) = (m-n)x^m y^n$ and $F(x^m y^n) = mx^{m-1}y^{n+1}$. Therefore, $(H(E(x^m y^n)) - E(H(x^m y^n)) =$

$$(n(m-n+2) - (m-n)n)x^{m+1}y^{n-1} = 2nx^{m+1}y^{n-1} = 2E(x^m y^n).$$

Hence $[H,E] = 2E$. Similarly, $(H(F(x^m y^n)) - F(H(x^m y^n)) =$

$$(m(m-n-2) - (m-n)m)x^{m-1}y^{n+1} = -2mx^{m-1}y^{n+1} = -2F(x^m y^n).$$

So $[H,F] = -2F$. Finally, $(E(F(x^m y^n)) - F(E(x^m y^n)) =$

$$((n+1)m - n(m+1))x^m y^n = (m-n)x^m y^n = H(x^m y^n).$$

So $[E,F] = H$ and we are done.

(v) The subspace P_m has basis $\{x^{m-k}y^k \mid 0 \leq k \leq m\}$ and is preserved by the linera operators E, H, F. Hence it is an L-submodule of P. Since $H(x^{m-k}y^k) = (m-2k)x^{m-k}y^k$, the weights of the L-module P_m are $\{m - 2k \mid 0 \leq k \leq m\} = \{m, m-2, \ldots, -m+2, -m\}$.

Two hours

UNIVERSITY OF MANCHESTER

GREEN'S FUNCTIONS, INTEGRAL EQUATIONS AND APPLICATIONS : **SOLUTIONS**

Answer ALL six questions (100 marks in total)

Electronic calculators may be used, provided that they cannot store text.

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1. This question is intended to be easy. It tests the students on basic facts that they should have memorised.
(a) The statement is false. **Marking guide:** Marked all or nothing.

(b) The formula is

$$u(x) = \int_a^b G(x, x_0)\ f(x_0)\ \mathrm{d}x_0.$$

Marking guide: Marked all or nothing.

(c) Reciprocity for the Green's function means $G(x, x_0) = \overline{G(x_0, x)}$. The Green's function has reciprocity if and only if the boundary value problem is self-adjoint. **Marking guide:** 3 marks for the reciprocity, and 3 marks for saying BVP is self-adjoint. 1/3 if they omit the conjugate in the reciprocity.

(d) The condition is that $p' = r$. **Marking guide:** Marked all or nothing.

2. This question is intended to be an easy application of the method to find Green's functions in one dimension that we studied in class. First we find that a complementary solution is

$$u(x) = a_1 \cos(x) + a_2 \sin(x).$$

Note that $u_1(x) = \cos(x)$ satisfies the right boundary condition $u_1'(0) = 0$, and $u_2(x) = \sin(x)$ satisfies the right boundary condition $u_2'(\pi/2) = 0$. The Wronskian of these two is

$$W = \cos^2(x) + \sin^2(x) = 1.$$

Thus the Green's function is

$$\begin{aligned} G(x, x_0) &= \frac{u_1(x)u_2(x_0)}{W(x_0)} H(x_0 - x) + \frac{u_1(x_0)u_2(x)}{W(x_0)} H(x - x_0) \\ &= \cos(x)\sin(x_0)H(x_0 - x) + \cos(x_0)\sin(x)H(x - x_0). \end{aligned}$$

Marking guide: A rough guide for allocation of marks will be:

- Complementary solution: 3 marks.
- Checking functions in complementary solution satisfy appropriate BCs: 2 marks.
- Wronskian: 2 marks.
- Correct final formula: 3 marks.

3. (a) The adjoint operator and boundary conditions are found by integrating by parts, or using the general formulae we covered in class. The adjoint operator is

$$\mathcal{L}^* v = v'' + v' - 6v.$$

For the adjoint boundary conditions we can apply Green's second identity which gives

$$\begin{aligned} \langle v, \mathcal{L}u \rangle_{L^2} - \langle \mathcal{L}^* v, u \rangle_{L^2} &= \Big[u'\overline{v} - u\overline{v}' - u\overline{v} \Big]_0^L \\ &= u'(L)\overline{v}(L) - u(L)\overline{v}'(L) - u(L)\overline{v}(L) - u'(0)\overline{v}(0) + u(0)\overline{v}'(0) + u(0)\overline{v}(0). \end{aligned}$$

Setting this equal to zero and assuming that u satisfies the original boundary conditions we have

$$0 = -u(L)\overline{(v'(L) - 3v(L))} - u'(0)\overline{v}(0)$$

from which we conclude that the adjoint boundary conditions are

$$v'(L) - 3v(L) = 0, \quad v(0) = 0.$$

Marking guide: 2 marks for adjoint operator, and 2 marks for each adjoint boundary condition.

(b) For this we apply the Fredholm alternative which states that there is a unique solution of the original boundary value problem for every f if and only if the only solution of the homogeneous adjoint problem is $v(x) = 0$. In this case we have found in part (a) that the homogeneous adjoint problem is

$$\begin{cases} v''(x) + v'(x) - 6v = 0, & x \in (0, L), \\ v'(L) - 3v(L) = 0, & v(0) = 0. \end{cases} \tag{1}$$

We must determine whether there is a nonzero solution to this problem. First, the general solution of the ODE is

$$v(x) = a_1 e^{2x} + a_2 e^{-3x},$$

and

$$v'(x) = 2a_1 e^{2x} - 3a_2 e^{-3x}.$$

Thus the boundary conditions are

$$0 = a_1 + a_2, \quad 0 = a_1 e^{2L} + a_2 6e^{-3L},$$

or in matrix form

$$\begin{pmatrix} 1 & 1 \\ e^{2L} & 6e^{-3L} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

There is a nonzero solution if and only if the determinant is zero:

$$6e^{-3L} - e^{2L} = 0 \Leftrightarrow 6 = e^{5L} \Leftrightarrow L = \frac{1}{5}\ln(6).$$

From the Fredholm alternative we thus have that there exists a unique solution of the original problem if and only if $L \neq \ln(6^{1/5})$. **Marking guide:**

- Show they understand what is required (find solutions of homogeneous adjoint problem) (2 marks).
- Find general solution of homogeneous adjoint problem (1 mark).
- Apply BCs (2 marks).
- Reach correct conclusion (1 mark).

(c) This is another application of the Fredholm alternative. In the case $L = \ln(6^{1/5})$, we can see from the work on part (b) that a nonzero solution of the homogeneous adjoint problem (1) is

$$v(x) = e^{2x} - e^{-3x}.$$

Therefore, when $L = \ln(6^{1/5})$ the Fredholm alternative states that there are infinitely many solutions of the original BVP if

$$\int_0^L (e^{2x} - e^{-3x}) f(x)\ \mathrm{d}x = 0.$$

Marking guide:

- Show they know the condition for the general case from the Fredholm alternative (2 marks).
- Find correct nonzero solution of adjoint problem (1 mark).
- Correct condition on f (1 mark).
- Saying there are infinitely many solutions (1 mark).

4. (a) Putting the harmonic time dependence into the equation we have

$$u''(x)e^{-i\omega t} = -\frac{\omega^2}{c(x)^2}u(x)e^{-i\omega t} + f(x)e^{-i\omega t} \Rightarrow u''(x) + k(x)^2 u(x) = f(x).$$

Marking guide: Should be marked all or nothing.

(b) Since $k(x) = \omega/c_0 = k_0$ for $|x|$ sufficiently large, we have that for x sufficiently positive

$$u(x) = a_1 e^{ik_0 x} + a_2 e^{-ik_0 x}.$$

Since the time dependence of the actual displacement $U(x,t)$ is $e^{-i\omega t}$ the term $e^{ik_0 x}$ gives a wave moving to the right, and $e^{-ik_0 x}$ gives a wave moving to the left. If no wave is coming from positive infinity then for x sufficiently positive it must be the case that there is no wave moving to the left, or $a_2 = 0$. Thus we require

$$0 = a_2 = -\frac{u'(x) - ik_0 u(x)}{2ik_0} \Leftrightarrow u'(x) - ik_0 u(x) = 0$$

for all x sufficiently positive. We express this as

$$\lim_{x\to\infty} u'(x) - ik_0 u(x) = 0.$$

Similarly for x sufficiently negative we have

$$u(x) = b_1 e^{ik_0 x} + b_2 e^{-ik_0 x},$$

and to ensure that there are no waves moving to the right (i.e. coming from negative infinity) we need $b_1 = 0$. Thus we require

$$0 = b_1 = \frac{u'(x) + ik_0 u(x)}{2ik_0} \Leftrightarrow u'(x) + ik_0 u(x) = 0$$

for all x sufficiently negative. We express this as

$$\lim_{x\to-\infty} u'(x) + ik_0 u(x) = 0.$$

Putting them together the radiation conditions are

$$\lim_{x\to\infty} u'(x) - ik_0 u(x) = 0,$$

$$\lim_{x\to-\infty} u'(x) + ik_0 u(x) = 0.$$

Marking guide:

- 3 marks for the conditions themselves.
- 3 marks for the explanation.

If they have the signs wrong in the radiation conditions then 2/3 for that part.

(c) The boundary value problem is

$$\begin{cases} u''(x) + k_0^2 u(x) = f(x), \quad x \in \mathbb{R} \\ \lim_{x\to\infty} u'(x) - ik_0 u(x) = 0, \quad \lim_{x\to-\infty} u'(x) + ik_0 u(x) = 0. \end{cases}$$

The Green's functions $G(x, x_0)$ must satisfy

$$\frac{\mathrm{d}^2 G}{\mathrm{d}x^2}(x, x_0) + k_0^2 G(x, x_0) = 0$$

for $x \neq x_0$, and therefore must take the form

$$G(x, x_0) = \begin{cases} b_1(x_0)e^{ik_0x} + b_2(x_0)e^{-ik_0x} & x < x_0 \\ a_1(x_0)e^{ik_0x} + a_2(x_0)e^{-ik_0x} & x > x_0. \end{cases}$$

The radiation conditions imply that $b_1 = a_2 = 0$, and so in fact

$$G(x, x_0) = \begin{cases} b_2(x_0)e^{-ik_0x} & x < x_0 \\ a_1(x_0)e^{ik_0x} & x > x_0. \end{cases}$$

We know that $G(x, x_0)$ must be continuous at $x = x_0$ which gives

$$b_2(x_0)e^{-ik_0x_0} = a_1(x_0)e^{ik_0x_0},$$

and the derivative $\mathrm{d}G/\mathrm{d}x(x, x_0)$ must jump by 1 at x_0 which gives

$$ik_0a_1(x_0)e^{ik_0x_0} + ik_0b_2(x_0)e^{-ik_0x_0} = 1.$$

Solving these equations yields

$$a_1(x_0) = \frac{e^{-ik_0x_0}}{2ik_0}, \quad b_2(x_0) = \frac{e^{ik_0x_0}}{2ik_0}$$

and so

$$G(x, x_0) = \begin{cases} \frac{e^{ik_0(x_0-x)}}{2ik_0} & x < x_0 \\ \frac{e^{ik_0(x-x_0)}}{2ik_0} & x > x_0. \end{cases}$$

or

$$G(x, x_0) = \frac{e^{ik_0|x-x_0|}}{2ik_0}.$$

Marking guide:

- Setting up equation for G when $x \neq x_0$ (1 mark)
- Applying radiation conditions (2 marks)
- Applying continuity and jump condition (2 marks)
- Final formula (2 marks)

(d) Putting $u(x) = u_{in}(x) + u_{sc}(x)$ into the equation and rearranging a bit we have

$$\underbrace{u_{in}''(x) + k_0^2 u_{in}(x)}_{=0} + u_{sc}''(x) + k_0^2 u_{sc}(x) + k_0^2 M\delta(x - x_0)(u_{in}(x) + u_{sc}(x)) = 0.$$

This implies that

$$u_{sc}''(x) + k_0^2 u_{sc}(x) = -k_0^2 M\delta(x - x_0)(u_{in}(x) + u_{sc}(x)),$$

and u_{sc} satisfies the radiation conditions. Therefore, using the Green's function from part (c)

$$u_{sc}(x) = -k_0^2 M \int_{-\infty}^{\infty} G(x, x_1)(u_{in}(x_1) + u_{sc}(x_1))\delta(x_1 - x_0)\ \mathrm{d}x_0 = -k_0^2 M G(x, x_0)(u_{in}(x_0) + u_{sc}(x_0)).$$

Putting $x = x_0$ into this equation we can solve for $u_{sc}(x_0)$ to get

$$u_{sc}(0) = \frac{iMk_0}{2 - iMk_0}.$$

Now putting this back into the previous equation we have

$$u_{sc}(x) = e^{ik_0|x - x_0|} e^{ik_0 x_0} \frac{iMk_0}{2 - iMk_0}.$$

Marking guide:

- Find ODE satisfied by u_{sc} (3 marks)
- Apply Green's function to get equation for u_{sc} involving $u_{sc}(0)$ (2 marks)
- Find $u_{sc}(0)$ (2 marks)
- Final formula for $u_{sc}(x)$ (2 marks)

5. (a) Start by setting

$$c = \int_0^1 e^{y^2} u(y)\mathrm{d}y.$$

Then by multiplying the integral equation by e^{x^2} and integrating from 0 to 1 we have

$$c = \lambda \left(\int_0^1 e^{x^2} x\ \mathrm{d}x \right) c + \int_0^1 e^{x^2} f(x)\ \mathrm{d}x.$$

or, after evaluating the integral,

$$c\left(1-\lambda\frac{e-1}{2}\right)=\int_0^1 e^{x^2}f(x)\ \mathrm{d}x.$$

This will have a unique solution c if and only if

$$1-\lambda\frac{e-1}{2}\neq 0 \Leftrightarrow \lambda\neq\frac{2}{e-1}.$$

In the case that $\lambda\neq\frac{2}{e-1}$ we solve for c to get

$$c=\frac{1}{1-\lambda\frac{e-1}{2}}\int_0^1 e^{x^2}f(x)\ \mathrm{d}x,$$

and putting this back into the original equation we find that

$$u(x)=\frac{x}{1-\lambda\frac{e-1}{2}}\int_0^1 e^{y^2}f(y)\ \mathrm{d}y+f(x)$$

must be the unique solution of the integral equation. To answer (a) plainly then, there is a unique solution for every continuous f if and only if $\lambda\neq 2/(e-1)$. **Marking guide:** Note that the marking for parts (a) and (b) overlaps somewhat.

- Define c (1 mark)
- Find equation for c from integral equation (1 mark)
- Correct conclusion. (1 mark)

(b) The answer to part (b) has already been found in the work above for part (a). The formula for the solution is

$$u(x)=\frac{x}{1-\lambda\frac{e-1}{2}}\int_0^1 e^{y^2}f(y)\ \mathrm{d}y+f(x).$$

Marking guide: Note that the marking for parts (a) and (b) overlaps somewhat.

- Define c (1 mark)
- Find equation for c from integral equation (2 marks)
- Solve for c (1 mark)
- Write down equation for u correctly (2 marks)

(c) Looking back at the work on part (a), we see that when $\lambda=2/(e-1)$ we must have

$$\int_0^1 e^{x^2}f(x)\ \mathrm{d}x=0$$

for there to be a solution. In this case c can be anything, and so putting c back into the original equation yields the general solution

$$u(x)=cx+f(x)$$

where we have absorbed λ into the arbitrary constant c. **Marking guide:**

- Correct condition (2 marks)
- Correct general solution (1 mark)

6. (a) The boundary value problem for $G(\mathbf{x}, \mathbf{x}_0)$ is

$$\begin{cases} \nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0), \quad \text{for all } \mathbf{x},\ \mathbf{x}_0 \in \mathcal{B}_1, \\ G(\mathbf{x}, \mathbf{x}_0) = 0, \quad \text{for all } \mathbf{x} \in \partial\mathcal{B}_1,\ \mathbf{x} \in \mathcal{B}_1. \end{cases}$$

Marking guide:

- $\nabla^2 G(x, x_0) = \delta(x - x_0)$ (2 marks)
- Boundary conditions (2 marks)
- Specifying x, x_0 in the correct sets everywhere (1 mark)

(b) This follows from equation (1) on the front of the exam if we set $D = \mathcal{B}_1$, $f(\mathbf{x}) = u(\mathbf{x})$, and $g(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0)$. Doing this we find

$$\int_{\mathcal{B}_1} f(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_0)\ \mathrm{d}\mathbf{x} = \int_{\partial\mathcal{B}_1} u(\mathbf{x}) \nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x})\ \mathrm{d}s(\mathbf{x})$$

which gives after switching $\mathbf{x}$ and $\mathbf{x}_0$ and using the fact that the Green's function is symmetric

$$f(\mathbf{x}) = \int_{\partial\mathcal{B}_1} h(\mathbf{x}_0) \nabla_{\mathbf{x}_0} G(\mathbf{x}, \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}_0)\ \mathrm{d}s.$$

To find the formula given in the exam note that for $\mathbf{x}_0$ in $\partial\mathcal{B}_1$

$$\mathbf{n}(\mathbf{x}_0) = \mathbf{x}_0.$$

Marking guide:

- Knowing to use formula (1) from front of exam (2 marks)
- Properly specifying f and g in formula (1 mark)
- Getting $\mathbf{x}$ and $\mathbf{x}_0$ in the proper places (1 mark)
- $\mathbf{n}(\mathbf{x}_0) = \mathbf{x}_0$ (2 marks)

(c) We apply the method of images in accordance with the given hint. This requires finding $a \in \mathbb{R}$ and $\widetilde{\mathbf{x}}_0 \in \mathbb{R}^3 \setminus \mathcal{B}_1$ such that

$$G_{3\infty}(\mathbf{x}, \mathbf{x}_0) = -aG_{3\infty}(\mathbf{x}, \widetilde{\mathbf{x}}_0)$$

for all $\mathbf{x}$ with $|\mathbf{x}| = 1$. This is equivalent to

$$|\mathbf{x} - \widetilde{\mathbf{x}}_0|^2 = a^2|\mathbf{x} - \mathbf{x}_0|^2 \Leftrightarrow |\mathbf{x}|^2 + |\widetilde{\mathbf{x}}_0|^2 - 2\mathbf{x} \cdot \widetilde{\mathbf{x}}_0 = a^2(|\mathbf{x}|^2 + |\mathbf{x}_0|^2 - 2\mathbf{x} \cdot \mathbf{x}_0). \tag{2}$$

Rearranging and using $|\mathbf{x}| = 1$ we have

$$a^2(1 + |\mathbf{x}_0|^2) - (1 + |\widetilde{\mathbf{x}}_0|^2) = 2\mathbf{x} \cdot (a^2\mathbf{x}_0 - \widetilde{\mathbf{x}}_0).$$

Since this must hold for every $\mathbf{x}$ with $|\mathbf{x}| = 1$, and the left hand side does not depend on $\mathbf{x}$, in fact both sides must equal zero. Therefore

$$a^2\mathbf{x}_0 = \widetilde{\mathbf{x}}_0, \tag{3}$$

and

$$a^2(1 + |\mathbf{x}_0|^2) - (1 + |\widetilde{\mathbf{x}}_0|^2) = 0.$$

Putting the first of these equations into the second yields

$$a^2(1 + |\mathbf{x}_0|^2) - a^4|\mathbf{x}_0|^2 - 1 = 0. \tag{4}$$

This is a quadratic equation in a^2 which can be solved using the quadratic formula to get

$$a^2 = \frac{(1 + |\mathbf{x}_0|^2) \pm \sqrt{(1 + |\mathbf{x}_0|^2)^2 - 4|\mathbf{x}_0|^2}}{2|\mathbf{x}_0|^2} = \frac{(1 + |\mathbf{x}_0|^2) \pm (1 - |\mathbf{x}_0|^2)}{2|\mathbf{x}_0|^2}.$$

So we find

$$a^2 = \frac{1}{|\mathbf{x}_0|^2} \quad \text{or} \quad a^2 = 1.$$

We do not take $a^2 = 1$ because in that case we would have $\mathbf{x}_0 = \widetilde{\mathbf{x}}_0$. Thus we find

$$a = -\frac{1}{|\mathbf{x}_0|}, \quad \widetilde{\mathbf{x}}_0 = \frac{\mathbf{x}_0}{|\mathbf{x}_0|^2}.$$

Note that a must be chosen to be negative since $G_{3\infty}(\mathbf{x}, \mathbf{x}_0) < 0$ everywhere. The Green's function is given by

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi}\left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} - \frac{1}{|\mathbf{x}_0||\mathbf{x} - \frac{\mathbf{x}_0}{|\mathbf{x}_0|^2}|}\right).$$

From this formula it is not clear what is the value of G at $\mathbf{x}_0 = 0$. We can rearrange the last formula to handle this, and also make the symmetry clear:

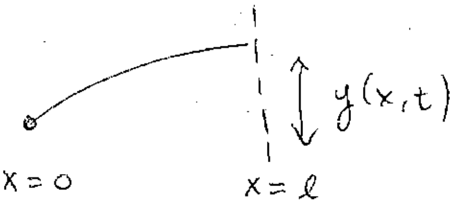
$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi}\left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} - \frac{1}{\sqrt{|\mathbf{x}|^2|\mathbf{x}_0|^2 - 2\mathbf{x} \cdot \mathbf{x}_0 + 1}}\right).$$

Marking guide: I anticipate this will be a difficult question, and so weight the marks quite a bit towards the first steps in the calculation.

- Evidence they know where to start (2 marks)
- Getting to equation (2) (2 marks)
- Separating into equation (3), and the one below it. (3 marks)
- Getting to equation (4) (1 mark)
- All the rest (2 marks).

[Seen similar - different end condition]

Q1



$\frac{\partial y}{\partial x}(x=\ell, t) = 0$ (Given)

(i) $y(x=0,t)=0$ (fixed pt.) (1)

$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ (1-D wave eq$^{\underline{n}}$) (1)

(ii) $y = X(x)T(t) \Rightarrow c^2 X'' T = X T''$

So $\underbrace{\frac{c^2 X''}{X}}_{\text{fn of } x} = \underbrace{\frac{T''}{T}}_{\text{fn of } t}$ $\Bigg\} \Rightarrow \begin{array}{l} T'' + \alpha^2 T = 0 \\ X'' + \frac{\alpha^2}{c^2} X = 0 \end{array}$ for sep. coeff^t α (1)

N.B: we can't satisfy the BC$^{\underline{ns}}$ if we choose $-\alpha^2$, or zero as the separation coeff^t. (2)

$T(t) = A\cos\alpha t + B\sin\alpha t$ $\quad X(x) = C\cos\frac{\alpha x}{c} + D\sin\frac{\alpha x}{c}$

$y(0,t)=0 \Rightarrow X(0)=0 \Rightarrow C=0$

$\frac{\partial y}{\partial x}(\ell,t)=0 \Rightarrow X'(\ell)=0 \Rightarrow \frac{\alpha D}{c}\cos\frac{\alpha \ell}{c} = 0$

$D=0$ (trivial sol$^{\underline{n}}$) or $\frac{\alpha \ell}{c} = \frac{n\pi}{2}$ (n, odd) (4)

So $\alpha = \frac{n\pi c}{2\ell}$

Released from rest $\Rightarrow \dot{T}(0) = 0$ so $B=0$

Superposition gives $y = \sum_{n \text{ odd}} A_n \sin\left(\frac{n\pi x}{2\ell}\right)\cos\left(\frac{n\pi c t}{2\ell}\right)$. (1)

/10

Q2. [seen]

$$\phi = C \cosh(\hat{k}(z-h)) \cos(\hat{k}x - \omega t) \qquad C \text{ const.}$$

where $\hat{k} \tanh \hat{k}h = \omega^2/g$.

For a particle at $\underline{r} = \binom{x}{z}$ we know $\frac{d\underline{r}}{dt} = \underline{u} = \nabla\phi$ (1)

So
$$\frac{dx}{dt} = \phi_x = C \cosh(\hat{k}(z-h))(-\hat{k})\sin(\hat{k}x - \omega t)$$
$$\frac{dz}{dt} = \phi_z = C\hat{k}\sinh(\hat{k}(z-h))\cos(\hat{k}x - \omega t)$$
(*) (2)

For small motion we can expand using a Taylor series:

eg. $\frac{\partial\phi}{\partial x} = \left.\frac{\partial\phi}{\partial x}\right|_{x_0, z_0}$ + smaller terms, $\frac{\partial\phi}{\partial z} = \left.\frac{\partial\phi}{\partial z}\right|_{z_0, x_0}$ + smaller (1)

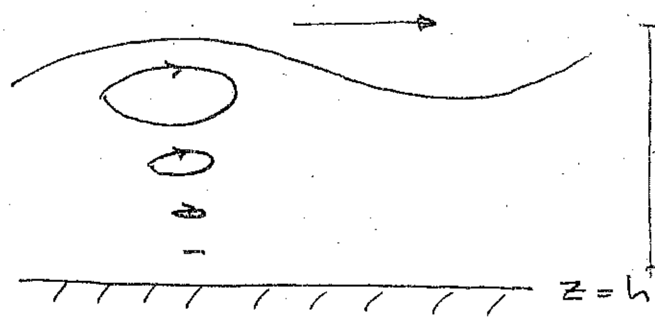
where (x_0, z_0) is a mean location.

We can therefore integrate (*) above:

$$x(t) = x_0 - \frac{\hat{k}C}{\omega}\cosh(\hat{k}(z-h))\cos(\hat{k}x - \omega t)$$
$$z(t) = z_0 - \frac{\hat{k}C}{\omega}\sinh(\hat{k}(z-h))\sin(\hat{k}x - \omega t)$$
(2)

$$\Rightarrow \frac{(x(t)-x_0)^2}{\cosh^2(\cdot)} + \frac{(z(t)-z_0)^2}{\sinh^2(\cdot)} = \frac{\hat{k}^2 C^2}{\omega^2}$$ (2)

equation of an ellipse



(2)

10

3. (related to discussion of internal gravity waves in notes).

$$\omega^2 = \frac{A^2 m^2}{|\underline{k}|^2} \qquad A = \text{const.}$$

$$\underline{k} = (k, l, m).$$

i)

clearly $\dfrac{m^2}{|\underline{k}|^2} \leq 1$ so $\omega^2 \leq A^2$ (2)

$$\Rightarrow \quad T = \frac{2\pi}{\omega} \geq \frac{2\pi}{A}.$$

ii) $\underline{c}_g = \left(\partial\omega/\partial k, \ \partial\omega/\partial l, \ \partial\omega/\partial m \right)$ $\qquad \omega^2 = A^2 m^2 (k^2+l^2+m^2)^{-1}$

$$\Rightarrow \quad 2\omega \frac{\partial \omega}{\partial k} = -A^2 m^2 (k^2+l^2+m^2)^{-2} \cdot 2k$$

$$2\omega \frac{\partial \omega}{\partial l} = -A^2 m^2 (k^2+l^2+m^2)^{-2} \cdot 2l.$$ (4)

$$2\omega \frac{\partial \omega}{\partial m} = -A^2 m^2 (k^2+l^2+m^2)^{-2} \cdot 2m + 2A^2 m (k^2+l^2+m^2)^{-1}$$

$$\underline{c}_g = \frac{-A^2 m^2}{(k^2+l^2+m^2)^2} \left(2k, \ 2l, \ 2m - \frac{2(k^2+l^2+m^2)}{m} \right)$$

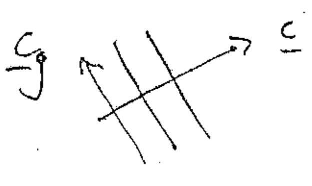
& $\underline{c} = \dfrac{\omega}{|\underline{k}|^2} \underline{k}$

So $\underline{c}_g \cdot \underline{c} \propto \left(k, \ l, \ \dfrac{-k^2-l^2}{m} \right) \cdot (k, l, m)$ (4)

$$= k^2 + l^2 - k^2 - l^2 = 0.$$

$\Rightarrow$ group vel. is $\perp$'r to dir^{n} of crest/trough propagation

$\Rightarrow$ energy propagates at right angles to crest/trough propagation



/10

[bookwork]

Q4. Wave eq^n is $\dfrac{\partial^2\phi}{\partial t^2} = c^2\nabla^2\phi = c^2\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\phi}{\partial r}\right).$

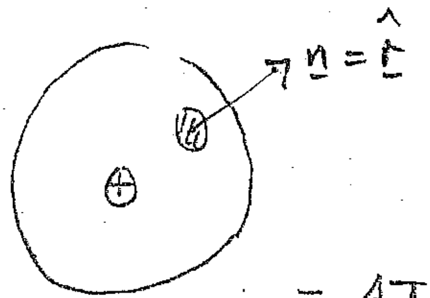
$$= c^2\cdot\frac{1}{r^2}\left(2r\frac{\partial\phi}{\partial r} + r^2\frac{\partial^2\phi}{\partial r^2}\right) = \frac{c^2}{r}\left(\frac{2}{\cancel{r}}\frac{\partial\phi}{\partial r} + r\frac{\partial^2\phi}{\partial r^2}\right)$$ 2

$$= \frac{c^2}{r}\frac{\partial^2}{\partial r^2}(r\phi)$$

So $\dfrac{\partial^2}{\partial t^2}(r\phi) = c^2\dfrac{\partial^2}{\partial r^2}(r\phi)$ the 1D wave eq^n 1

D'Alembert $\Rightarrow r\phi = \underbrace{f(t - r/c)}_{\text{outgoing}} + \underbrace{g(t + r/c)}_{\text{incoming}}.$ 2

So $\phi = -\dfrac{1}{4\pi r}m(t - r/c)$, $f = -\dfrac{1}{4\pi}m$.



$$Q(t) = \int_{S'}\underline{u}\cdot\underline{n}\,dS' = \int_{S'}\left.\frac{\partial\phi}{\partial r}\right|_{r=R} dS'$$ 5

$$= 4\pi R^2\left\{\frac{1}{4\pi r^2}m(t - r/c) + \frac{1}{4\pi r}m'(t - r/c)\right\}_{r=R}$$

$$= m(t - R/c) + Rm'(t - R/c)$$

$$\longrightarrow m(t) \text{ as } R \longrightarrow 0.$$

10

Q5. (modification to include surface tension & finite depth)

$$\eta_t = \phi_z \; ; \; -\rho\phi_t = -\rho g\eta + T\eta_{xx} \quad \text{on } z=0$$

(i) Eliminate η.

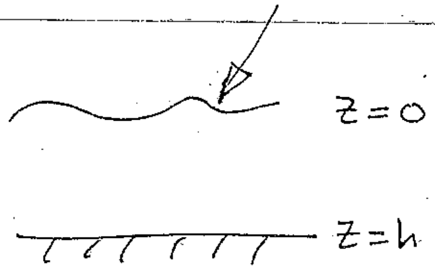
$$-\rho\phi_{tt} = -\rho g\eta_t + T\eta_{txx}$$

$$\Rightarrow \quad -\rho\phi_{tt} = -\rho g\phi_z + T\phi_{zxx}$$

$$\phi_{tt} = g\phi_z - \frac{T}{\rho}\phi_{zxx} \quad \text{on } z=0.$$

3

(ii)



$\nabla^2\phi = 0$ in the fluid

$\frac{\partial\phi}{\partial z} = 0$ on "sea bed".

2

$$\phi(x,z,t) = \Phi(z)\cos(kx-\omega t)$$

so $\Phi'' - k^2\Phi = 0$ in the fluid

$$\Phi'(h) = 0 \quad \& \quad -\omega^2\Phi = g\Phi' - \frac{T}{\rho}(-k^2)\Phi' \quad \text{on } z=0$$

clearly $\Phi = A\sinh kz + B\cosh kz$.

Given that we want $\Phi'(h)=0$ it is easier to rewrite as

$$\Phi = \hat{A}\cosh(k(z-\hat{B}))$$

then $\hat{B} = h$ & $\hat{A}$ is undetermined (since linear)

3

The dispersion relation then comes from the free surface cond^n:

$$-\omega^2\cosh(-kh) = gk\sinh(-kh) + \frac{Tk^2}{\rho}\cdot k\sinh(-kh)$$

2.

$$\omega^2 = (gk + Tk^3/\rho)\tanh(kh)$$ (2)

(iii) $\gamma = kh \Rightarrow \omega^2 = \left(\frac{g\gamma}{h} + \frac{T\gamma^3}{h^3\rho}\right)\tanh\gamma$ [unseen]

For $\gamma << 1$ $\tanh\gamma = \gamma - \frac{\gamma^3}{3} + O(\gamma^5)$

(by Taylor series, if not known)

$$\text{so } \omega^2 = \left(\frac{g\gamma}{h} + \frac{T\gamma^3}{h^3\rho}\right)\left(\gamma - \frac{\gamma^3}{3} + O(\gamma^5)\right)$$ (5)

$$\Rightarrow c^2 = \frac{\omega^2}{k^2} = \frac{h^2\omega^2}{\gamma^2} = gh + h^2\left(\frac{T}{h^3\rho} - \frac{g}{3h}\right)\gamma^2 + O(\gamma^4).$$

$$c = \sqrt{gh}\left(1 + \frac{\cancel{h}}{g}\left(\frac{T}{h^{\cancel{3}2}\rho} - \frac{g}{3\cancel{h}}\right)\gamma^2 + O(\gamma^4)\right)^{1/2}$$

$$= \sqrt{gh}\left(1 + \frac{1}{2g}\left(\frac{T}{h^2\rho} - \frac{g}{3}\right)\gamma^2 + O(\gamma^4)\right)$$

so $c_0 = \sqrt{gh}$ $c_1 = \frac{1}{2}\sqrt{h/g}\,(T/h^2\rho - g/3)$

(iv) $c_g = \frac{d\omega}{dk} = \frac{d\omega}{d\gamma}.h$ and $\omega^2 = c^2k^2 = c^2\gamma^2/h^2$

$$\text{so } \cancel{2}\omega\frac{d\omega}{d\gamma} = \frac{1}{h^2}\left(\cancel{2}\gamma c^2 + \gamma^2.\cancel{2}c\frac{dc}{d\gamma}\right)$$

$$\frac{c\gamma}{h}\frac{d\omega}{d\gamma} = \frac{1}{h^2}(\gamma c)\left(c + \gamma\frac{dc}{d\gamma}\right)$$ (3)

$$\Rightarrow c_g = c + \gamma\frac{dc}{d\gamma}.$$

$$\frac{dc}{d\gamma} = \sqrt{gh}.\frac{1}{g}\left(\frac{T}{h^2\rho} - \frac{g}{3}\right)\gamma + O(\gamma^3) = 0 \text{ when } h = \sqrt{\frac{3T}{\rho g}}.$$ (2)

so $c_g \simeq c + O(\gamma^4)$ at this depth and energy propagates at the phase speed to $O(\gamma^4)$. (20)

Q6 [unseen].

$$\frac{\partial p_1}{\partial t} + k_1 \frac{\partial u_1}{\partial x} = 0$$
$$\frac{\partial u_1}{\partial t} + \frac{1}{\rho_1}\frac{\partial p_1}{\partial x} = 0 \qquad \Big\} \quad k_1, \rho_1 > 0.$$

$$\Rightarrow \frac{\partial^2 p_1}{\partial t^2} + k_1 \frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial t}\right) = 0, \qquad p_{1,tt} - \frac{k_1}{\rho_1} p_{1,xx} = 0$$

$$\frac{\partial^2 p_1}{\partial t^2} = \frac{k_1}{\rho_1}\frac{\partial^2 p_1}{\partial x^2}$$ — 1D wave eq$^{\underline{n}}$ for p_1

define $c_1^2 = \dfrac{k_1}{\rho_1}$ (which is the wave speed). — 4

D'Alembert's solution:

$$p_1 = \underbrace{f(t - x/c_1)}_{\substack{\text{right} \\ \text{trav.}}} + \underbrace{g(t + x/c_1)}_{\substack{\text{left} \\ \text{trav.}}}$$ — 2

(if they don't mention D'Alembert, then must verify this is a sol$^{\underline{n}}$).

Given this p_1, $u_1 = A_1 f(t - x/c_1) + A_2 g(t + x/c_1)$ is an obvious candidate, where

$$1 + k_1 . A_1 . \left(-\frac{1}{c_1}\right) = 0 \quad \Rightarrow \quad A_1 = \frac{c_1}{k_1} = \frac{1}{k_1}\sqrt{\frac{k_1}{\rho_1}} = \frac{1}{\sqrt{k_1\rho_1}}$$

by subst into eq$^{\underline{ns}}$ above. — 4

Similarly, $A_2 + \dfrac{1}{\rho_1}\left(+\dfrac{1}{c_1}\right) \Rightarrow A_2 = \dfrac{-1}{\rho_1 c_1} = \dfrac{-1}{\sqrt{k_1\rho_1}}$

Similarly in region 2: $p_2 = h(t - x/c_2)$ $\quad c_2^2 = \frac{k_2}{\rho_2}$

$u_2 = B_1 h(t - x/c_2)$

where $1 + k_2\left(-\frac{1}{c_2}\right) \cdot B_1 = 0 \Rightarrow B_1 = \frac{c_2}{k_2} = \frac{1}{\sqrt{k_2\rho_2}}$ (1)

At $x=0$, continuity of velocity & pressure:

$$\frac{1}{\sqrt{k_1\rho_1}}\left(f(t) - g(t)\right) = \frac{1}{\sqrt{k_2\rho_2}} h(t)$$

$$f(t) + g(t) = h(t)$$

eliminate $h(t)$: $\frac{1}{\sqrt{k_1\rho_1}}(f-g) = \frac{1}{\sqrt{k_2\rho_2}}(f+g)$

$$f \cdot \left(\frac{1}{\sqrt{k_1\rho_1}} - \frac{1}{\sqrt{k_2\rho_2}}\right) = g \cdot \left(\frac{1}{\sqrt{k_1\rho_1}} + \frac{1}{\sqrt{k_2\rho_2}}\right)$$ (6)

$g/f = \frac{Y_1 - Y_2}{Y_1 + Y_2}$ where $Y_1 = \frac{1}{\sqrt{k_1\rho_1}}$ $\quad Y_2 = \frac{1}{\sqrt{k_2\rho_2}}$

sim $h/f = 1 + \frac{Y_1 - Y_2}{Y_1 + Y_2} = \frac{2Y_1}{Y_1 + Y_2}$

if $k_1 = k_2 = 9$, $\rho_1 = 1$, $\rho_2 = 9$, then $Y_1 = \frac{1}{3}$, $Y_2 = \frac{1}{9}$

So $g/f = \frac{1/3 - 1/9}{1/3 + 1/9} = \frac{2/9}{4/9} = \frac{1}{2}$ (reflected) (3)

and $h/f = \frac{2.(1/3)}{1/3 + 1/9} = \frac{2/3}{4/9} = \frac{6}{4} = \frac{3}{2}$ (transmitted).

20

MATH35032: Mathematical Biology
Solutions to the June exam, 2014

A1. *This problem appeared on an old exam that isn't currently available to students, so should be new to them. Similar single-species population models are studied at length in lecture and in the Problem Sets.*

(a) *[3 marks]* The first term in the ODE

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \frac{hN}{A+N}$$

is the standard logistic growth law, while the second term models the impact of fishing, which is approximately linear in population for $N \ll A$, but tends to the constant rate h for $N \gg A$. The parameters are

r the intrinsic, per-animal *growth rate* of the population: it has units of 1/time. It is also the maximal growth rate or, equivalently, the rate at which the population grows when N is small, so that fishing and self-limitation are small effects.

K the *carrying capacity* of the environment. In this problem it has units of "fish". In the absence of fishing (if $h = 0$) the population would have a stable steady state with $\lim_{t\to\infty} N(t) = K$.

h The *maximal rate of extraction* from fishing: it has units of fish/time.

A The fish population at which *extraction reaches half its maximal rate.*

(b) *[3 marks]* A suitable change of variables is

$$u = \frac{N}{K} \qquad \text{and} \qquad \tau = rt \;\text{ or }\; t = \frac{\tau}{r}.$$

Then one can compute

$$\begin{aligned}
\frac{du}{d\tau} &= \frac{du}{dN}\frac{dt}{d\tau}\frac{dN}{dt} \\
&= \frac{1}{K}\left(\frac{1}{r}\right)\left[rN\left(1 - \frac{N}{K}\right) - \frac{hN}{A+N}\right] \\
&= \frac{N}{K}\left(1 - \frac{N}{K}\right) - \left(\frac{h}{rK}\right)\frac{(N/K)}{(A/K)+(N/K)} \\
&= u\,(1-u) - \frac{\gamma u}{a+u}
\end{aligned}$$

The last line is the expression we were aiming for and, by comparing it to the line above, one can see that

$$\gamma = \frac{h}{rK} \qquad \text{and} \qquad a = \frac{A}{K}.$$

(c) *[4 marks]* If $a = 1$ the equilibrium condition is

$$u(1-u) = \frac{\gamma u}{1+u}$$

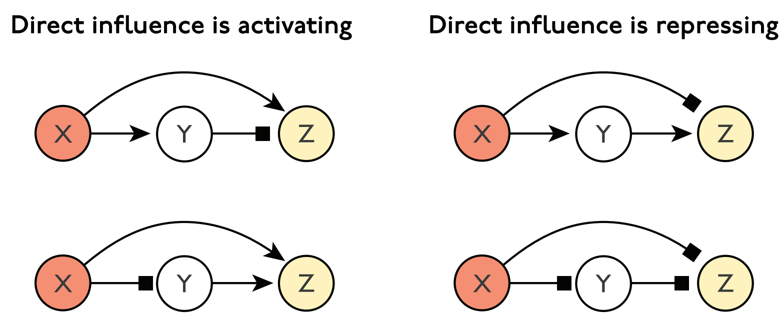
which has roots $u_\star = 0$ and $u_\star = \pm\sqrt{1-\gamma}$. Only the positive square root is a sensible population and, even then, only when $\gamma < 1$. And for γ in this range it's easy to see that $du/dt > 0$ for $0 < u < u_\star$, while $du/dt < 0$ for $u > u_\star$, so the origin is unstable and $u_\star = \sqrt{1-\gamma}$ is stable.

A2. *This is similar to problems I did in example classes, but not identical.*

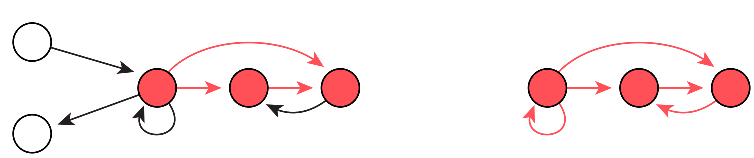
Motifs were introduced by Uri Alon and his collaborators[1], while graphlets were proposed as an alternative by Nataša Pržulj and her colleagues[2].

- *[1 mark]* A *motif* is a small, directed, weakly connected subgraph of a regulatory network that has no parallel edges and no loops.

- *[2 marks]* A *3-node feed-forward loop* (FFL) is a regulatory motif in which one gene, say X, controls the expression of another, Z in two ways: directly, and indirectly, through the expression of some intermediate gene Y. The loop is *incoherent* if the direct and indirect influences of X on Z oppose each other (that is, one is repressing and the other is enhancing). There are four incoherent three-node FFLs (all illustrated below) but a correct answer to the question need only include one of them.



- *[2 marks]* A *graphlet* is similar to a motif, but with the distinction that a graphlet must be an *induced* subgraph of the network. An induced subgraph is one formed by taking a subset of the vertices and *all* the edges that run between them. The example below illustrates the distinction.



  If one were counting motifs, the network at left would be considered to contain a copy of the three-node feed-forward loop (FFL) whose vertices and edges are highlighted in red. It would, however, *not* contain the FFL when regarded as a graphlet because the subgraph induced by the red vertices (shown at right) includes two extra edges that aren't part of the FFL.

[1]R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, and U. Alon (2002), Network motifs: simple building blocks of complex networks, *Science*, **298**:824–827. DOI: 10.1126/science.298.5594.824

[2]N. Pržulj, D. G. Corneil, and I. Jurisica (2004), Modeling interactome: scale-free or geometric?, *Bioinformatics*, **20**:3508–3515. DOI: 10.1093/bioinformatics/bth436

The remaining 5 marks are for the following argument.
Consider the adjacency matrix of the regulatory network. If the network is to contain the graphlet from the exam then there must be some group of five vertices whose mutual interactions look exactly like those pictured in the diagram. The presence of the graphlet thus fixes $5 \times 5 = 25$ of the entries in the network's adjacency matrix—one for each entry in the graphlet's adjacency matrix. On the one hand, if we define K_g to be the number of N-node, E-edge networks that include the graphlet then

$$K_g = \binom{N}{5} \times \binom{5}{1} \times \binom{N^2 - 25}{E - 4}.$$

Here the first factor counts the number of ways to choose the 5 vertices that appear in the graphlet, the second factor counts the number of ways to choose from among those five the single vertex that has four outgoing edges and the last factor counts the number of ways to place the remaining $(E - 4)$ edges.

On the other hand, there are a total of

$$T = \binom{N^2}{E}.$$

possible N-node, E-edge regulatory networks and so, assigning equal probability to each, we find that the desired probability is

$$p = \frac{K_g}{T} = \frac{\binom{N}{5}\binom{5}{1}\binom{N^2 - 25}{E - 4}}{\binom{N^2}{E}}$$

For the case $N = 8$, $E = 9$ this is

$$p = \frac{\binom{8}{5}\binom{5}{1}\binom{39}{5}}{\binom{64}{9}} = \frac{56 \times 5 \times 1712304}{27540584512} = \frac{1070190}{61474519} \approx 0.0174,$$

but an answer in terms of binomial coefficients will receive full credit.

A3. *This problem relates to Lewis Wolpert's "French Flag" model of developmental patterning. A similar problem, but with an arbitrary power-law degradation kinetics, M^k instead of the M^3 used here, appeared in the problem sets.*

(a) *[4 marks]* The steady-state concentration profile obeys the ODE

$$\frac{d^2M}{dx^2} = \frac{\alpha}{D}M^3.$$

Substituting the proposed form into this yields

$$\frac{\gamma\nu(\nu+1)}{(x+\epsilon)^{\nu+2}} = \left(\frac{\alpha}{D}\right)\frac{\gamma^3}{(x+\epsilon)^{3\nu}}$$

which in turn implies

$$\nu + 2 = 3\nu \qquad \text{or} \qquad \nu = 1$$

and

$$2\gamma = \left(\frac{\alpha}{D}\right)\gamma^3 \qquad \text{or} \qquad \gamma = \sqrt{\frac{2D}{\alpha}}.$$

Finally, one can use the boundary condition to set ϵ. As $M(0) = M_0$ we have

$$M_0 = \frac{\gamma}{0+\epsilon} \qquad \text{or} \qquad \epsilon = \frac{\gamma}{M_0} = \sqrt{\frac{2D}{\alpha M_0^2}}$$

(b) *[3 marks]* The position $x_1^\star$ that forms the boundary between cells of types A and B satisfies $M(x_1^\star) = \theta_1$ so $\theta_1 = \gamma/(x_1^\star + \epsilon)$ and thus

$$x_1^\star \;=\; \frac{\gamma}{\theta_1} - \epsilon \;=\; \frac{\gamma}{\theta_1} - \frac{\gamma}{M_0}.$$

(c) *[3 marks]* By direct calculation

$$\Delta \;=\; x_2^\star - x_1^\star \;=\; \left(\frac{\gamma}{\theta_2} - \frac{\gamma}{M_0}\right) - \left(\frac{\gamma}{\theta_1} - \frac{\gamma}{M_0}\right) \;=\; \frac{\gamma}{\theta_2} - \frac{\gamma}{\theta_1}$$

which is clearly independent of M_0. This means that when M_0 varies—as it might if, for example, rates of protein synthesis did—both boundaries $x_j^\star$ shift by the same amount, preserving the length of the region of cells of type B.

B4. *Similar questions about two-species population models have appeared in problem sets, but this one should be new to students.*

The object of study here is the model

$$\frac{dx_1}{d\tau} = x_1\left[1 - \frac{x_1}{1+\beta_2 x_2}\right] \qquad \text{and} \qquad \frac{dx_2}{d\tau} = x_2\left[1 - \frac{x_2}{1+\beta_1 x_1}\right]. \tag{4.1}$$

(a) *[3 marks]* The ODEs in (4.1) look similar to the dimensionless forms of the logistic growth law

$$\frac{dx_1}{d\tau} = x_1(1-x_1).$$

But in (4.1) the x_1 appearing in the population-limiting factor $(1-x_1)$ is scaled by $(1+\beta_2 x_2) > 1$. Thus the presence of species two acts to *increase* the effective carrying capacity for species one. Species one exerts a similar beneficial influence on $dx_2/d\tau$ and so this model represents two species that interact to produce mutual benefit.

If either species is left to develop on its own it would have a stable attracting equilibrium population at $x_j = 1$.

(b) *[6 marks]* A null cline is a locus on which one or the other of the derivatives vanish. Here

$$\frac{dx_1}{d\tau} = 0 \;\Rightarrow\; x_1 = 0 \text{ or } x_1 = 1 + \beta_2 x_2,$$

while

$$\frac{dx_2}{d\tau} = 0 \;\Rightarrow\; x_2 = 0 \text{ or } x_2 = 1 + \beta_1 x_1.$$

Figure B4.1 includes all the sets of null clines requested in the problem.

(c) *[5 marks]* If we define $g_j(x_1, x_2)$ so that $dx_j/d\tau = g_j(x_1, x_2)$ then

$$A = \begin{pmatrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} \\ \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \left(1 - \dfrac{2x_1}{1+\beta_2 x_2}\right) & \dfrac{\beta_2 x_1^2}{(1+\beta_2 x_2)^2} \\ \dfrac{\beta_1 x_2^2}{(1+\beta_1 x_1)^2} & \left(1 - \dfrac{2x_2}{1+\beta_1 x_1}\right) \end{pmatrix} \tag{4.2}$$

An equilibrium with $x_1^\star$, $x_2^\star > 0$ arises from the intersection of the null clines

$$x_1 = 1 + \beta_2 x_2 \qquad \text{and} \qquad x_2 = 1 + \beta_1 x_1,$$

which implies that

$$\frac{x_1^\star}{1+\beta_2 x_2^\star} = 1 \qquad \text{and} \qquad \frac{x_2^\star}{1+\beta_1 x_1^\star} = 1.$$

Putting these relationships into (4.2) yields the desired result:

$$A = \begin{pmatrix} -1 & \beta_2 \\ \beta_1 & -1 \end{pmatrix}.$$

(d) *[8 marks]* The equilibria of (4.1) are (0, 0), (0,1), (1,0) and

$$(x_1^\star,\, x_2^\star) \;=\; \left(\frac{1+\beta_2}{1-\beta_1\beta_2},\, \frac{1+\beta_1}{1-\beta_1\beta_2}\right). \tag{4.3}$$

As the β_j are positive, the only way that the equilibrium populations in (4.3) can be positive is if $\beta_1\beta_2 < 1$. The stabilities of all these equilibria are summarised in Table B4.1.

(e) *[3 marks]* If the $\beta_1\beta_2 \geq 1$ then, eventually, any solution with initial data $x_1(0) > 0$ and $x_2(0) > 0$ will enter the region between the two null clines that aren't coincident with the coordinate axes. In this region $x_1(\tau)$ and $x_2(\tau)$ are monotone increasing and, as the system has no stable fixed points when $\beta_1\beta_2 \geq 1$, both populations increase without bound. Robert May[3] has described this as "an orgy of mutual benefaction".

[3] R. M. May *et al.* (1981), *Theoretical Ecology. Principles and Applications*, 2nd edition, Blackwell Scientific Publications. ISBN 0878935150.

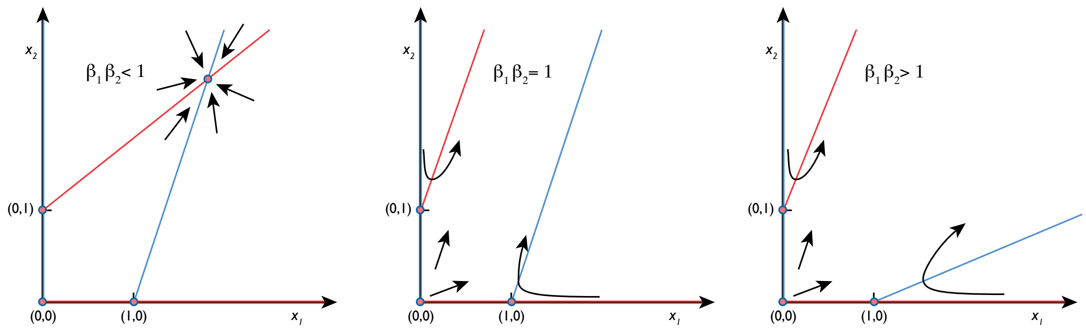


Figure B4.1: *The panel at left shows the pattern of null clines—blue for $dx_1/d\tau = 0$ and red for $dx_2/d\tau = 0$—for the case where $\beta_1\beta_2 < 1$, while the middle panel gives the story when $\beta_1\beta_2 = 1$: in the latter case the null clines include a pair of parallel lines running through the equilibria at $(0,1)$ and $(1,0)$. Finally, the panel at right illustrates the situation when $\beta_1\beta_2 > 1$.*

$(x_1^\star,\, x_2^\star)$	*Linearization*	*Eigenvalues*	*Classification*
$(0,\, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	both positive	unstable node
$(1,\, 0)$	$\begin{bmatrix} -1 & \beta_2 \\ 0 & 1 \end{bmatrix}$	one positive and one negative	saddle
$(0,\, 1)$	$\begin{bmatrix} 1 & 0 \\ \beta_1 & -1 \end{bmatrix}$	one positive and one negative	saddle
$\left(\frac{1+\beta_2}{1-\beta_1\beta_2},\, \frac{1+\beta_1}{1-\beta_1\beta_2}\right)$	$\begin{bmatrix} -1 & \beta_2 \\ \beta_1 & -1 \end{bmatrix}$	both negative when $\beta_1\beta_2 < 1$.	stable when present

Table B4.1: *The equilibria of (4.1) along with the linearisation (4.2) evaluated at $(x_1^\star,\, x_2^\star)$ and the resulting classifications. Note that the stability types of the first three points, which are always present, do not depend on the parameters at all.*

B5. *This year's coursework will involve an SIR model, but with ODEs rather than the discrete-state Markov process considered here. The material on metabolic flux analysis (parts (a)–(c)) appears in the problem sets, although the specific problem below does not.*

(a) *[4 marks]* The stoichiometric matrix N has one row for each "species" and one column for each reaction: here the species are the disease states S, I & R. The entry N_{ij} is the number of molecules of species i produced when reaction j occurs: if the reaction consumes species i— if species i appears on the left-hand side of the reaction—this number may be negative. If we arrange the disease states in the order $\{S, I, R\}$ and the reactions in the order {infection, recovery} the stoichiometric matrix is

$$N = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Of course, other orderings on the reactions and disease states will give rise to permuted forms of N, any one of which will be regarded as correct if explained properly.

(b) *[5 marks]* The desired conserved quantity is $S + I + R$. One can obtain it by performing row-reduction on a copy of N that is augmented at right with a copy of the identity,

$$N = \left[\begin{array}{rr|rrr} -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rrr} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right],$$

and then noting that the bottom row consists entirely of 1's.

(c) *[4 marks]* The rank is the number of linearly independent rows and a suitable decomposition is

$$N \;=\; LN_R \;=\; \left[\frac{I_r}{L_0}\right] N_R \;=\; \left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \\ \hline -1 & -1 \end{array}\right] \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

where N_R consists of a pair of linearly independent rows from N. Here I've used the first two rows, but students will receive full credit for any answer in which N_R consists of a pair of linearly independent rows from N, L has the specified form and $N = LN_R$.

(d) *[3 marks]* The table below lists all possible states consistent with the initial condition.

S	I	R
1	1	0
1	0	1
0	2	0
0	1	1
0	0	2

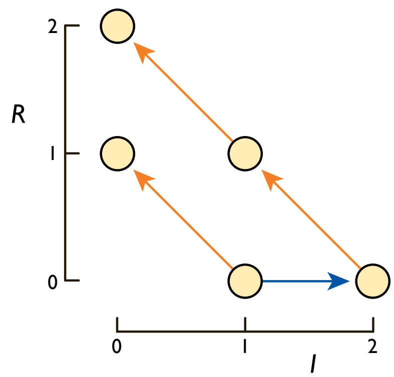


Figure B5.1: *The nodes here show correspond to all possible states of the system enumerated in part (4). The single blue arrow shows the only possibility for further infection, while the orange arrows show the transitions that occurs when an infected person recovers. Note that the states with $I = 0$ are all absorbing states.*

(e) *[3 marks]* Given that $S + I + R$ is conserved, we can specify the state of the system by giving the number of infected and recovered persons. The graph in Figure B5.1 shows all five possible states as well as those transitions between them which have nonzero rates.

(f) *[4 marks]* The desired ODEs can be written in matrix form

$$\frac{d}{dt}\begin{bmatrix} \pi_{10} \\ \pi_{20} \\ \pi_{11} \\ \pi_{01} \\ \pi_{02} \end{bmatrix} = \begin{bmatrix} -(\beta+\gamma) & 0 & 0 & 0 & 0 \\ \beta & -2\gamma & 0 & 0 & 0 \\ 0 & 2\gamma & -\gamma & 0 & 0 \\ \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \end{bmatrix}\begin{bmatrix} \pi_{10} \\ \pi_{20} \\ \pi_{11} \\ \pi_{01} \\ \pi_{02} \end{bmatrix} \tag{5.1}$$

where β and γ are the reaction rates.

(g) *[2 marks]* The asymptotic probabilities form an eigenvector with eigenvalue 0 for the matrix R in (5.1). It's clear that the states in which no one is infected are absorbing, and all others are transient, so eventually the only nonzero components of π will be π_{01} and π_{02}. Further, as the sum of the π_{jk} is conserved, we know that with $\lim_{t\to\infty} \pi_{01}(t) + \pi_{02}(t) = 1$.

B6. *This is a cut-down version of a harder homework problem that treated the case of a ring of cells of arbitrary size.*

(a) *[2 marks]* If $X_j = X_\star$ and $Y_j = Y_\star$ for all $1 \leq j \leq N$ then

$$\frac{dX_j}{dt} = f(X_\star, Y_\star) + \mu(X_\star - 2X_\star - X_\star) = 0$$

and a similar equation holds for dY_j/dt. This solution is spatially uniform because it is independent of j.

(b) *[7 marks]* Entries in the linearisation consist of derivatives

$$\frac{\partial}{\partial X_k}\left(\frac{dX_j}{dt}\right) = \begin{cases} \partial_x f - 2\mu & \text{if } j = k \\ \mu & \text{if } k = j \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\frac{\partial}{\partial Y_k}\left(\frac{dX_j}{dt}\right) = \begin{cases} \partial_y f & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

Similar expressions hold for the partial derivatives of dY_j/dt. In these latter expressions the role of $f(X_j, Y_j)$ is played by $g(X_j, Y_j)$ and μ is replaced by ν. The linearisation thus has the specified form with

$$\begin{array}{ll} a = \partial_x f|_{X_\star, Y_\star} & b = \partial_y f|_{X_\star, Y_\star} \\ c = \partial_x g|_{X_\star, Y_\star} & d = \partial_y g|_{X_\star, Y_\star} \end{array}$$

(c) *[2 marks]*

$$\mathcal{D}_3 u = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 3 \end{bmatrix} = -3u$$

so u is an eigenvector with eigenvalue $\lambda_u = -3$. A further eigenvector-eigenvalue pair is $[1, 1, 1]^T$ with eigenvalue zero.

(d) *[8 marks]* From the results in part (c) we have, on the one hand, that

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \theta(t)u \\ \eta(t)u \end{bmatrix} = \begin{bmatrix} \frac{d\theta}{dt}u \\ \frac{d\eta}{dt}u \end{bmatrix}.$$

On the other hand, the linearisation is

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \left(\begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} + \begin{bmatrix} \mu\mathcal{D}_3 & 0 \\ 0 & \nu\mathcal{D}_3 \end{bmatrix}\right)\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ &= \\ &= \begin{bmatrix} (a\theta(t) + b\eta(t) - 3\mu\theta(t))\,u \\ (c\theta(t) + d\eta(t) - 3\nu\eta(t))\,u \end{bmatrix}. \end{aligned}$$

This in turn implies

$$\begin{aligned}
\frac{d\theta}{dt} &= (a - 3\mu)\theta(t) + b\eta(t) \\
\frac{d\eta}{dt} &= c\theta(t) + (d - 3\nu)\eta(t),
\end{aligned}$$

which is the result we sought.

(e) *[6 marks]* The system of ODEs governing $\theta(t)$ and $\eta(t)$ is thus linear, so is unstable provided that the matrix

$$B = \begin{bmatrix} (a - 3\mu) & b \\ c & (d - 3\nu) \end{bmatrix}$$

has at least one positive eigenvalue. This happens *unless*

$$\det(B) = (a - 3\mu)(d - 3\nu) - cd > 0 \;\text{ and }\; \mathrm{Tr}(B) = a + d - 3(\mu + \nu) < 0.$$

Suitable values thus include $a = d = 4$, $\mu = \nu = 1/3$ and $c = b = 0$.

MATH36022 Solutions: Numerical Analysis 2 Exam 2015

SECTION A

A1. **This is all bookwork.**

a) $a_{ij} = \int_a^b w(x)\phi_i(x)\phi_j(x)\,dx$ and $f_i = \int_a^b f(x)\phi_i(x), dx, \quad i,j = 0,1,\ldots,n$. [**2 marks**].

b) Suppose there is a nonzero vector z such that $Az = 0$. Then we have

$$\begin{aligned} 0 = z^T Az &= \sum_{i=0}^{n}\sum_{j=0}^{n} z_i a_{ij} z_j = \sum_{i=0}^{n}\sum_{j=0}^{n}\int_a^b w(x) z_i\phi_i(x)\phi_j(x) z_j\,dx \\ &= \int_a^b w(x)\left(\sum_{i=0}^{n} z_i\phi_i(x)\right)\left(\sum_{j=0}^{n} z_j\phi_j(x)\right)dx \\ &= \int_a^b w(x)\left(\sum_{k=0}^{n} z_k\phi_k(x)\right)^2 dx = \left\|\sum_{k=0}^{n} z_k\phi_k(x)\right\|_{2,w}^2. \end{aligned}$$

Since the ϕ_i are linearly independent and $z \neq 0$, $\sum_{k=0}^{n} z_k\phi_k(x)$ is non-zero and hence so is its norm. Hence $z^T Az \neq 0$, a contradiction. Therefore A is nonsingular. [**6 marks**].

c) A good choice is to let ϕ_i be a polynomial of degree i with $\{\phi_i(x)\}$ chosen to be orthogonal with respect to the weight function $w(x)$. Then, A is diagonal ($a_{ij} = 0$ whenever $i \neq j$) and the normal equations are easy and cheap to solve. [**2 marks**].

A2. **Same exercise with different choices of $r(x)$ was set on an Examples Sheet. The second part is bookwork.**

We are looking for an approximation of the form

$$r_{21}(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x} = \frac{p_{21}(x)}{q_{21}(x)}$$

and we require that $\exp(2x)q_{21}(x) - p_{21}(x) = O(x^4)$. Hence,

$$\left(1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + O(x^4)\right)(1 + b_1 x) - \left(a_0 + a_1 x + a_2 x^2\right) = O(x^4).$$

Equating the coefficients of x^0, x^1, x^2, x^3 to zero gives:

$$1 - a_0 = 0, \quad b_1 + 2 - a_1 = 0, \quad 2b_1 + 2 - a_2 = 0, \quad 2b_1 + \frac{4}{3} = 0.$$

Hence, $a_0 = 1$, $b_1 = -2/3$, $a_1 = 2 - 2/3 = 4/3$ and $a_2 = 2 - 4/3 = 2/3$ so

$$r_{21}(x) = \frac{1 + 4x/3 + 2x^2/3}{1 - 2x/3}.$$

[**6 marks**]. Polynomials cannot have asymptotes and they are always finite on the finite real axis and tend to $\pm\infty$ as $x \to \pm\infty$. They also have a tendency to oscillate. A Padé approximant is a ratio of polynomials. A rational function with equal degree numerator and denominator stays bounded as $x \to \pm\infty$. A rational function also has poles (the roots of the denominator polynomial). Rationals can also be free of oscillations. [**2 marks for any 2 of these reasons**].

A3. **This question is taken straight from an examples sheet - where it was posed with a general error tolerance, rather than 10^{-3}.**

Let $f_i = f(x_i)$ and set $\bar{f}_i = f_i + \epsilon_i$, with $|\epsilon_i| \leq 10^{-3}$. Writing the closed rule (Simpson's rule) as $J(f) = \frac{1}{6}(f_0 + 4f_{\frac{1}{2}} + f_1)$ we have

$$|J(f) - J(\bar{f})| = \frac{1}{6}|\epsilon_0 + 4\epsilon_{\frac{1}{2}} + \epsilon_1| \leq \frac{1}{6}(10^{-3} + 4 \cdot 10^{-3} + 10^{-3}) = 10^{-3}.$$

[**2 marks**]. Similarly, for the open rule,

$$|J(f) - J(\bar{f})| \leq \frac{1}{3}(2 \cdot 10^{-3} + 10^{-3} + 2 \cdot 10^{-3}) = \frac{5}{3} \cdot 10^{-3}.$$

[**2 marks**]. So, for the closed rule, where all the weights are positive, errors of size at most 10^{-3} in the f_i values change the approximation to the integral by at most 10^{-3}. But for a rule with weights of both sign, the change in the rule can exceed 10^{-3}. [**2 marks**].

A4. **The statement of the condition is bookwork. Very similar examples have been set as an exercise on an Examples Sheet.**

Let $f(x, y)$ be continuous for $x \in [0, 1]$ and for all $y \in \mathbb{R}$. If f satisfies

$$|f(x, u) - f(x, v)| \leq L|u - v| \quad \text{for all } x \in [0, 1] \text{ and all } u, v \in \mathbb{R},$$

where L is a finite constant, then a unique solution to the IVP exists. [**2 marks**].

For (1) we have

$$|f(x, u) - f(x, v)| = |3x + 2u^2 - \left(3x + 2v^2\right)| = |2(u^2 - v^2)| = 2|(u + v)(u - v)|.$$

Since the term $(u + v)$ is unbounded, the condition is not satisfied. [**2 marks**]. For (2), since $x \in [0, 1]$, we have

$$|f(x, u) - f(x, v)| = |e^{-x}(\sin(u) - \sin(v))| \leq e^0|\sin(u) - \sin(v)|.$$

Applying the mean value theorem gives $|f(x, u) - f(x, v)| \leq |(u - v)\cos(\theta)|$ for some $\theta \in [0, 1]$ and so the condition holds with $L = 1$. [**3 marks**]

A5. **This is bookwork and similar to part of a Section B question from 2012**.

(a) For the ℓ-step method, we can derive the methods by replacing the integrand in

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))\, dx,$$

by the polynomial $p_k(x)$ of degree k which interpolates f at the $k + 1$ values

- $x_n, x_{n-1}, \ldots, x_{n-k}$ (Bashforth)
- $x_{n+1}, x_{n-1}, \ldots, x_{n-k+1}$ (Moulton)

where $k = \ell - 1$ for A–B and $k = \ell$ for A–M. The Adams-Bashforth method is explicit (so has $b_0 = 1$) and has order ℓ. The Adams-Moulton method is implicit but has order $\ell + 1$. Hence the A–B methods are easier to implement but less accurate; the A–M methods are harder to implement (as a nonlinear equation needs to be solved for y_{n+1}) but have better accuracy. [**5 marks**]

(b) Using the two ℓ-step schemes, a simple predictor–corrector method is given by

i. Predict: Compute $y_{n+1}^{(0)}$ using the (explicit) A-B method

ii. Evaluate: $f_{n+1}^{(0)} = f(x_{n+1}, y_{n+1}^{(0)})$

iii. Correct: Use the (implicit) A-M method to compute $y_{n+1}^{(1)}$ but using the 'predicted value' $f_{n+1}^{(0)}$ in place of f_{n+1}.

Combining the methods as a predictor-corrector pair maintains the accuracy of the (implicit) Adams–Moulton method, but no nonlinear equations need to be solved.[**4 marks**]

SECTION B

B6. **This is all bookwork. For part (c), the best approximation was constucted in class via geometric arguments and not via the Equioscillation Theorem.**

(a) The leading term of $T_{n+1}(x)$ is $2^n x^{n+1}$ [**1 mark**] and the extremal values are attained at the $n+2$ points

$$x_i = \cos\left(\frac{i\pi}{n+1}\right), i = 0, 1, \ldots, n+1,$$

(and the sign alternates). [**2 marks**]

(b) An alternant is a set of (at least) $n+2$ points $x_0, x_1, \ldots, x_{n+1}$, with $a \le x_0 < x_1 < \cdots < x_{n+1} \le b$, such that

$$|f(x_i) - p_n(x_i)| = \|f - p_n\|_\infty, \qquad i = 0 : n+1$$

and

$$f(x_i) - p_n(x_i) = -\big(f(x_{i+1}) - p_n(x_{i+1})\big), \qquad i = 0 : n.$$

[**3 marks**]

(c) Let $m = \min_{x\in[-1,1]} f(x)$ and $M = \max_{x\in[-1,1]} f(x)$. $p_0 = c = \frac{1}{2}(m+M)$. [**1 mark**]

With this choice $\|f - p_0\|_\infty = c - m = M - c = \frac{1}{2}(M - m)$. At any point x where $f(x) = m$, we have $f(x) - c = \frac{1}{2}(m - M) = -\|f - p_0\|_\infty$ and at any point x where $f(x) = M$, we have $f(x) - c = \frac{1}{2}(M - m) = +\|f - p_0\|_\infty$. There is at least one point in $[-1, 1]$ where $f(x) = m$ and at least one point where $f(x) = M$. Hence, $p_0 = c$ has a 2-point alternant. [**3 marks**]

(d) We have $f(x) - q_n(x) = 2^{-n}T_{n+1}(x)$. Since $|T_{n+1}(x)| \le 1$, we have $\|f - q_n\|_\infty = 2^{-n}$. This is attained at points where $T_{n+1}(x) = \pm 1$. By part (a), we know that there exist $n+2$ points where $T_{n+1}(x) = \pm 1$ and hence there are $n+2$ points where $f(x) - q_n(x) = \|f - q_n\|_\infty = 2^{-n}$. The sign of $f(x) - q_n(x)$ also alternates at these points and so q_n is a best L_∞ approximation. [**4 marks**]

(e) Since $x^{n+1} - q_n(x) = 0 - 2^{-n}T_{n+1}(x)$ in part (d), saying that $q_n(x)$ is the best L_∞ approximation to x^{n+1} is equivalent to saying that $2^{-n}T_{n+1}(x)$ has the smallest L_∞ norm, of all monic polynomials of degree $n+1$. [**2 marks**]

Consider $\|f - p_n\|_\infty$. Unfortunately, we can't control the term with ξ_x but we can try to position the interpolation points x_i so that

$$\| \Pi_{i=0}^{n} (x - x_i) \|_\infty$$

is as small as possible. Since $\Pi_{i=0}^{n} (x - x_i)$ is also a monic polynomial of degree $n+1$, part (d) tells us that we should choose $\{x_i\}_{i=0}^{n}$ as the $n+1$ roots of the polynomial $T_{n+1}(x)$. **[2 marks]** With uniformly spaced points, $\| \Pi_{i=0}^{n} (x - x_i) \|_\infty$ may blow up as $n \to \infty$. However, if we use Chebyshev points then

$$\| \Pi_{i=0}^{n} (x - x_i) \|_\infty = 2^{-n} \to 0 \text{ as } n \to \infty.$$

[2 marks]

B7. **All bookwork. Questions like (c) with more complicated weight functions have been set as exercises. Hint is given in (a) so that (b) is still accessible.**

(a) In Gauss quadrature, the n weights are determined by making the rule exact for polynomials of degree up to $n-1$. So, if $p_{n-1}(x)$ is a polynomial of degree $n-1$ or less, then, expressing $p_{n-1}(x)$ in its Lagrange interpolating form, we have (with $f_i = p_{n-1}(x_i)$)

$$\int_a^b w(x) p_{n-1}(x)\, dx = \int_a^b w(x) \sum_{i=1}^{n} f_i l_i(x)\, dx = \sum_{i=1}^{n} f_i \int_a^b w(x) l_i(x)\, dx,$$

where $l_i(x)$ is the Lagrange interpolating polynomial of degree $n-1$ satisfying $l_i(x_j) = \delta_{ij}$. This shows that we should take

$$w_i = \int_a^b w(x) l_i(x)\, dx.$$

[4 marks]

(b) Let f be a polynomial of degree $\leq 2n-1$. Write $f(x) = q(x)\phi_n(x) + r(x)$, where q and r are polynomials of degrees $\leq n-1$. Then

$$I(f) = \underbrace{\int_a^b w(x) q(x) \phi_n(x)\, dx}_{\substack{= 0 \text{ by orthogonality} \\ \text{since } q(x) = \sum_{i=0}^{n-1} \alpha_i \phi_i}} + \int_a^b w(x) r(x)\, dx,$$

$$G_n(f) = \underbrace{\sum_{i=1}^{n} w_i q(x_i) \phi_n(x_i)}_{\substack{= 0 \text{ because the } x_i \\ \text{are the zeros of } \phi_n}} + \sum_{i=1}^{n} w_i r(x_i).$$

Now $\int_a^b w(x) r(x)\, dx = \sum_{i=1}^{n} w_i r(x_i)$, by the choice of the weights w_i, and since r has degree $\leq n-1$, so $I(f) = G_n(f)$, as required. **[8 marks]**

(c) We have

$f(x)$	$I(f)$	$G_2(f)$	
1	2	$w_1 + w_2$	(1)
x	0	$w_1x_1 + w_2x_2$	(2)
x^2	$2/3$	$w_1x_1^2 + w_2x_2^2$	(3)
x^3	0	$w_1x_1^3 + w_2x_2^3$	(4)

[3 marks]. Let $\phi_2(x) = (x - x_1)(x - x_2) \equiv x^2 + ax + b$. Then the linear combination $(3) + a(2) + b(1)$ gives

$$\frac{2}{3} + 2b = w_1\phi(x_1) + w_2\phi(x_2) = 0 \quad \Rightarrow \quad b = -\frac{1}{3}.$$

Similarly, $(4) + a(3) + b(2)$ gives $\frac{2}{3}a = 0$, so $a = 0$. Thus $\phi_2(x) = x^2 - \frac{1}{3}$ so $x_1 = -1/\sqrt{3}$ and $x_2 = 1/\sqrt{3}$ **[3 marks]**. Then (2) gives $w_1 = w_2$ and (1) yields $w_1 = w_2 = 1$.
[2 marks].

B8. **This is all mainly bookwork. Part (d) is unseen in that absolutely stability was talked about for a specific method with specific constants, not the general one.**

(a) The truncation error is the remainder when the exact solution $y(x_n)$ is substituted for y_n in the numerical method. If $\tau(h) = O(h^{p+1})$ then we say the method is order p.
[3 marks]

(b) Substituting $y(x_n)$ for y_n and subtracting the right-hand side from the left-hand side gives:

$$\tau(h) = y(x_{n+1}) - y(x_n) - hb_1f(x_n, y(x_n)) - hb_2f(x_n + c_2h, y(x_n) + ha_{21}f(x_n, y(x_n))).$$

[2 marks] Noting that $f(x_n, y(x_n)) = y'(x_n)$ then gives

$$\tau(h) = y(x_{n+1}) - y(x_n) - hb_1y'(x_n) - hb_2f(x_n + c_2h, y(x_n) + ha_{21}y'(x_n)).$$ **[1 mark]**

(c) We need to show that $\tau(h)$ is $\mathcal{O}(h^3)$ (i.e., that the terms in h^0, h and h^2 cancel out). Using a Taylor series expansion for $y(x_{n+1})$ and the hint gives

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \mathcal{O}(h^3) \\ &= y(x_n) + hy'(x_n) + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)\Big|_{(x_n, y(x_n))} + \mathcal{O}(h^3). \end{aligned}$$

[3 marks] Next, use a Taylor series in two dimensions:

$$\begin{aligned} f(x_n + c_2h, y(x_n) + ha_{21}y'(x_n)) &= y'(x_n) + c_2h\frac{\partial f}{\partial x}(x_n, y(x_n)) \\ &\quad + ha_{21}y'(x_n)\frac{\partial f}{\partial y}(x_n, y(x_n)) + \mathcal{O}(h^2). \end{aligned}$$

[3 marks] Substituting both expansions into the expression for $\tau(h)$ in part (b) gives

$$\begin{aligned} \tau(h) = &\left[y(x_n) + hy'(x_n) + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)\Big|_{(x_n, y(x_n))} + \mathcal{O}(h^3)\right] - y(x_n) - hb_1y'(x_n) \\ &-hb_2\left(y'(x_n) + c_2h\frac{\partial f}{\partial x}(x_n, y(x_n)) + ha_{21}y'(x_n)\frac{\partial f}{\partial y}(x_n, y(x_n)) + \mathcal{O}(h^2)\right). \end{aligned}$$

Rearranging then gives

$$\tau(h) = h\left(1 - b_1 - b_2\right) y'(x_n) + h^2\left(\frac{1}{2} - b_2 c_2\right)\frac{\partial f}{\partial x}(x_n, y(x_n))$$
$$+ h^2\left(\frac{1}{2} - b_2 a_{12}\right)\left(\frac{\partial f}{\partial y} f\right)\big|_{(x_n, y(x_n))} + \mathcal{O}(h^3)$$

Equating the coefficients in front of h and h^2 to zero then gives the result. **[4 marks]**

(d) Appling the RK method to the test problem gives

$$\begin{aligned} y_{n+1} &= y_n + hb_1\lambda y_n + hb_2\lambda\left(y_n + ha_{21}\lambda y_n\right) \\ &= y_n\left(1 + \lambda h\left(b_1 + b_2\right) + (\lambda h)^2 b_2 a_{21}\right). \end{aligned}$$

For an order two method, using the result from part (c) gives

$$y_{n+1} = y_n\left(1 + \lambda h + (\lambda h)^2/2\right).$$

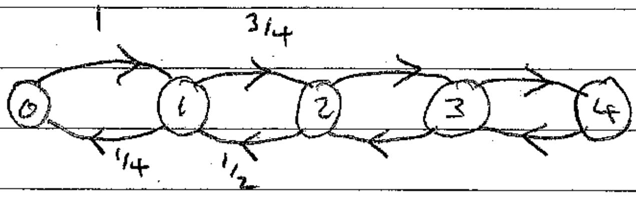
Hence, the method is absolutely stable if $|p(\lambda h)| < 1$ where

$$p(\lambda h) = 1 + \lambda h + (\lambda h)^2/2.$$

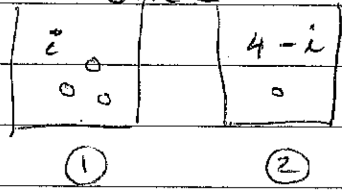
[4 marks]

1. a)

2 (i)



In state i, there are $4-i$ balls in urn 2



3 (ii) $p_{01} = P(\text{ball from urn } 2) = \frac{4}{4} = 1$

$p_{12} = P(\text{" " " "}) = \frac{3}{4}$

$p_{23} = P(\text{" " " "}) = \frac{2}{4} = 1/2$

$p_{34} = P(\text{" " " "}) = 1/4$

Similarly, when the ball selected is from urn 1, we get

$p_{43} = 1, \quad p_{32} = 3/4, \quad p_{21} = \frac{1}{2}, \quad p_{10} = 1/4$

This gives $P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Seen general model

3 (iii) The s.d. (exists and is unique since finite and irreducible) satisfies $\underline{\pi} P = \underline{\pi} \iff$

$$\frac{1}{4}\pi_1 = \pi_0$$
$$\pi_0 + \frac{1}{2}\pi_2 = \pi_1$$
$$\frac{3}{4}\pi_1 + \frac{3}{4}\pi_3 = \pi_2$$
$$\frac{1}{2}\pi_2 + \pi_4 = \pi_3$$
$$\frac{1}{4}\pi_3 = \pi_4$$

Solving gives $\pi_1 = 4\pi_0$, $\pi_2 = 6\pi_0$, $\frac{1}{2}\pi_2 = 3\pi_4 \Rightarrow \pi_4 = \pi_0$

$\pi_3 = 4\pi_0$. Then $\sum \pi_i = 1 \Rightarrow \pi_0 + 4\pi_0 + 6\pi_0 + 4\pi_0 + \pi_0 = 1$

$\Rightarrow \pi_0 = \frac{1}{16}$

Therefore the s.d is $\left(\frac{1}{16}, \frac{4}{16}, \frac{6}{16}, \frac{4}{16}, \frac{1}{16}\right)$

2 (iv) Return to 0 is possible after 2, 4, 6, 8, ... steps.

Hence period is 2. Irreducible, hence all states have period 2.

mentioned in passing for general model

2/

1.(a)

3 (~~iv~~) Let T_i = steps to visit 2 from i. Condition on first step.

$$E[T_0] = E[T_0 / \text{first step to } 1] \times 1 = E[T_1] + 1$$

Seen this method on problem sheet

$$E[T_1] = \{E[T_0] + 1\}\tfrac{1}{4} + 1 . \tfrac{3}{4} = \tfrac{1}{4}E[T_0] + 1$$

Solving: $E[T_0] = \frac{1}{4}E[T_0] + 1 + 1 \Rightarrow \frac{3}{4}E[T_0] = 2 \Rightarrow E[T_0] = 8/3$

b) ~~Since~~ Partitioning according to time n of first visit to j, we have

(i) $$p_{ij}(n) = P(X_n = j / X_0 = i) = \sum_{k=1} P(X_n = j, \text{first visit } j \text{ at } k / X_0 = i)$$

4 $$= \sum_{k=1}^{n} P(X_n = j / \text{first visit } j \text{ at } k, X_0 = i) P(\text{first visit } j \text{ at } k / X_0 = i)$$

BW which, by Markov property

$$= \sum_{k=1}^{n} P(X_n = j / X_k = j) P(\text{first visit } j \text{ at } k / X_0 = i).$$

$$= \sum_{k=1}^{n} p_{jj}(n-k) f_{ij}(k)$$

(ii) $$p_{ij}(n+m) = \sum_{k \in S} P(X_{m+n} = j, X_m = k / X_0 = i)$$

BW $$= \sum_{k \in S} P(X_{m+n} = j / X_m = k, X_0 = i) P(X_m = k / X_0 = i)$$

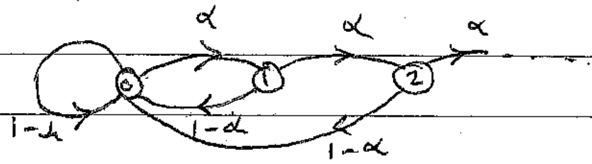
3 which, by Markov property

$$= \sum_{k \in S} p(X_{m+n} = j / X_m = k) P(X_m = k / X_0 = i)$$

$$= \sum_{k \in S} p_{kj}(n) p_{ik}(m) \quad \text{as given}$$

2.

a)



(i)

$$1 - f_{00} = \lim_{n\to\infty} \alpha^n = 0 \Rightarrow f_{00} = 1 \Rightarrow 0 \text{ is recurrent}$$

Seen similar on problem sheet

Irreducible hence all states are recurrent

(ii) A state is +ve recurrent $\Leftrightarrow$ expected # steps to first return is finite

Let T = # steps to 1st return for zero. Then

$$P(T \geq k) = p_{01}\, p_{12} \cdots p_{k-2,k-1} = \alpha^{k-1} \qquad k \geq 2$$
$$= 1 \qquad k = 1$$

seen on problem sheet

Therefore $E[T] = \sum_{k=1}^{\infty} P(T \geq k) = \frac{1}{1-\alpha} < \infty$ for $0 < \alpha < 1$

and so zero is +ve recurrent.

(iii) The s.d. (exists, unique since irreducible & +ve recurrent) satisfies $\underline{\pi} P = \underline{\pi}$ where

$$P = \begin{bmatrix} 1-\alpha & \alpha & 0 & 0 & 0 & \cdots \\ 1-\alpha & 0 & \alpha & 0 & 0 & \cdots \\ 1-\alpha & 0 & 0 & \alpha & 0 & \cdots \\ 1-\alpha & 0 & 0 & 0 & \alpha & \cdots \\ \vdots & & & & & \ddots \end{bmatrix}$$

giving $(1-\alpha)\pi_0 + (1-\alpha)\pi_1 + (1-\alpha)\pi_2 + \ldots = \pi_0$

$$\pi_0 \alpha = \pi_1$$
$$\pi_1 \alpha = \pi_2$$
$$\vdots$$

as given. Setting $\sum \pi_i = 1$, we get $\pi_0 = 1-\alpha$,
Then $\pi_1 = \alpha\pi_0 = \alpha(1-\alpha)$, $\pi_2 = \alpha^2(1-\alpha)$, ...
Therefore the s.d is $(1-\alpha, \alpha(1-\alpha), \alpha^2(1-\alpha), \ldots)$

2 b)

i) For $k \geq 2$ we now have

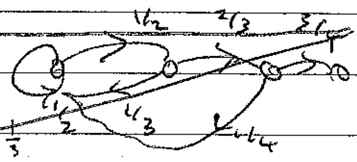
$$P(T \geq k) = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{k-1}{k} = \frac{1}{k}$$

unseen

Now $E[T] = \sum_{k=1}^{\infty} P(T \geq k) = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Therefore null recurrent or transient and no s.d. exists.

~~ii) We have that~~ $\cancel{f_{00}(1) = \frac{1}{2}}$



$$\cancel{f_{00}(2) = \frac{1}{2 \cdot 3}, \quad f_{00}(3) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4}, \quad f_{00}(4) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5}}$$

$$\cancel{f_{00}(n) = \frac{(n-1)!}{(n+1)!} \quad n \geq 2, \quad = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}}$$

c) Suppose i is positive recurrent, then for any state j $\exists\, r, s$ st. $p_{ji}(r) > 0$ & $p_{ij}(s) > 0$

BW

Also, i positive recurrent $\Rightarrow p_{ii}(n) \to \frac{1}{\mu_i} > 0$ as $n \to \infty$

Now, $\forall n$, $p_{jj}(n+r+s) \geq p_{ji}(r)\, p_{ii}(n)\, p_{ij}(s)$
$\geq \alpha\, p_{ii}(n)$ some $\alpha > 0$

Therefore $\lim p_{jj}(n) = \alpha \lim_{n \to \infty} p_{ii}(n) = \frac{\alpha}{\mu_i} > 0$

Hence $\mu_j < \infty$ and j is positive recurrent

Have seen on problem sheet but λ_i and μ_i were given and not derived from the model

3. (i) This is a birth-death process where upward transitions are not possible, so that $\lambda_i \equiv 0$.
We have $S = \{0, 1, 2, \ldots, M\}$

The time until the first death in state i is the minimum of i independent $\exp(\mu)$ r.vs.
Therefore, for $1 \le i \le M$, we have $\mu_i = i\mu$

Considering $P'(t) = P(t)Q$ where $Q = \begin{bmatrix} 0 & 0 & - & - & - & 0 \\ \mu & -\mu & 0 & - & - & - \\ 0 & 2\mu & -2\mu & 0 & - & 0 \\ 0 & & \ddots & \ddots & & \\ \vdots & & & & \ddots & -M\mu \end{bmatrix}$

we see that multiplying the M^{th} row of $P(t)$ by the cols of the matrix Q gives us the stated system of equations.

(ii) Taking the final equation $p'_{MM}(t) = -\mu M p_{MM}(t)$ and solving we get

$$p_{MM}(t) = Ae^{-M\mu t}$$

Then $p_{MM}(0) = 1 \Rightarrow A = 1$, giving the stated result in (ii).

(iii) This can be obtained by substituting the result of (ii) into the penultimate equation, giving

$$p'_{M,M-1}(t) = \mu M e^{-\mu M t} - \mu(M-1)\, p_{M,M-1}(t)$$

$$\Leftrightarrow \frac{d}{dt}\left\{ e^{\mu(M-1)t} p_{M,M-1}(t) \right\} = \mu M e^{\mu(M-1)t} e^{-M\mu t} = \mu M e^{-\mu t}$$

Hence $e^{\mu(M-1)t} p_{M,M-1}(t) = -\cancel{\mu} M e^{-\mu t} + A$

where $p_{M,M-1}(0) = 0 \Rightarrow \underline{A = M}$

3(iii)(continued)

Hence $\quad p_{M,M-1}(t) = M(e^{-\mu t})^{M-1}(1 - e^{-\mu t})$ as stated.

Alternatively

After time t, the # survivors is binomial with $p = e^{-\mu t}$.

(iv) As above, $P(T > t) = e^{-\mu t}$ Independence gives

$$p_{M,i}(t) = \binom{M}{i}(e^{-\mu t})^{i}(1 - e^{-\mu t})^{M-i} \quad 0 \leq i \leq M$$

Let $M(t) = E[X(t)]$. This is the expectation of a bin$(M, e^{-\mu t})$ dist^{n} hence

$$M(t) = Me^{-\mu t}$$

(v) Let E = "time to extinction"; E_i = "time from i to $i-1$ survivors".

Then using l.o.m property of exponential

unseen

$$E = E_M + E_{M-1} + \ldots + E_1$$

Now $E_i = \min(X_1, \ldots, X_i)$ where X_i are iid $\exp(\mu)$

Therefore $E[E_i] = \frac{1}{i\mu}$ and

$$E[E] = \frac{1}{\mu}\left[\frac{1}{M} + \frac{1}{M-1} + \ldots + \frac{1}{2} + 1\right]$$

4.

a) (i) $X(t)$ denotes the # machines out of service

When $X(t)$ is in state $i \le m-1$ there are $m-i$ working machines. Hence time to first breakdown is the minimum of $m-i$ i.i.d. $\exp(\lambda)$ r.v's, and so has the $\exp[\lambda(m-i)]$ distn.

Therefore $\lambda_i = \lambda(m-i)$. Also, $\lambda_m = 0$.

When $X(t)$ is in state $1 \le i \le r$ there are i machines undergoing repair. Hence time to first completing is the minimum of i $\exp(\mu)$ r.vs, and so has the $\exp[i\mu]$ dist.

Therefore $\mu_i = i\mu \quad 1 \le i \le r$

In states $r < i \le m$, there are r machines undergoing repair. Hence $\mu_i = r\mu$. Also $\mu_0 = 0$

Seen on problem sheets

ii) We have, for $1 \le i \le m$, the unique s.d satisfies

$$\pi_i = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} \pi_0$$

$$= \frac{\lambda m \lambda(m-1) \cdots \lambda(m-i+1)}{\mu 1 \mu 2 \cdots \mu i} \pi_0 \quad \text{for } i \le r$$

and $$= \frac{\lambda m \lambda(m-1) \cdots \lambda(m-i+1)}{\mu 1 \mu 2 \cdots \mu r \underbrace{\mu r \cdots \mu r}_{i-r \text{ terms}}} \pi_0 \quad \text{for } r < i$$

The above simplify to $$\frac{\lambda^i \frac{m!}{(m-i)!}}{\mu^i i!} \pi_0 = \binom{m}{i}\left(\frac{\lambda}{\mu}\right)^i \pi_0$$

and $$\frac{\lambda^i \frac{m!}{(m-i)!}}{\mu^i r! \, r^{i-r}} \pi_0 \quad \text{as given}$$

When the s.d exists for a b.d. process, it is the limiting distn.

4. b)

(i) This is an example of the telephone exchange model with 5 operators. This is a birth-death process with

unseen

$$\lambda_i = \lambda = 2 \quad \text{for} \quad i = 0, 1, 2, 3, 4$$
$$\mu_i = i\mu = \frac{60}{27} i \quad\quad i = 1, 2, 3, 4, 5$$

The proportion of time spent in state i is given by the unique s.d.

$$\pi_i = \frac{\lambda^i}{i!\,\mu^i}\pi_0 = \left(\frac{2}{\frac{60}{27}}\right)^i \frac{1}{i!}\pi_0 \quad 0 \le i \le 5$$
$$= \frac{(0.9)^i}{i!}\pi_0$$

Therefore, for $0 \le i \le 5$, $\pi_i = \dfrac{\frac{(0.9)^i}{i!}}{\left(1 + \frac{0.9}{1!} + \frac{0.9^2}{2!} + \dots + \frac{0.9^5}{5!}\right)}$

Since the denominator ~~...~~ is less that $e^{0.9}$, we have $\pi_i \geq \dfrac{(0.9)^i/i!}{e^{0.9}} = \dfrac{e^{-0.9}(0.9)^i}{i!}$

The proportion of cars unserviced is the proportion which arrive while the process is in state 5. Hence the result.

Reducing the capacity to 2 gives $\pi_2 = \dfrac{(0.9)^2/2!}{1 + 0.9 + \frac{0.9^2}{2}} = 0.1757$

For this option to be cost effective, considering change in profit per hr, we want cost saving − average lost income > 0

Seen similar using π_i time = prop in i

$\Leftrightarrow 20 + (\pi_5 - \pi_2)\, 2 \times 40 > 0$

We can use i) to get LHS $\geq 20 - 13.088 > 0$

Hence the new option is cost effective

THE UNIVERSITY OF MANCHESTER

Time Series Analysis MATH48032

2014/2015

Solutions

SECTION A
Answer ALL four questions

A1.

a) The mean is $a+bt+ct^2$ which depends on t, so not stationary. (bonus mark if mentions $b, c \neq 0$ for this conclusion). [2 marks]

b)

$$\begin{aligned} Y_t &= X_t - X_{t-1} \\ &= b + c(2t-1) + \varepsilon_t - \varepsilon_{t-1} \end{aligned}$$

So,

$$\begin{aligned} (1-\mathbf{B})^2 X_t &= Y_t - Y_{t-1} \\ &= b + c(2t-1) + \varepsilon_t - \varepsilon_{t-1} - (b + c(2(t-1)-1) + \varepsilon_{t-1} - \varepsilon_{t-2}) \\ &= 2c + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}) \\ &= 2c + (1-\mathbf{B})^2 \varepsilon_t. \end{aligned}$$

So, $\{(1-\mathbf{B})^2 X_t\}$ is MA(2). [4 marks]

c) The moving average polynomial has roots equal to 1, so is not invertible. [2 marks]

Qu. Total
8 marks

A2.

a) $(1-\mathbf{B}^{12})(1-\phi\mathbf{B})X_t = (1+\theta\mathbf{B}^{12})\varepsilon_t$. [3 marks]

b) $X_t - \phi X_{t-1} - X_{t-12} + \phi X_{t-13} = \varepsilon_t + \theta\varepsilon_{t-12}$. [3 marks]

Qu. Total
6 marks

A3. $(1-\mathbf{B})(1-0.2\mathbf{B})X_t = \varepsilon_t$

Only the first two ψ's are required. I have given more for my reference.
We have $(1-\mathbf{B})(1-0.2\mathbf{B}) = 1 - 1.2\mathbf{B} + 0.2\mathbf{B}^2$,

$$\begin{aligned} 1 &= (1 - 1.2\mathbf{B} + 0.2\mathbf{B}^2)(1 + \psi_1\mathbf{B} + \psi_2\mathbf{B}^2 + \psi_3\mathbf{B}^3 + \cdots) \\ &= (1 + \psi_1\mathbf{B} + \psi_2\mathbf{B}^2 + \psi_3\mathbf{B}^3 + \cdots) \\ &\quad - 1.2\mathbf{B}(1 + \psi_1\mathbf{B} + \psi_2\mathbf{B}^2 + \psi_3\mathbf{B}^3 + \cdots) \\ &\quad + 0.2\mathbf{B}^2(1 + \psi_1\mathbf{B} + \psi_2\mathbf{B}^2 + \psi_3\mathbf{B}^3 + \cdots) \\ &= (1 + \psi_1\mathbf{B} + \psi_2\mathbf{B}^2 + \psi_3\mathbf{B}^3 + \cdots) \\ &\quad - 1.2(\mathbf{B} + \psi_1\mathbf{B}^2 + \psi_2\mathbf{B}^3 + \cdots) \\ &\quad + 0.2(\mathbf{B}^2 + \psi_1\mathbf{B}^3 + \cdots). \end{aligned}$$

Comparison of coefficients at $\mathbf{B}^k$ gives $\psi_1 - 1.2 = 0$, $\psi_2 - 1.2\psi_1 + 0.2 = 0$, and, for $k \geq 3$, $\psi_k - 1.2\psi_{k-1} + 0.2\psi_{k-2} = 0$. So,

$$\begin{aligned} \psi_1 &= 1.2 \\ \psi_2 &= 1.2\psi_1 - 0.2 = 1.2^2 - 0.2 = 1.24 \\ \psi_3 &= 1.2\psi_2 - 0.2\psi_1 = 1.2 \times 1.24 - 0.2 \times 1.2 = 1.248 \end{aligned}$$

Horizon:	$h = 1$	$h = 2$	$h = 3$
Variance:	4	9.76	15.91040

Qu. Total
8 marks

A4.

a) We have

$$Y_t = (1 - \mathbf{B})X_t = X_t - X_{t-1}.$$

So,

$$\mathrm{E}\,Y_t = \mathrm{E}(1 - \mathbf{B})X_t = \mathrm{E}\,X_t - \mathrm{E}\,X_{t-1} = \mu - \mu = 0.$$

Then

$$\begin{aligned}\mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{E}(X_t - X_{t-1})(X_{t-k} - X_{t-k-1}) \\ &= \gamma_k - \gamma_{k-1} + \gamma_k - \gamma_{k+1} \\ &= 2\gamma_k - (\gamma_{k-1} + \gamma_{k+1}),\end{aligned}$$

as required.

b) Obviously, $\mathrm{E}\,Y_t = 0$, a constant. In part (a) we saw that $\mathrm{Cov}(Y_t, Y_{t-k})$ does not depend on t. Hence, $\{Y_t\}$ is stationary.

c) Since $Y_t = (1 - \mathbf{B})X_t$, it follows that $\phi(\mathbf{B})Y_t = (1 - \mathbf{B})\phi(\mathbf{B})X_t = (1 - \mathbf{B})\theta(\mathbf{B})\varepsilon_t$ So,

$$\phi(\mathbf{B})Y_t = (1 - \mathbf{B})\theta(\mathbf{B})\varepsilon_t,$$

i.e. $\{Y_t\}$ is ARMA with the same autoregression part as that of $\{X_t\}$ and moving average part $(1 - \mathbf{B})\theta(\mathbf{B})$.

Qu. Total
10 marks

SECTION B
Answer 2 of the 3 questions

B5.

a) i) (Bookwork) A stationary process, $\{X_t\}$, with mean $\mu = \mathrm{E}\,X_t$ is said to be an *autoregressive process of order* p, AR(p), if it can be represented as

$$X_t - \mu = \sum_{i=1}^{p} \phi_i (X_{t-i} - \mu) + \varepsilon_t. \tag{1}$$

where $\{\varepsilon_t\}$ is WN$(0, \sigma^2)$, $\mathrm{E}\,X_t\varepsilon_s = 0$ whenever $t < s$, the parameters ϕ_i are such that all roots of the polynomial

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$$

have moduli greater than one. [4 marks]

ii) (Bookwork) $\phi(z)$ above. [2 marks]

iii) (Bookwork) The innovations are orthogonal to past values of the process, i.e. $\mathrm{E}\,X_t\varepsilon_s = 0$ whenever $t < s$, [2 marks]

iv) (Bookwork) For an autoregression of order p, $\beta_k = 0$ for $k > p$. [2 marks]

v) (Bookwork) (There are various ways to do this.) As a possible predictor of X_t from $j \geq p + 1$ past values, consider the linear combination

$$\tilde{X}_t = \mu + \sum_{i=1}^{p} \phi_i (X_{t-i} - \mu) + \sum_{i=p+1}^{j} 0 \times (X_{t-i} - \mu).$$

We have $X_t = \tilde{X}_t + \varepsilon_t$. The orthogonality of ε_t to past X_s's (see above) means that ε_t is orthogonal to all predictor variables used in $\tilde{X}_t$. By the orthogonality property of the prediction error it follows that $\tilde{X}_t$ is the optimal linear predictor. [6 marks]

vi) Partial autocorrelation function can be used to identify AR models. If the sample pacf is small beyond some lag p (cut-off property), then this suggests AR(p). Useful tool, especially as starting point but should be used together with other tools. Also, usage straightforward for AR processes only. [2 marks]

b) $\beta_1 = \rho_1 = \frac{\theta}{1+\theta^2}$.
$\beta_2 = b$ where b is the coefficient at X_{t-2} in the linear predictor of X_t from X_{t-1}, X_{t-2}, i.e. solution of the 2nd order Yule-Walker equations. (Can be obtained also from first principles.)

$$\rho_1 - a - b\rho_1 = 0$$
$$\rho_2 - a\rho_1 - b = 0$$

Solving we get, $\beta_2 = b = \frac{-\rho_1^2}{1-\rho_1^2}$. [6 marks]

Qu. Total
24 marks

B6.

a) $X_t - 2X_{t-1} + X_{t-2} = \varepsilon_t - 0.81\varepsilon_{t-1} + 0.38\varepsilon_{t-2}$ or $X_t = 2X_{t-1} - X_{t-2} + \varepsilon_t - 0.81\varepsilon_{t-1} + 0.38\varepsilon_{t-2}$ [2 marks]

b) $I(2)$ since two differences are needed to make it stationary. [2 marks]

c) For $t = T+k$ the above equation gives $X_{T+k} = 2X_{T+k-1} - X_{T+k-2} + \varepsilon_{T+k} - 0.81\varepsilon_{T+k-1} + 0.38\varepsilon_{T+k-2}$, which gives

$$\hat{X}_{T+k|T,\ldots,1} = 2\hat{X}_{T+k-1|T,\ldots,1} - \hat{X}_{T+k-2|T,\ldots,1},$$

since the remaining terms are orthogonal to the past.
This is a homogeneous linear difference equation of order two. Its characteristic polynomial is $(1-z)^2$ which has a repeated root equal to 1. So the general solution is

$$\hat{X}_{T+k|T,\ldots,1} = a + bt,$$

with initial values

$$\begin{aligned}
\hat{X}_{T+3|T,\ldots,1} &= 2v - u \\
\hat{X}_{T+4|T,\ldots,1} &= 2\hat{X}_{T+3|T,\ldots,1} - v \\
&= 2(2v - u) - v \\
&= 3v - 2u,
\end{aligned}$$

where $u = \hat{X}_{T+1|T,\ldots,1}$, $v = \hat{X}_{T+2|T,\ldots,1}$. So,

$$\begin{aligned}
a + 3b &= 2v - u \\
a + 4b &= 3v - 2u
\end{aligned}$$

Solving we get $a = 2u - v$, $b = -u + v$.
This can be solved also by writing down the first few predictors and carefully examining them. [8 marks]

d) A straight line, this was found above. [2 marks]

e)

$$\begin{aligned}
\hat{X}_{95+1|95,\ldots,1} &= 2X_{95} - X_{94} - 0.81\varepsilon_{95} + 0.38\varepsilon_{94} \\
&= 2 \times 15.9 - 15.2 - 0.81 \times 0.586 + 0.38 \times (-1.286) \\
&= 15.6367
\end{aligned}$$

$$\begin{aligned}
\hat{X}_{95+2|95,\ldots,1} &= 2\hat{X}_{95+1|95,\ldots,1} - X_{95} + 0.38\varepsilon_{95} \\
&= 2 \times 15.6367 - 15.9 + 0.38 \times (0.586) \\
&= 15.5961
\end{aligned}$$

$$\begin{aligned}
\hat{X}_{95+3|95,\ldots,1} &= 2\hat{X}_{95+2|95,\ldots,1} - \hat{X}_{95+1|95,\ldots,1} \\
&= 215.5961 - 15.6367 \\
&= 15.5555
\end{aligned}$$

For the variances, we need the first few coefficients of the infinite MA representation, $X_t = \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + \cdots$. Consider,

$$(1 - 2z + z^2)(1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots) = 1 - 0.81z + 0.38z^2,$$

expand left-hand side,

$$1 + (\psi_1 - 2)z + (1 - 2\psi_1 + \psi_2)z^2 + (\psi_3 - 2\psi_2 + \psi_1)z^3 + \cdots = 1 - 0.81z + 0.38z^2.$$

Comparing coefficients gives

$$\begin{aligned}\psi_1 &= 2 - 0.81\\ \psi_2 &= 2\psi_1 - 1 + 0.38\\ \psi_3 &= 2\psi_2 - \psi_1\end{aligned}$$

So, $\psi_1 = 1.19$, $\psi_2 = 1$, $\psi_3 = 0.81$.
Hence the variances of the prediction errors for the $k = 1, 2, 3$ are 1, $1 + 1.19^2 = 2.4161$, and $1 + 1.19^2 + 1^2 = 3.4161$.
(ψ_3 is redundant, don't need it.) [10 marks]

Qu. Total
24 marks

B7.

a) The raw time series shows a trend similar to that of random walk. So, differencing is necessary. No seasonality. The autocorrelation function decreases slowly (after four years still relatively large correlation), supporting the need for differencing.
Differenced series seems to have constant level. There is one very small value in 1992, probably the difference between the third and second quarters. Autocorrelations are small, except for $\hat{\rho}_1$ which is marginally significant on the 5% level.
So, ARIMA(0,1,0) and ARIMA(0,1,1) models seem plausible. [4 marks]

b) i) The acf's of the residuals are small (non-significant). The Ljung-Box test supports the white noise-ness of the residuals (large p-values, whatever its parameter). The plot of the residuals shows one outlying value which may be influencing the fit and the model choice. There is a hint for clustering of positive and negative values in the residuals.
The standard errors of the MA coefficients are about half of their magnitudes, not bad although the MA(2) is just on the border of a 95% CI. No clear evidence of overfitting. [4 marks]

ii) Overall, ARIMA(0,1,1) seems best among the models with 1 difference Its aic = -45.35 and σ^2 estimated as 0.01587.
Overall, ARIMA(0,2,2) seems best among the models with 2 differences with aic = -38.73 and σ^2 estimated as 0.01678.
Comparison of the AICs of these two models should be made with caution since they represent different orders of nonstationarity. The ARIMA(0,1,1) model gives also a smaller residual variance and is more parsimonious. So we select it. [4 marks]

iii) The big through in the differenced series and the residuals suggest that improvements are possible. One may drop the observation giving the outlier in the residuals (and maybe all preceding observations) and refit the model.
One might also try to fit a model to the data with the offending stretch dropped. If that does not help, then another class of moels should be tried since it is clear that ARIMA cannot be improved further. [2 marks]

c) Quarter 4, 2000 is just after the last obervation. So, the point prediction is $2 \times 3.5310 - 3.3522 = 3.7098$. From the output for this model, $\hat{\sigma}^2 = 0.02415$. So, a 95% prediction interval is $3.7098 \pm 1.96\sqrt{0.02415} = (3.405211, 4.014389)$. [4 marks]

d) i)

$$(1 - \mathbf{B})X_t = b_{t-1} + \varepsilon_t \tag{2}$$

$$(1 - 0.833\mathbf{B})b_{t-1} = 0.167(1 - \mathbf{B})X_{t-1}. \tag{3}$$

[1 mark]

ii) From the above,

$$b_{t-1} = 0.167(1 - \mathbf{B})(1 - 0.833\mathbf{B})^{-1}X_{t-1}.$$

Put this into the first eq. above and simplify

$$\begin{aligned}(1 - \mathbf{B})X_t &= b_{t-1} + \varepsilon_t \\ &= 0.167(1 - \mathbf{B})(1 - 0.833\mathbf{B})^{-1}X_{t-1} + \varepsilon_t \\ &= 0.167(1 - \mathbf{B})(1 - 0.833\mathbf{B})^{-1}\mathbf{B}X_t + \varepsilon_t\end{aligned}$$

So,

$$(1 - \mathbf{B})X_t - 0.167(1 - \mathbf{B})(1 - 0.833\mathbf{B})^{-1}\mathbf{B}X_t = \varepsilon_t$$

So,

$$\begin{aligned}\varepsilon_t &= (1 - \mathbf{B})(1 - 0.167(1 - 0.833\mathbf{B})^{-1}\mathbf{B}X_t) \\ &= (1 - \mathbf{B})(1 - 0.833\mathbf{B} - 0.167\mathbf{B})(1 - 0.833\mathbf{B})^{-1}X_t \\ &= (1 - \mathbf{B})^2(1 - 0.833\mathbf{B})^{-1}X_t.\end{aligned}$$

Hence,

$$(1 - \mathbf{B})^2 X_t = (1 - 0.833\mathbf{B})\varepsilon_t,$$

as required. [5 marks]

Qu. Total
24 marks

SECTION C
Answer ALL questions

C8.

a) (Bookwork) $\{\eta_t\}$ is i.i.d.(0,1) and such that η_t is independent of the past of $\{X_t\}$ (i.e. of $\mathcal{F}_{t-1}$). [4 marks]

b) Using the independence of η_t from the past we get:

$$\mathrm{E}(X_{t+h}|\mathcal{F}_t) = \phi\,\mathrm{E}(X_{t+h-1}|\mathcal{F}_t) + \mathrm{E}(\varepsilon_{t+h}|\mathcal{F}_t) = \phi\,\mathrm{E}(X_{t+h-1}|\mathcal{F}_t) = \cdots = \phi^h X_t.$$

[4 marks]

c) (Bookwork) Let $h \geq 1$. Then

$$\begin{aligned}
\mathrm{E}(\varepsilon_{t+h}^2|\mathcal{F}_t) &= \mathrm{E}(\sigma_{t+h}^2\eta_{t+h}^2|\mathcal{F}_t) \qquad \text{(using the GARCH equation)}\\
&= \mathrm{E}(\mathrm{E}(\sigma_{t+h}^2\eta_{t+h}^2|\mathcal{F}_{t+h-1})|\mathcal{F}_t) \qquad \text{(by iterated expectations rule)}\\
&= \mathrm{E}(\sigma_{t+h}^2\,\mathrm{E}(\eta_{t+h}^2|\mathcal{F}_{t+h-1})|\mathcal{F}_t) \qquad (\text{since } \sigma_{t+h}^2 \in \mathcal{F}_{t+h-1})\\
&= \mathrm{E}(\sigma_{t+h}^2(\mathrm{E}\,\eta_{t+h}^2)|\mathcal{F}_t) \qquad (\text{since } \eta_{t+h} \text{ is independent of } \mathcal{F}_{t+h-1})\\
&= \mathrm{E}(\sigma_{t+h}^2|\mathcal{F}_t) \qquad (\text{since } \mathrm{E}\,\eta_{t+h}^2 = 1),
\end{aligned}$$

as required. [4 marks]

d) Taking conditional expectation on both sides of the volatility equation we get

$$\mathrm{E}(\sigma_{t+h}^2|\mathcal{F}_t) = \omega + \alpha_1\,\mathrm{E}(\varepsilon_{t+h-1}^2|\mathcal{F}_t) + \alpha_2\,\mathrm{E}(\varepsilon_{t+h-2}^2|\mathcal{F}_t).$$

For fixed t, this is a difference equation with $\mathrm{E}(\varepsilon_t^2|\mathcal{F}_t) = (X_t - \phi X_{t-1})^2$ and $\mathrm{E}(\varepsilon_{t-1}^2|\mathcal{F}_t) = (X_{t-1} - \phi X_{t-2})^2$. [4 marks]

e) (Bookwork) Take expected values on both sides of the volatility equation, use stationarity and c) to get

$$\sigma^2 = \omega + \alpha_1\sigma^2 + \alpha_2\sigma^2.$$

Hence, $\sigma^2 = \omega/(1 - \alpha_1 - \alpha_2)$. [4 marks]

Qu. Total
20 marks

C9.

a) i) The Ljung-Box statistics show that there is no serial correlation in the log return; Q(12) = 9.49 with p-value 0.66.

ii) There is, however, significant ARCH effect because Q(12) = 32.17 with p-value 0.001 for the squares (i.e. the squares are correlated).

iii) The expected log return is not zero, because t-test gives t = 2.93 with p-value 0.004.

iv) The t-test for the mean is derived under assumption for independence, which is violated since the squares are correlated. (This is not the only possible answer.)

[4 marks]

b) The fitted model is

$$\begin{aligned} X_t &= 0.015 + \varepsilon_t \\ \varepsilon_t &= \sigma_t \eta_t \\ \eta_t &\sim N(0,1) \\ \sigma_t^2 &= 0.000253 + 0.136\varepsilon_{t-1}^2 + 0.844\sigma_{t-1}^2. \end{aligned}$$

The Jarque-Berra and Shapiro-Wilks tests clearly suggest that the conditional distribution is not normal.
Except for the normality assumption, the model seems adequate. See Ljung-Box tests for standardized residual series and its squared series.
Alll coefficients significant at 5% level.

[6 marks]

c) i) The fitted model is

$$\begin{aligned} X_t &= 0.0126 + \varepsilon_t \\ \varepsilon_t &= \sigma_t \eta_t \\ \eta_t &\sim \text{skew-}t \text{ with 10 d.f. and skew } \hat{\xi} = 0.888 \\ \sigma_t^2 &= 0.000291 + 0.108\varepsilon_{t-1}^2 + 0.8637\sigma_{t-1}^2. \end{aligned}$$

ii) Similarly to the previous model, the standardised residuals and their squares are uncorrelated. The adequateness of the conditional distribution cannot be inferred from the given information. The skewness is not significantly different from one (see below).

iii) For symmetric distribution the skew parameter is equal to one. Based on results, we have $t = (0.888 - 1)/0.06 = 1.87$, whose absolute value is less than 1.96 (the 0.975 quantile of N(0,1)). Therefore, we cannot reject the null hypothesis that the log return series has a symmetric distribution.

iv) I would produce a qq-plot of the standardised residuals against the quantiles of the fitted skew-t distribution.

[7 marks]

d) The fitted model (not requested) is

$$\begin{aligned} X_t &= 0.0128 + \varepsilon_t \\ \varepsilon_t &= \sigma_t \eta_t \\ \eta_t &\sim N(0,1) \\ \sigma_t^2 &= 0.000292 + 0.1256(|\varepsilon_{t-1}| - 0.23\varepsilon_{t-1})^2 + 0.8395\sigma_{t-1}^2. \end{aligned}$$

The parameter interpreted as leverage is γ. (This is another way of modelling skewness.) From the output, its estimate is $\hat{\gamma} = 0.23$. The p-value shows significance at the 5% level. [3 marks]

Qu. Total
20 marks

END OF EXAMINATION PAPER

THE UNIVERSITY OF MANCHESTER

Time Series Analysis MATH48032

2014/2015

Solutions

SECTION A
Answer ALL four questions

A1.

a) The mean is $a+bt+ct^2$ which depends on t, so not stationary. (bonus mark if mentions $b, c \neq 0$ for this conclusion). [2 marks]

b)

$$\begin{aligned} Y_t &= X_t - X_{t-1} \\ &= b + c(2t-1) + \varepsilon_t - \varepsilon_{t-1} \end{aligned}$$

So,

$$\begin{aligned} (1-\mathbf{B})^2 X_t &= Y_t - Y_{t-1} \\ &= b + c(2t-1) + \varepsilon_t - \varepsilon_{t-1} - (b + c(2(t-1)-1) + \varepsilon_{t-1} - \varepsilon_{t-2}) \\ &= 2c + \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}) \\ &= 2c + (1-\mathbf{B})^2 \varepsilon_t. \end{aligned}$$

So, $\{(1-\mathbf{B})^2 X_t\}$ is MA(2). [4 marks]

c) The moving average polynomial has roots equal to 1, so is not invertible. [2 marks]

Qu. Total
8 marks

A2.

a) $(1-\mathbf{B}^{12})(1-\phi\mathbf{B})X_t = (1+\theta\mathbf{B}^{12})\varepsilon_t$. [3 marks]

b) $X_t - \phi X_{t-1} - X_{t-12} + \phi X_{t-13} = \varepsilon_t + \theta\varepsilon_{t-12}$. [3 marks]

Qu. Total
6 marks

A3. $(1-\mathbf{B})(1-0.2\mathbf{B})X_t = \varepsilon_t$

Only the first two ψ's are required. I have given more for my reference.
We have $(1-\mathbf{B})(1-0.2\mathbf{B}) = 1 - 1.2\mathbf{B} + 0.2\mathbf{B}^2$,

$$\begin{aligned} 1 &= (1 - 1.2\mathbf{B} + 0.2\mathbf{B}^2)(1 + \psi_1\mathbf{B} + \psi_2\mathbf{B}^2 + \psi_3\mathbf{B}^3 + \cdots) \\ &= (1 + \psi_1\mathbf{B} + \psi_2\mathbf{B}^2 + \psi_3\mathbf{B}^3 + \cdots) \\ &\qquad - 1.2\mathbf{B}(1 + \psi_1\mathbf{B} + \psi_2\mathbf{B}^2 + \psi_3\mathbf{B}^3 + \cdots) \\ &\qquad + 0.2\mathbf{B}^2(1 + \psi_1\mathbf{B} + \psi_2\mathbf{B}^2 + \psi_3\mathbf{B}^3 + \cdots) \\ &= (1 + \psi_1\mathbf{B} + \psi_2\mathbf{B}^2 + \psi_3\mathbf{B}^3 + \cdots) \\ &\qquad - 1.2(\mathbf{B} + \psi_1\mathbf{B}^2 + \psi_2\mathbf{B}^3 + \cdots) \\ &\qquad + 0.2(\mathbf{B}^2 + \psi_1\mathbf{B}^3 + \cdots). \end{aligned}$$

Comparison of coefficients at $\mathbf{B}^k$ gives $\psi_1 - 1.2 = 0$, $\psi_2 - 1.2\psi_1 + 0.2 = 0$, and, for $k \geq 3$, $\psi_k - 1.2\psi_{k-1} + 0.2\psi_{k-2} = 0$. So,

$$\begin{aligned} \psi_1 &= 1.2 \\ \psi_2 &= 1.2\psi_1 - 0.2 = 1.2^2 - 0.2 = 1.24 \\ \psi_3 &= 1.2\psi_2 - 0.2\psi_1 = 1.2 \times 1.24 - 0.2 \times 1.2 = 1.248 \end{aligned}$$

Horizon:	$h=1$	$h=2$	$h=3$
Variance:	4	9.76	15.91040

Qu. Total
8 marks

A4.

a) We have

$$Y_t = (1-\mathbf{B})X_t = X_t - X_{t-1}.$$

So,

$$\mathrm{E}\,Y_t = \mathrm{E}(1-\mathbf{B})X_t = \mathrm{E}\,X_t - \mathrm{E}\,X_{t-1} = \mu - \mu = 0.$$

Then

$$\begin{aligned}\mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{E}(X_t - X_{t-1})(X_{t-k} - X_{t-k-1})\\ &= \gamma_k - \gamma_{k-1} + \gamma_k - \gamma_{k+1}\\ &= 2\gamma_k - (\gamma_{k-1} + \gamma_{k+1}),\end{aligned}$$

as required.

b) Obviously, $\mathrm{E}\,Y_t = 0$, a constant. In part (a) we saw that $\mathrm{Cov}(Y_t, Y_{t-k})$ does not depend on t. Hence, $\{Y_t\}$ is stationary.

c) Since $Y_t = (1-\mathbf{B})X_t$, it follows that $\phi(\mathbf{B})Y_t = (1-\mathbf{B})\phi(\mathbf{B})X_t = (1-\mathbf{B})\theta(\mathbf{B})\varepsilon_t$ So,

$$\phi(\mathbf{B})Y_t = (1-\mathbf{B})\theta(\mathbf{B})\varepsilon_t,$$

i.e. $\{Y_t\}$ is ARMA with the same autoregression part as that of $\{X_t\}$ and moving average part $(1-\mathbf{B})\theta(\mathbf{B})$.

Qu. Total
10 marks

SECTION B
Answer 2 of the 3 questions

B5.

a) i) (Bookwork) A stationary process, $\{X_t\}$, with mean $\mu = \mathrm{E}\,X_t$ is said to be an *autoregressive process of order* p, AR(p), if it can be represented as

$$X_t - \mu = \sum_{i=1}^{p} \phi_i (X_{t-i} - \mu) + \varepsilon_t. \tag{1}$$

where $\{\varepsilon_t\}$ is WN$(0, \sigma^2)$, $\mathrm{E}\,X_t\varepsilon_s = 0$ whenever $t < s$, the parameters ϕ_i are such that all roots of the polynomial

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$$

have moduli greater than one. [4 marks]

ii) (Bookwork) $\phi(z)$ above. [2 marks]

iii) (Bookwork) The innovations are orthogonal to past values of the process, i.e. $\mathrm{E}\,X_t\varepsilon_s = 0$ whenever $t < s$, [2 marks]

iv) (Bookwork) For an autoregression of order p, $\beta_k = 0$ for $k > p$. [2 marks]

v) (Bookwork) (There are various ways to do this.) As a possible predictor of X_t from $j \geq p + 1$ past values, consider the linear combination

$$\tilde{X}_t = \mu + \sum_{i=1}^{p} \phi_i (X_{t-i} - \mu) + \sum_{i=p+1}^{j} 0 \times (X_{t-i} - \mu).$$

We have $X_t = \tilde{X}_t + \varepsilon_t$. The orthogonality of ε_t to past X_s's (see above) means that ε_t is orthogonal to all predictor variables used in $\tilde{X}_t$. By the orthogonality property of the prediction error it follows that $\tilde{X}_t$ is the optimal linear predictor. [6 marks]

vi) Partial autocorrelation function can be used to identify AR models. If the sample pacf is small beyond some lag p (cut-off property), then this suggests AR(p). Useful tool, especially as starting point but should be used together with other tools. Also, usage straightforward for AR processes only. [2 marks]

b) $\beta_1 = \rho_1 = \frac{\theta}{1+\theta^2}$.
$\beta_2 = b$ where b is the coefficient at X_{t-2} in the linear predictor of X_t from X_{t-1}, X_{t-2}, i.e. solution of the 2nd order Yule-Walker equations. (Can be obtained also from first principles.)

$$\rho_1 - a - b\rho_1 = 0$$
$$\rho_2 - a\rho_1 - b = 0$$

Solving we get, $\beta_2 = b = \frac{-\rho_1^2}{1-\rho_1^2}$. [6 marks]

Qu. Total
24 marks

B6.

a) $X_t - 2X_{t-1} + X_{t-2} = \varepsilon_t - 0.81\varepsilon_{t-1} + 0.38\varepsilon_{t-2}$ or $X_t = 2X_{t-1} - X_{t-2} + \varepsilon_t - 0.81\varepsilon_{t-1} + 0.38\varepsilon_{t-2}$ [2 marks]

b) $I(2)$ since two differences are needed to make it stationary. [2 marks]

c) For $t = T+k$ the above equation gives $X_{T+k} = 2X_{T+k-1} - X_{T+k-2} + \varepsilon_{T+k} - 0.81\varepsilon_{T+k-1} + 0.38\varepsilon_{T+k-2}$, which gives

$$\hat{X}_{T+k|T,\ldots,1} = 2\hat{X}_{T+k-1|T,\ldots,1} - \hat{X}_{T+k-2|T,\ldots,1},$$

since the remaining terms are orthogonal to the past.
This is a homogeneous linear difference equation of order two. Its characteristic polynomial is $(1-z)^2$ which has a repeated root equal to 1. So the general solution is

$$\hat{X}_{T+k|T,\ldots,1} = a + bt,$$

with initial values

$$\begin{aligned}\hat{X}_{T+3|T,\ldots,1} &= 2v - u \\ \hat{X}_{T+4|T,\ldots,1} &= 2\hat{X}_{T+3|T,\ldots,1} - v \\ &= 2(2v-u) - v \\ &= 3v - 2u,\end{aligned}$$

where $u = \hat{X}_{T+1|T,\ldots,1}$, $v = \hat{X}_{T+2|T,\ldots,1}$. So,

$$\begin{aligned}a + 3b &= 2v - u \\ a + 4b &= 3v - 2u\end{aligned}$$

Solving we get $a = 2u - v$, $b = -u + v$.
This can be solved also by writing down the first few predictors and carefully examining them. [8 marks]

d) A straight line, this was found above. [2 marks]

e)

$$\begin{aligned}\hat{X}_{95+1|95,\ldots,1} &= 2X_{95} - X_{94} - 0.81\varepsilon_{95} + 0.38\varepsilon_{94} \\ &= 2 \times 15.9 - 15.2 - 0.81 \times 0.586 + 0.38 \times (-1.286) \\ &= 15.6367\end{aligned}$$

$$\begin{aligned}\hat{X}_{95+2|95,\ldots,1} &= 2\hat{X}_{95+1|95,\ldots,1} - X_{95} + 0.38\varepsilon_{95} \\ &= 2 \times 15.6367 - 15.9 + 0.38 \times (0.586) \\ &= 15.5961\end{aligned}$$

$$\begin{aligned}\hat{X}_{95+3|95,\ldots,1} &= 2\hat{X}_{95+2|95,\ldots,1} - \hat{X}_{95+1|95,\ldots,1} \\ &= 215.5961 - 15.6367 \\ &= 15.5555\end{aligned}$$

For the variances, we need the first few coefficients of the infinite MA representation, $X_t = \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + \cdots$. Consider,

$$(1 - 2z + z^2)(1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots) = 1 - 0.81z + 0.38z^2,$$

expand left-hand side,

$$1 + (\psi_1 - 2)z + (1 - 2\psi_1 + \psi_2)z^2 + (\psi_3 - 2\psi_2 + \psi_1)z^3 + \cdots = 1 - 0.81z + 0.38z^2.$$

Comparing coefficients gives

$$\begin{aligned}
\psi_1 &= 2 - 0.81 \\
\psi_2 &= 2\psi_1 - 1 + 0.38 \\
\psi_3 &= 2\psi_2 - \psi_1
\end{aligned}$$

So, $\psi_1 = 1.19$, $\psi_2 = 1$, $\psi_3 = 0.81$.
Hence the variances of the prediction errors for the $k = 1, 2, 3$ are 1, $1 + 1.19^2 = 2.4161$, and $1 + 1.19^2 + 1^2 = 3.4161$.
(ψ_3 is redundant, don't need it.) [10 marks]

Qu. Total
24 marks

B7.

a) The raw time series shows a trend similar to that of random walk. So, differencing is necessary. No seasonality. The autocorrelation function decreases slowly (after four years still relatively large correlation), supporting the need for differencing.
Differenced series seems to have constant level. There is one very small value in 1992, probably the difference between the third and second quarters. Autocorrelations are small, except for $\hat{\rho}_1$ which is marginally significant on the 5% level.
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Overall, ARIMA(0,2,2) seems best among the models with 2 differences with aic = -38.73 and σ^2 estimated as 0.01678.
Comparison of the AICs of these two models should be made with caution since they represent different orders of nonstationarity. The ARIMA(0,1,1) model gives also a smaller residual variance and is more parsimonious. So we select it. [4 marks]

iii) The big through in the differenced series and the residuals suggest that improvements are possible. One may drop the observation giving the outlier in the residuals (and maybe all preceding observations) and refit the model.
One might also try to fit a model to the data with the offending stretch dropped. If that does not help, then another class of moels should be tried since it is clear that ARIMA cannot be improved further. [2 marks]

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$$(1 - 0.833\mathbf{B})b_{t-1} = 0.167(1 - \mathbf{B})X_{t-1}. \tag{3}$$

[1 mark]

ii) From the above,

$$b_{t-1} = 0.167(1 - \mathbf{B})(1 - 0.833\mathbf{B})^{-1}X_{t-1}.$$

Put this into the first eq. above and simplify

$$\begin{aligned}(1 - \mathbf{B})X_t &= b_{t-1} + \varepsilon_t \\ &= 0.167(1 - \mathbf{B})(1 - 0.833\mathbf{B})^{-1}X_{t-1} + \varepsilon_t \\ &= 0.167(1 - \mathbf{B})(1 - 0.833\mathbf{B})^{-1}\mathbf{B}X_t + \varepsilon_t\end{aligned}$$

So,

$$(1-\mathbf{B})X_t - 0.167(1-\mathbf{B})(1-0.833\mathbf{B})^{-1}\mathbf{B}X_t = \varepsilon_t$$

So,

$$\begin{aligned}\varepsilon_t &= (1-\mathbf{B})(1-0.167(1-0.833\mathbf{B})^{-1}\mathbf{B}X_t)\\ &= (1-\mathbf{B})(1-0.833\mathbf{B}-0.167\mathbf{B})(1-0.833\mathbf{B})^{-1}X_t\\ &= (1-\mathbf{B})^2(1-0.833\mathbf{B})^{-1}X_t.\end{aligned}$$

Hence,

$$(1-\mathbf{B})^2X_t = (1-0.833\mathbf{B})\varepsilon_t,$$

as required. [5 marks]

Qu. Total
24 marks

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Answer ALL questions

C8.

a) (Bookwork) $\{\eta_t\}$ is i.i.d.(0,1) and such that η_t is independent of the past of $\{X_t\}$ (i.e. of $\mathcal{F}_{t-1}$). [4 marks]

b) Using the independence of η_t from the past we get:

$$\mathrm{E}(X_{t+h}|\mathcal{F}_t) = \phi\,\mathrm{E}(X_{t+h-1}|\mathcal{F}_t) + \mathrm{E}(\varepsilon_{t+h}|\mathcal{F}_t) = \phi\,\mathrm{E}(X_{t+h-1}|\mathcal{F}_t) = \cdots = \phi^h X_t.$$

[4 marks]

c) (Bookwork) Let $h \geq 1$. Then

$$\begin{aligned}
\mathrm{E}(\varepsilon_{t+h}^2|\mathcal{F}_t) &= \mathrm{E}(\sigma_{t+h}^2\eta_{t+h}^2|\mathcal{F}_t) \qquad \text{(using the GARCH equation)}\\
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&= \mathrm{E}(\sigma_{t+h}^2\,\mathrm{E}(\eta_{t+h}^2|\mathcal{F}_{t+h-1})|\mathcal{F}_t) \qquad \text{(since } \sigma_{t+h}^2 \in \mathcal{F}_{t+h-1})\\
&= \mathrm{E}(\sigma_{t+h}^2(\mathrm{E}\,\eta_{t+h}^2)|\mathcal{F}_t) \qquad \text{(since } \eta_{t+h} \text{ is independent of } \mathcal{F}_{t+h-1})\\
&= \mathrm{E}(\sigma_{t+h}^2|\mathcal{F}_t) \qquad \text{(since } \mathrm{E}\,\eta_{t+h}^2 = 1),
\end{aligned}$$

as required. [4 marks]

d) Taking conditional expectation on both sides of the volatility equation we get

$$\mathrm{E}(\sigma_{t+h}^2|\mathcal{F}_t) = \omega + \alpha_1\,\mathrm{E}(\varepsilon_{t+h-1}^2|\mathcal{F}_t) + \alpha_2\,\mathrm{E}(\varepsilon_{t+h-2}^2|\mathcal{F}_t).$$

For fixed t, this is a difference equation with $\mathrm{E}(\varepsilon_t^2|\mathcal{F}_t) = (X_t - \phi X_{t-1})^2$ and $\mathrm{E}(\varepsilon_{t-1}^2|\mathcal{F}_t) = (X_{t-1} - \phi X_{t-2})^2$. [4 marks]

e) (Bookwork) Take expected values on both sides of the volatility equation, use stationarity and c) to get

$$\sigma^2 = \omega + \alpha_1\sigma^2 + \alpha_2\sigma^2.$$

Hence, $\sigma^2 = \omega/(1 - \alpha_1 - \alpha_2)$. [4 marks]

Qu. Total
20 marks

C9.

a) i) The Ljung-Box statistics show that there is no serial correlation in the log return; Q(12) = 9.49 with p-value 0.66.

ii) There is, however, significant ARCH effect because Q(12) = 32.17 with p-value 0.001 for the squares (i.e. the squares are correlated).

iii) The expected log return is not zero, because t-test gives t = 2.93 with p-value 0.004.

iv) The t-test for the mean is derived under assumption for independence, which is violated since the squares are correlated. (This is not the only possible answer.)

[4 marks]

b) The fitted model is

$$\begin{aligned} X_t &= 0.015 + \varepsilon_t \\ \varepsilon_t &= \sigma_t \eta_t \\ \eta_t &\sim N(0,1) \\ \sigma_t^2 &= 0.000253 + 0.136\varepsilon_{t-1}^2 + 0.844\sigma_{t-1}^2. \end{aligned}$$

The Jarque-Berra and Shapiro-Wilks tests clearly suggest that the conditional distribution is not normal.
Except for the normality assumption, the model seems adequate. See Ljung-Box tests for standardized residual series and its squared series.
Alll coefficients significant at 5% level. [6 marks]

c) i) The fitted model is

$$\begin{aligned} X_t &= 0.0126 + \varepsilon_t \\ \varepsilon_t &= \sigma_t \eta_t \\ \eta_t &\sim \text{skew-}t \text{ with 10 d.f. and skew } \hat{\xi} = 0.888 \\ \sigma_t^2 &= 0.000291 + 0.108\varepsilon_{t-1}^2 + 0.8637\sigma_{t-1}^2. \end{aligned}$$

ii) Similarly to the previous model, the standardised residuals and their squares are uncorrelated. The adequateness of the conditional distribution cannot be inferred from the given information. The skewness is not significantly different from one (see below).

iii) For symmetric distribution the skew parameter is equal to one. Based on results, we have $t = (0.888 - 1)/0.06 = 1.87$, whose absolute value is less than 1.96 (the 0.975 quantile of N(0,1)). Therefore, we cannot reject the null hypothesis that the log return series has a symmetric distribution.

iv) I would produce a qq-plot of the standardised residuals against the quantiles of the fitted skew-t distribution.

[7 marks]

d) The fitted model (not requested) is

$$\begin{aligned} X_t &= 0.0128 + \varepsilon_t \\ \varepsilon_t &= \sigma_t \eta_t \\ \eta_t &\sim N(0,1) \\ \sigma_t^2 &= 0.000292 + 0.1256(|\varepsilon_{t-1}| - 0.23\varepsilon_{t-1})^2 + 0.8395\sigma_{t-1}^2. \end{aligned}$$

The parameter interpreted as leverage is γ. (This is another way of modelling skewness.) From the output, its estimate is $\hat{\gamma} = 0.23$. The p-value shows significance at the 5% level. [3 marks]

Qu. Total
20 marks

END OF EXAMINATION PAPER

MATH3/4/68052 Solutions

A1 (a) The NB(2, p) distribution with pmf [Unseen]

$$\begin{aligned} P(Y=y) &= (y+1)p^2(1-p)^y \\ &= \exp\{y\log(1-p)+2\log p+\log(y+1)\} \\ &\in \text{exponential family} \end{aligned}$$

with parameters $\theta = \log(1-p)$ and $\phi = 1$. The three functions are

$$b(\theta) = -2\log p = -2\log(1-e^\theta),\ a(\phi)=\phi, \text{ and } c(y,\phi)=\log(y+1).$$

[3]

(b) Property 1 of the distribution: $\mathrm{E}[Y] = b'(\theta)$, $\mathit{Var}\{Y\} = b''(\theta)a(\phi)$.

Applying the formulas to $b(\theta) = -2\log(1-e^\theta)$ and $a(\phi)=\phi=1$,

$$\mathrm{E}[Y] = -2\frac{-e^\theta}{1-e^\theta} = \frac{2e^\theta}{1-e^\theta} = \frac{2(1-p)}{p},$$

$$\mathit{Var}\{Y\} = \left(\frac{2e^\theta}{1-e^\theta}\right)' = 2\frac{e^\theta(1-e^\theta)-e^\theta(-e^\theta)}{(1-e^\theta)^2} = \frac{2e^\theta}{(1-e^\theta)^2} = \frac{2(1-p)}{p^2}.$$

[4]

(c) The role of the link function g is to transform the mean response μ so that $g(\mu)=\eta$, the linear predictor. The canonical link is the same function of μ as θ is. [Bookwork]

[4]

(d) From (b), $\mu = 2(1-p)/p$, thus

$$p\mu = 2(1-p),\quad p=\frac{2}{\mu+2},\quad 1-p=\frac{\mu}{\mu+2}.$$

Then from (a)

$$\theta = \log(1-p) = \log\frac{\mu}{\mu+2}.$$

Therefore the canonical link is $g(\mu) = \log\frac{\mu}{\mu+2}$. [4]

(e) Canonical link means $g(\mu)=\theta$, thus $g'(\mu)\frac{\mathrm{d}\mu}{\mathrm{d}\theta} = 1$. Because $\mu = b'(\theta)$, $\frac{\mathrm{d}\mu}{\mathrm{d}\theta} = b''(\theta) = V(\mu)$. Thus $g'(\mu)V(\mu)=1$. The Fisher scores are

$$\frac{\partial\ell}{\partial\beta_j} = \frac{1}{a(\phi)}\sum_{i=1}^n \frac{x_{ij}}{V(\mu_i)g'(\mu_i)}(y_i-\mu_i) = \frac{1}{a(\phi)}\sum_{i=1}^n x_{ij}(y_i-\mu_i),\ j=1,\dots,p.$$

When differenciating again wrt β_k, y_i disappears and the result remains the same after taking expectation because it is not random. Thus the expected and observed Fisher info are identical. [5]

A2 (a) Poisson response log linear model with intercept: [Bookwork]

$$y_i = \mu_i + \varepsilon_i \sim Pois(\mu_i) \text{ independent,}$$

$$\log(\mu_i) = \beta_0 + \beta_1 x_i, \quad i = 1, \ldots, n.$$

[3]

(b) The data matrix is

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}.$$

The weight matrix W is diagonal with elements

$$w_i = \frac{1}{V(\mu_i) g'(\mu_i)^2} = \frac{1}{\mu_i / \mu_i^2} = \mu_i = \lambda_i, \; i = 1, \ldots, n.$$

Therefore

$$W = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

[3]

(c) The log-likelihood function is

$$\begin{aligned} \ell(\underset{\sim}{\beta}) &= \sum_{i=1}^n \log\left(\frac{\mu_i^{y_i} e^{-\mu_i}}{y_i!}\right) \\ &= \sum_{i=1}^n (y_i \log \mu_i - \mu_i) - \log y_i! \\ &= \sum_{i=1}^n y_i(\beta_0 + \beta_1 x_i) - \sum_{i=1}^n e^{\beta_0 + \beta_1 x_i} - \log y_i!. \end{aligned}$$

Differentiating w.r.t. β_0 and β_1,

$$\frac{\partial \ell}{\partial \beta_0} = \sum_{i=1}^n y_i - \sum_{i=1}^n e^{\beta_0 + \beta_1 x_i}, \quad \frac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i e^{\beta_0 + \beta_1 x_i}.$$

Setting the partial derivatives to zero, the MLE of (β_0, β_1) must satisfy

$$\begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 - \mu_1 \\ \vdots \\ y_n - \mu_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $\mu_i = e^{\beta_0 + \beta_1 x_i}$, $i = 1, \ldots, n$. In matrix form,

$$X'(\underset{\sim}{y} - \underset{\sim}{\mu}) = \underset{\sim}{0}.$$

Then

$$X'W\underset{\sim}{\xi} = X'W(X\underset{\sim}{\beta} + W^{-1}(\underset{\sim}{y} - \underset{\sim}{\mu}) = X'WX\underset{\sim}{\beta} + X'(\underset{\sim}{y} - \underset{\sim}{\mu}) = X'WX\underset{\sim}{\beta}.$$

[6]

(d) Differentiating again, the second derivatives [Bookwork]

$$\frac{\partial^2 \ell}{\partial \beta_0^2} = -\sum_{i=1}^{n} e^{\beta_0+\beta_1 x_i}, \; \frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_1} = -\sum_{i=1}^{n} x_i e^{\beta_0+\beta_1 x_i}, \; \frac{\partial^2 \ell}{\partial \beta_1^2} = -\sum_{i=1}^{n} x_i^2 e^{\beta_0+\beta_1 x_i}.$$

are not random. Thus by definition, the expected/observed Fisher information matrix is

$$\begin{aligned} I(\underset{\sim}{\beta}) &= -\begin{bmatrix} \frac{\partial \ell}{\partial \beta_0} & \frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_1} \\ \frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_1} & \frac{\partial^2 \ell}{\partial \beta_1^2} \end{bmatrix} \\ &= \sum_{i=1}^{n} \begin{bmatrix} 1 & x_i \\ x_i & x_i^2 \end{bmatrix} e^{\beta_0+\beta_1 x_i}. \\ &= \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mu_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \\ &= X'WX. \end{aligned}$$

[4]

(e) Using the expression in (b) for the log-likelihood, we have by definition [Bookwork]

$$\begin{aligned} \text{Deviance} &= 2\sum_{i=1}^{n}(y_i \log y_i - y_i) - 2\sum_{i=1}^{n}(y_i \log \hat{y}_i - \hat{y}_i) \\ &= 2\sum_{i=1}^{n}(y_i \log \frac{y_i}{\hat{y}_i} + \hat{y}_i - y_i), \end{aligned}$$

where $\hat{y}_i = e^{\hat{\beta}_0+\hat{\beta}_1 x_i}$, $i = 1, \dots, n$. The fitted values are exactly y_i under the saturated model. [2]

Because the residuals $y_i - \hat{y}_i$ add up to 0 (part c), the deviance is simply

$$\text{Deviance} = 2\sum_{i=1}^{n} y_i \log \frac{y_i}{\hat{y}_i}.$$

[2]

B1 (a) Putting the $\Gamma(\mu, 2)$ density in exponential family form, [Seen more general]

$$\begin{aligned} f(y;\ \mu) &= \exp\left\{-\frac{2y}{\mu} - 2\log\mu + \log(4y)\right\} \\ &= \exp\left\{\frac{y(-1/\mu) - \log\mu}{1/2} + \log(4y)\right\} \end{aligned}$$

with $\theta = -1/\mu$, $\phi = 2$. The three functions are

$$a(\phi) = 1/\phi,\ b(\theta) = \log\mu = -\log(-\theta) \text{ and } c(y, \phi) = \log(4y).$$

[3]

(b) The mean response equals $b'(\theta) = -1/(-\theta) \times (-1) = -1/\theta = \mu$. [2]

The variance function is $V(\mu) = (-1/\theta)' = -(-1)/\theta^2 = 1/\theta^2 = \mu^2$. [2]

(c) (i) The scaled deviance is $15.36/(1/2) = 30.72$ on $25 - 2 = 23$ is less than the upper tail critical value $\chi^2_{0.05;\ 23} = 35.172$. It is not significant at level 5%. Thus the model provides adequate fit. [4]

(ii) At $x = 15$, the linear predictor is calculated as

$$\hat{\eta} = 0.1676 - 0.000364 \times 15 = 0.16214,$$

and the fitted tensile strength is $\hat{y} = 1/0.16214 = 6.1675$. [4]

(iii) Standard error of estimated linear predictor

$$se(\hat{\eta}) = \sqrt{0.05^2 + 15^2 \times 0.0036^2 + 2 \times 15 \times (-0.8678) \times 0.05 \times 0.0036} = 0.0270.$$

95% confidence interval for η:

$$0.16214 \pm 1.96 \times 0.0270 = (0.10922, 0.21506).$$

95% confidence interval for μ:

$$(1/0.21506, 1/0.10922) = (4.6499, 9.1558).$$

[5]

B2 (a) The difference is that fit1 has an interaction term γ_{ij} in the linear predictor $\eta_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$ while fit0 does not. [2]

(b) When the interaction term is added, the change in deviance is 12.193 on 9 df. The corresponding P-value is $0.20 > 0.10$. Thus the interaction between `age` and `car` is not significant at 10%. [4]

(c) When `car` is added to model with `age` in it, the change in deviance is 105.284 on 3 df with P-value$<$ 2.2×10^{-16}. Thus the effect of `car` is significant at 1% in the presence of `age`. [4]

(d) The additive model with deviance 12.193 on 9 df is not significant at 10% sig. level (P-value=$0.20 > 0.10$). It is adequate for the data as far as deviance is concerned. Cannot be simplified further as all the parameters are significant at 5% (P-values < 0.05). [4]

(e) For an older policy holder (> 35 years) driving a medium engine sized car (1.5-2.0L),

$$\hat{\eta} = -1.59528 - 0.62887 + 0.45760 = -1.76655$$

$$\hat{\pi} = \frac{e^{\hat{\eta}}}{1 + e^{\hat{\eta}}} = 0.1460 \quad \text{(probability of claim).}$$

[2]

The variance of $\hat{\eta}$ (linear predictor) is

$$\begin{aligned}
&0.006902304 - 0.0053429877 - 0.0018941565 \\
&- 0.005342988 + 0.0060772730 - 0.0002045798 \\
&- 0.001894156 - 0.0002045798 + 0.0034908773 \\
&= 0.001587007.
\end{aligned}$$

An approximate 95% c.i. for η is

$$-1.76655 \pm 1.96\sqrt{0.001587007} = (-1.844631, -1.688469),$$

and one for π is

$$\left(\frac{1}{1 + e^{1.844631}}, \frac{1}{1 + e^{1.688469}} \right) = (0.1365, 0.1560).$$

[4]

B3 (a) The distribution is multinomial with probability function [Bookwork]

$$P(Y_{11} = y_{11}, \ldots, Y_{IJ} = y_{IJ}) = \frac{n!}{y_{11}! \cdots y_{IJ}!} \pi_{11}^{y_{11}} \cdots \pi_{IJ}^{y_{IJ}},$$

where π_{ij} are cell probabilities with $\sum_{ij} \pi_{ij} = 1$.

[3]

(b) For independent $Y_{ij} \sim Pois(n\pi_{ij})$, $Y_{..} = Y_{11} + \cdots + Y_{IJ} \sim Pois(n)$. [Bookwork]

$$\begin{aligned} P(Y_{11} = y_{11}, \ldots, Y_{IJ} = y_{IJ} | Y_{..} = n) &= \frac{P(Y_{11} = y_{11}, \ldots, Y_{IJ} = y_{IJ})}{P(Y_{..} = n)} \\ &= \frac{\prod_{ij} (n\pi_{ij})^{y_{ij}} \exp(-n\pi_{ij})/y_{ij}!}{n^n \exp(-n)/n!} \\ &= \frac{n!}{\prod_{ij} y_{ij}!} \prod_{ij} \pi_{ij}^{y_{ij}}, \end{aligned}$$

if $y_{11} + \cdots + y_{IJ} = n$. Thus the conditional distribution is multinomial.

[5]

(c) (i) The additive model has deviance 16.236 on 4 df, which is significant at the 5% sig. level as it is greater than $\chi^2_{0.05;\,4} = 9.488$. Thus significance evidence to reject independence between row and column classifications. [4]

(ii) Vehicle condition cannot be removed from the model because of its significant interaction with vehicle type. One can also say because of the lack of fit of the additive model – it cannot be simplified further.

(iii) When vehicle age is added, the change in deviance is 9.6 on 1 df. This is significant at 5% since it is greater than $\chi^2_{0.05;\,1} = 3.841$. Thus vehicle age should be included.

[The model with vehicle age provides adequate fit as 6.636 on 3 df is not significant at 5%.]

[4]

A3 (MATH4/68052 only)

(a) Definitions: [Bookwork]

$$S(t) = P(T > t),\ t > 0.$$

$$h(t) = \lim_{\delta \to 0^+} \frac{P(T \leq t + \delta | T > t)}{\delta},\ t > 0.$$

Calculation of $h(t)$ from $S(t)$:

$$h(t) = -\frac{S'(t)}{S(t)}.$$

Calculation of $S(t)$ from $h(t)$:

$$S(t) = \exp\left(-\int_0^t h(t)\mathrm{d}t\right).$$

[4]

(b) (i) $f(t) = 2te^{-t^2},\ t > 0.$

$F(t) = \int_0^t 2te^{-t^2}\mathrm{d}t = -e^{-t^2}|_0^t = 1 - e^{-t^2},\ t > 0.$

$S(t) = 1 - F(t) = e^{-t^2},\ t > 0.$ [3]

(ii) $h(t) = f(t)/S(t) = 2t$, $t > 0$.

Straight line (slope=2) when plotted against t. [3]

(c) $h(t;\ x) = h_0(t)e^{\beta x},\ t > 0$

$h_0(t)$ is a hazard function ('baseline')

β is a constant. [4]

(d) The hazard $h(t;\ x)$ is proportional to $h_0(t)$ and the hazard ratio

$$\frac{h(t;\ x)}{h(t;\ x^*)} = e^{\beta(x - x^*)}$$

does not depend on t. [3]

(e) Partial likelihood is based on the order in which failures occur and relative risk. It is constructed as a product of risk $\psi = e^{\beta x}$ divided by total risk just before each failure.

When d observations are tied, their contribution becomes the product of the d risks divided by the sum of all possible products of d from the subset at risk.

[3]

C1 (MATH4/68052 only)

(a) Calculating Kaplan-Meier estimate of the survival function:

At $t = 8$, $r = 12$, $d = 1$, $\hat{S}(t) = 1 - \frac{1}{12} = 0.917$

At $t = 10$, $r = 11$, $d = 1$, $\hat{S}(t) = 0.9167 \times (1 - \frac{1}{11}) = 0.833$

At $t = 11$, $r = 10$, $d = 1$, $\hat{S}(t) = 0.8334 \times (1 - \frac{1}{10}) = 0.750$

At $t = 14$, $r = 7$, $d = 1$, $\hat{S}(t) = 0.7501 \times (1 - \frac{1}{7}) = 0.643$

At $t = 16$, $r = 5$, $d = 1$, $\hat{S}(t) = 0.6429 \times (1 - \frac{1}{5}) = 0.514$

At $t = 18$, $r = 4$, $d = 1$, $\hat{S}(t) = 0.5143 \times (1 - \frac{1}{4}) = 0.386$

At $t = 21$, $r = 2$, $d = 1$, $\hat{S}(t) = 0.3857 \times (1 - \frac{1}{2}) = 0.193$

At $t = 22$, $r = 1$, $d = 1$, $\hat{S}(t) = 0.1929 \times (1 - \frac{1}{1}) = 0$ [4]

The estimated survival function is

$$\hat{S}(t) = \begin{cases} 1, & 0 < t < 8 \\ 0.917, & 8 \le t < 10 \\ 0.833, & 10 \le t < 11 \\ 0.750, & 11 \le t < 14 \\ 0.643, & 14 \le t < 16 \\ 0.514, & 16 \le t < 18 \\ 0.386, & 18 \le t < 21 \\ 0.193, & 21 \le t < 22 \\ 0, & 22 \le t \end{cases}$$

[2]

(b) The estimated mean survival time is

$$1 \times 8 + 0.917 \times 2 + 0.833 \times 1 + 0.750 \times 3 + 0.643 \times 2 + 0.514 \times 2 + 0.386 \times 3 + 0.193 \times 1 = 16.582.$$

[3]

(c) Estimated median survival time is 18, because $\hat{S}(18) < 0.5$ and $\hat{S}(t) > 0.5$ when $t < 18$. [3]

(d) Estimated mean residual lifetime $\mathrm{E}[T - t|T > t] = \frac{\int_t^\infty S(t)\mathrm{d}t}{S(t)}$ at $t = 18$ is

$$(0.386 \times 3 + 0.193 \times 1)/0.386 = 3.5$$

Anyone who survives beyond 18 months is expected to live 3.5 months longer. [4]

(e) Nelson-Aalen estimate of cumulative hazard $H(t)$ at $t = 18$:

$$\frac{1}{12} + \frac{1}{11} + \frac{1}{10} + \frac{1}{7} + \frac{1}{5} + \frac{1}{4} = 0.867.$$

An approximate 95% confidence interval for $H(18) = -\log S(18)$ is

$$-\log(0.386) \pm 1.96 \times \sqrt{\frac{1}{12 \times 11} + \frac{1}{11 \times 10} + \frac{1}{10 \times 9} + \frac{1}{7 \times 6} + \frac{1}{5 \times 4} + \frac{1}{4 \times 3}} = (0.109,\ 1.795),$$

or $0.867 \pm 1.96 \times \sqrt{11/12^3 + 10/11^3 + 9/10^3 + 6/7^3 + 4/5^3 + 3/4^3} = (0.190, 1.544)$. [4]

C2 (MATH4/68052 only)

(a) The log rank test statistic takes the value $\chi^2 = 1.1$ on 1 df which is not significant at 10% as P-value$= 0.303 > 0.1$. Thus no significant difference between treatments A and B. [4]

(b) (i) Age is most significant with P-value < 0.01

Ecog.ps and resid.ds are the least significant with P-values 0.6 and 0.3 respectively. [3]

(ii) Age affects survival time significantly at 1%. P-value$= 0.0078 < 0.01$.

Survival time decreases with age significantly at 1%. P-value$= 0.0039 < 0.01$.

[4]

(iii) Hazard ratio $= e^{-0.914} = 0.40$ treatment 2 to treatment 1.

Less than 1, although not significantly so at 5%, P-value$= 0.08 > 0.05$. [3]

(c) The fitted Weibull model is also a proportional hazards model.

$$\hat{\lambda}_0 = e^{-10.6320} = 2.4131 \times 10^{-5}, \quad \hat{\alpha} = 1/0.52 = 1.9231,$$

Multiplying the estimated coefficients by -1.9231 gives estimates of Cox model coefficients.

age: $-0.0650 \times (-1.9231) = 0.1250$

resid.ds: $-0.5210 \times (-1.9231) = 1.0019$

rx: $0.5206 \times (-1.9231) = -1.0012$

ecog.ps: $-0.0668 \times (-1.9231) = 0.1285$ [4]

Additionally it gives $h_0(t)$ in parametric form:

$$\hat{h}_0(t) = 1.9231 \times e^{-10.6320 \times 1.9231} \times t^{0.9231}, \; t > 0.$$ [2]

UNIVERSITY OF MANCHESTER: MATH38152

Social Statistics

Tuesday 20th May 14:00 – Two Hours

Electronic calculators may be used provided that they cannot store text

Mathematical formula sheet provided

Answer **ALL** five questions in SECTION A (40 Marks)
Answer **TWO** of the three questions in SECTION B (20 marks each)

The total number of marks on the paper is 80.
A further 20 marks are available from coursework during the semester making a total of 100.

WITH **ANSWERS**

SECTION A

Answer ALL five questions

A1.

(a) Give an example of measurement error in a survey. Provide the example in terms of a particular survey question, what it measures, and what type of observations the measurement error could result in and why (2 marks)
(b) In a study of 'happiness' it is found that the happiness Y_{ij} of an individual $i = 1, \ldots, n$ when interviewed by an interviewer $j = 1, \ldots, m$ is given by $Y_{ij} = \mu + u_j + e_i$, where μ is a constant, $u_j \overset{\text{i.i.d}}{\sim} N(0, \tau^2)$, and independently thereof $e_i \overset{\text{i.i.d}}{\sim} N(0, \sigma^2)$. What is the variance $V(Y_{ij})$? (2 marks)
(c) What is the correlation between the response of two individuals i and j that have been interviewed by the same interviewer? (4 marks)
[8 marks total]
SOLUTION:

(a) For example: how many standard units of alcohol do you drink per week; measures alcohol consumption; under-reporting due to memory error or prestige bias
(b) $V(Y_{ij}) = V(u_j + e_i) = \tau^2 + \sigma^2$
(c) $\frac{Cov(Y_{ij}, Y_{kj})}{\sqrt{Y_{ij}Y_{kj}}}$, in which

$$\begin{aligned} E(Y_{ij}Y_{kj}) &= E[(\mu + u_j + e_i)(\mu + u_j + e_k)] \\ &= E[\mu^2] + 2E[\mu u_j] + E[\mu e_k] + E[u_j^2] + E[u_j e_k] + E[\mu e_i] + E[u_j e_i] + E[e_i e_k] \\ &= \mu^2 + E[u_j^2] = \mu^2 + \tau^2 \end{aligned}$$

and $E(Y_{ij})E(Y_{kj}) = \mu^2$, so that

$$\frac{Cov(Y_{ij}, Y_{kj})}{\sqrt{Y_{ij}Y_{kj}}} = \frac{\tau^2}{\tau^2 + \sigma^2}$$

A2.

(i) Write down one *advantage* and one *disadvantage* of cluster (or two-stage) sampling, compared with simple random sampling (SRS) with the same sample size, n. (2 marks).

A cluster sample was taken in which there were 10 equal sized clusters of size 25, the between cluster sum of squares of the variable of interest (SSB) is 30 and the within cluster sum of squares (SSW) is 270.

(ii) From this information, calculate the Intra-Cluster correlation, $\hat{\rho}$. (3 marks).

(iii) From this information, calculate the effective sample size. (3 marks).

[8 marks total]

SOLUTION:

(i) Advantage: Pragmatic, cost effective. Disadvantage: Less precise estimates than equivalent size SRS due to clustering of values of variable of interest within PSUs.

(ii)

$$\begin{aligned}\hat{\rho} &= 1 - \left(\frac{m}{m-1} * \frac{SSW}{SSW + SSB}\right) \\ &= 1 - \left(\frac{25}{24} * \frac{270}{300} = 0.0625\right)\end{aligned}$$

Where m is the cluster (PSU) size.

(iii)

$$\begin{aligned}DEFF &= 1 + \hat{\rho}(m-1) \\ &= 1 + (0.0625 \times 24) \\ &= 2.5 \\ \text{Effective sample size} &= \frac{m \times n}{DEFF} \\ &= \frac{250}{2.5} \\ &= 100.\end{aligned}$$

Table 1: Hospital Patient Waiting Time Data.

Hospital	**No. of patients**	**No. waiting over 4 hours for treatment**
1	100	10
2	300	12
3	400	15
4	200	20
5	500	5
6	600	10
7	200	5
8	100	10
9	200	10
10	400	12
Total	3000	109

A3.

(i) Explain briefly what is meant by *probability proportional to size* sampling (2 marks).

A local health authority in the north west collected the data in Table 1 for its 10 hospitals. A sample of size n=2 (with replacement) was drawn from the data in Table 1, using selection probabilities proportional to the number of patients in each hospital. This sample comprises hospitals 3 and 7. Using the data in Table 1:

1. Write down the selection probabilities for these two hospitals. Hence calculate the estimated total number of patients waiting more than 4 hours in all 10 hospitals using the Hansen-Hurwitz estimator. (3 marks).
2. Write down the sample inclusion probabilities for these two hospitals, and hence calculate the estimated total number of patients waiting more than 4 hours in all 10 hospitals using the Horvitz-Thompson estimator. (3 marks).

[**8 marks total**]

SOLUTION:

(i) Each sample unit has a probability of selection that is proportional to its size. Thus, for example, for a population of 10 hospitals, we can select a sample of them using selection probabilities on the basis of the number of patients in them, rather than giving them equal selection of 1/10 regardless of size.
(ii) $p_3 = 300/300 = 0.1$ and $p_7 = 200/3000 = 0.0667$, hence:

$$\hat{\tau}_{HH} = \frac{1}{2}\left(\frac{15}{0.1} + \frac{5}{0.0667}\right) = 112.4813.$$

(iii) In general in SSWR for sample of size n, the inclusion probability for unit i is:

$$\pi_i = 1 - (1 - p_i)^n$$

Hence, when $n = 2$:

$$\pi_3 = 1 - (1 - 0.1)^2 = 0.19$$

and

$$\pi_7 = 1 - (1 - 0.0667)^2 = 0.1290$$

Using these values we can estimate $\hat{\tau}_{HT}$ as:

$$\hat{\tau}_{HT} = \left(\frac{15}{0.19} + \frac{5}{0.1290}\right) = 117.7070$$

A4.
A variable Y has been measured for n independent subjects. Assume that observations $i = 1, \ldots, r$ have been fully observed and that observations $i = r + 1, \ldots, n$ are missing.
(a) Assume that, $n = 25$, $r = 15$, $\sum_{i=1}^{r} y_i = 8.7$ and $\sum_{i=1}^{r} y_i^2 = 23.49$ and that you impute using the mean. What is the sample mean and sample variance of the imputed variable? (2 marks)
(b) Is the sample mean based on mean imputation biased and if so how big is the bias? (2 marks)
(c) Is the sample variance as an estimator of $V(Y)$ (for $n - r$ missing as above) biased and if so how big is the bias? (4 marks)
[**8 marks total**]

SOLUTION:

(a) Un-imputed mean is $\bar{y}_{CC} = \frac{1}{15}\sum_{i=1}^{r} y_i = 8.7/15 = 0.58$ so the imputed mean $\bar{y}_{IMP} = \frac{1}{25}\sum_{i=1}^{r} y_i =$

$8.7/25 + 10\bar{y}_{CC}/25 = 0.58$. The sample variance is

$$\begin{aligned} s^2_{IMP} &= \frac{\left(\sum_{i=1}^r y_i^2 + (n-r)\bar{y}^2_{IMP}\right) - \frac{\left(\sum_{i=1}^r y_i + (n-r)\bar{y}_{IMP}\right)^2}{n}}{n-1} \\ &= \frac{(26.854) - \frac{(14.5)^2}{25}}{24} \\ &= \frac{(35.94) - \frac{210.25}{25}}{24} \\ &= 0.7685 \end{aligned}$$

(b)

$$E(\bar{Y}_{IMP}) = \frac{1}{n}E(\sum_{i=1}^r Y_i) + \frac{n-r}{rn}E(\sum_{i=1}^r Y_i) = \sum_{i=1}^r E(Y_i)\left(\frac{1}{n} + \frac{n-r}{nr}\right) = E(Y)$$

(c)

$$\begin{aligned} E(S^2_{IMP}) &= E\left(\frac{\left(\sum_{i=1}^r y_i^2 + (n-r)\{\sum_{i=1}^r y_i/r\}^2\right)}{n-1} - \frac{\left(\sum_{i=1}^r y_i + (n-r)/r\sum_{i=1}^r y_i\right)^2}{n(n-1)}\right) \\ &= \frac{1}{(n-1)}\sum_{r=1}^r E[Y_i^2] + \frac{n-r}{(n-1)}E\left[\bar{Y}^2\right] - \frac{n}{(n-1)}E(\bar{Y}^2) \\ &= \frac{1}{(n-1)}r\left[V(Y) + E(Y)\right] - \frac{r}{(n-1)}E(\bar{Y}^2) \\ &= \frac{1}{(n-1)}r\left[V(Y) + E(Y)^2\right] - \frac{r}{(n-1)}\left[V(Y)/r + E(Y)^2\right] \\ &= V(Y)\frac{r-1}{n-1} \end{aligned}$$

and consequently the bias is $E(S^2_{IMP} - V(Y)) = V(Y)\frac{r-n}{n-1}$

A5.

In a regression $Y = \alpha + \beta x + \epsilon$, making standard assumptions, the intercept was estimated to $\hat{\alpha} = 9.60$ and the slope to $\hat{\beta} = 2.94$. A third variable z is introduced. The three variables are plotted in Figure 1. .

(a) Estimates from a regression $Y = \alpha^* + \beta^* x + \gamma + z\epsilon$ were estimated. The values were 2.98, 0.05, -1.47. What parameter was estimated to what numerical value? (3 marks)

(b) The standard errors for $\hat{\beta}^*$ and $\hat{\gamma}$ were 0.019 and 0.010, respectively. Assume that x is the crime-rate of an area (on ward level), y is average life satisfaction of an area, and z is income-level. Perform necessary tests and interpret the results in terms of what 'causes' life satisfaction (3 marks)

(c) Using the definitions of variables in (c) above, how does X relate to Y causally and what additional models would we need to fit to investigate this? (2 marks)

[8 marks total]

SOLUTION:

(a) $\hat{\gamma}$ is clearly positive. Judging by the figure, it seems that y increases roughly 2 to 3 units for every unit increase in z, so 2.98 is more likely than 0.05, leaving $\hat{\alpha}^* = 0.05$. Given z there is a clear negative association between x and y so $\hat{\beta}^* = -1.47$ (which can also be confirmed by noting that a unit increase in

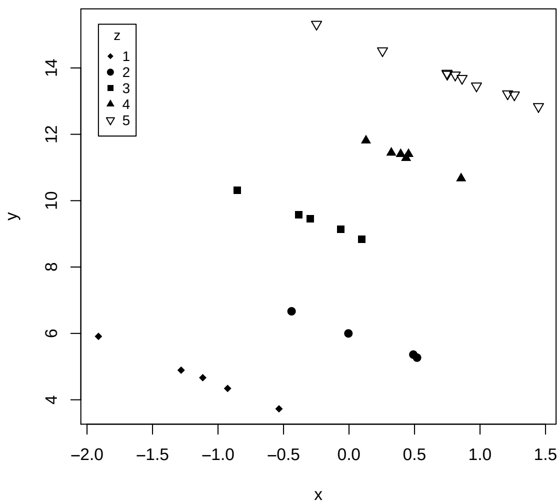


Figure 1: Scatter plot of three variables, x, y, and z

x, for given value of z, leads to a decrease of somewhere between 1 and 2 units in y)

(b) First equation: H0: $\beta = 0$, against H1: $\beta \neq 0$. The test statistic $T = \hat{\beta}/\widehat{s.e.}\hat{\beta} \sim t(28)$ when H0 is true. As the degrees of freedom are large we approximate the t-distribution with a standard normal distribution. We reject H0 if $|T| > 1.96$ on the 95%-level. Here $T = 4.7$, therefor we reject H0.

The causal model implied by the regression states that a unit increase in crime-rate in an area leads to an increase of 2.94 units of life satisfaction (in expectation).

Introducing z, H0: $\gamma = 0$, against H1: $\gamma \neq 0$. The test statistic $T = \hat{\gamma}/\widehat{s.e.}\hat{\gamma} \sim t(28)$ when H0 is true. As the degrees of freedom are large we approximate the t-distribution with a standard normal distribution. We reject H0 if $|T| > 1.96$ on the 95%-level. Here $T = 285$, therefor we reject H0. Test H0: $\beta^* = 0$, against H1: $\beta^* \neq 0$. The test statistic $T = \hat{\beta}^*/\widehat{s.e.}\hat{\beta}^* \sim t(28)$ when H0 is true. As the degrees of freedom are large we approximate the t-distribution with a standard normal distribution. We reject H0 if $|T| > 1.96$ on the 95%-level. Here $T = -76$, therefor we reject H0.

The causal model implied means that a unit increase of average wealth leads to a 2.98 increase in life satisfaction (in expectation) given the crime level and that the crime-level has a negative effect.

(c) It could be that wealth generates crime or the other way around. We can fit regressions to ascertain the strength of association but direction has to be decided based on logic and theory.

SECTION B

Answer TWO of the three questions

B1.

For a variable Y and treatment $T \in \{0, 1\}$ the *average treatment effect* (ATE) is defined as $E[Y(1) - Y(0)]$ where $Y(1)$ is defined as the outcome under treatment and $Y(0)$ is defined as the outcome under control.

(a) Is the equality $E[Y(1) - Y(0)] = E[Y(1)] - E[Y(0)]$ correct?(2 Marks)

(b) Define missing data indicators M_1 and M_0 for $Y(1)$ and $Y(0)$ respectively, given T(2 Marks)

(c) What property holds for the sum of M_1 and M_0 (1 Mark)

(d) For a sample of individuals $i = 1, \ldots, n$, let $\bar{Y}_t = \frac{1}{\sum_i T_i} \sum_{i:T_i=1} Y_i$ and $\bar{Y}_c = \frac{1}{\sum_i (1-T_i)} \sum_{i:T_i=0} Y_i$ be the sample averages of outcomes in the treatment and control groups respectively. Assuming independent observations on Y, prove that $E[Y(1)|T = 1] - E[Y(0)|T = 0]$ is an unbiased estimator of ATE if data are missing completely at random (MCAR) (5 Marks)
(e) Assume independent observations on Y for a collection $U = \{1, \ldots, N\}$ of individuals. Prove that selecting units $S \subset U$ to receive treatment using simple random sampling (without replacement) implies that observations are MCAR (5 Marks)
(f) Assume that the expected value of Y is 2 units higher for women than for men, everything else equal. Assume independent observations on Y for a collection $U = \{1, \ldots, N\}$ of individuals where half of the units are men and the rest women. Further assume that select units $S \subset U$ to receive treatment using a stratified random sampling. You select n men and $n + k$ women from each strata using simple random sampling (without replacement). Is the ATE estimator going to over- or underestimate the ATE? (5 Marks)

[20 marks total]
solution:

(a) YES. By linearity of expectations.
(b) Let $M_0 = 1$ if $Y(0)$ is unobserved and let $M_1 = 1$ if $Y(1)$ is unobserved and we can symbolically write $Y(0) = Y_{\text{obs}}$ and $Y(1) = Y_{\text{miss}}$. Given that $T = 1$, only the outcome in the treatment state $Y(1)$ is observed. $Y(0)$ is then the counterfactual and is unobserved, hence $(M_0, M_1) = (1, 0)$. Given that $T = 0$, only the outcome in the control state $Y(0)$ is observed . $Y(1)$ is then the counterfactual and is unobserved, hence $(M_0, M_1) = (0, 1)$. More compactly we can express this as $M_0 = T$ and $M_0 = 1 - T$.
(c) As $M_0 = 1 - M_1$ we have $M_0 + M_1 = 1$.
(d) The estimator $\bar{Y}_t$ is an unbiased estimator of $E[Y(1)|T = 1]$ and $\bar{Y}_c$ is an unbiased estimator of $E[Y(0)|T = 0]$. By definition, MCAR implies that $\Pr(M_0 = a, M_1 = b|Y_{\text{obs}}, Y_{\text{miss}}) = \Pr(M_0 = a, M_1 = b)$.The face-likelihood

$$\begin{aligned} f(y(1)|T = 1) &= \frac{f(y(1)) \Pr(T = 1|Y(1) = y(1))}{\int f(y(1)) \Pr(T = 1|Y(1) = y(1)) dy(1)} \\ &= \frac{f(y(1)) \Pr(T = 1)}{\Pr(T = 1) \int f(y(1)) dy(1)} \\ &= f(y(1)) \end{aligned}$$

Thus $E[Y(1)|T = 1] = \int y(1) f(y(1)) dy = E[Y(1)]$ and equivalently for $E[Y(0)|T = 0]$.
(e) Now the treatment indicator T_i serves the same role as the inclusion indicators. The missing data generating model is $\Pr(M_0 = 1, M_1 = 0|Y) = \Pr(T = 1) = n/N$ and $\Pr(M_0 = 0, M_1 = 1|Y) = \Pr(T = 1) = 1 - n/N$.
(f) Let the variable X be equal to 1 or zero according to whether a person in male or female. The expected value of the $\bar{Y}_t$ will be equal to $E[Y(1)|T = 1] = E[Y(1)|T = 1, X = 1]\frac{n}{2n+k} + E[Y(1)|T = 1, X = 0]\frac{n+k}{2n+k}$ (as there is simple random sampling in each group) and the expected value of $\bar{Y}_c$ will be equal to $E[Y(0)|T = 0] = E[Y(0)|T = 0, X = 1]\frac{n+k}{2n+k} + E[Y(0)|T = 0, X = 0]\frac{n}{2n+k}$. The higher proportion of females receiving treatment means that $\bar{Y}_t - \bar{Y}_c$ will overestimate the ATE.

B2.
(a) A Mathematics school in a University has the following four research groups (RGs) with non-overlapping membership, and knows how many staff are in each group, and how many papers were published in 2014 for each research group. These data are shown in Table 2.

For samples of size $n = 2$ groups without replacement from this population of $N = 4$ groups, the inclusion

Table 2: Research Groups (RGs) and Papers Published in 2014, School of Mathematics, University of Somewhere.

Research Group	**No. of Male Staff**	**No. of Female Staff**	**Total Staff**	**Proportion of total staff**	**Total Papers**
A: Algebra	12	8	20	0.2500	12
B: Geometry	8	12	20	0.2500	10
C: Applied Maths	15	10	25	0.3125	18
D: Logic	5	10	15	0.1875	15
Total:	40	40	80	1.0000	55

and joint inclusion probabilities based on the total number of staff in each research group are given below in Table .

Table 3: Inclusion probabilities, π_i, π_k, and joint inclusion probabilities π_{ik} for samples of size $n = 2$ groups that could be selected from the research groups (RGs) A-D in Table 2

		RG k				
		A	B	C	D	π_i
	A	-	0.1667	0.2178	0.1202	0.5047
	B	0.1667	-	0.2178	0.1202	0.5047
RG i	C	0.2178	0.2178	-	0.1573	0.5929
	D	0.1202	0.1202	0.1573	-	0.3977
	π_k	0.5047	0.5047	0.5929	0.3977	2.0000

(i) For a sample size of $n = 2$, write down an expression for the joint inclusion probabilities. Do this under the assumption that in a two-step selection process, the selection probability for the first unit is proportional to total number of staff and that the conditional selection probability of the second unit is proportional to the number of staff. (5 marks).

(ii) Use the Horvitz Thompson (H-T) Estimator to estimate the total number of papers published for a sample of n=2 research groups, comprising Applied Maths and Logic. (3 marks).

(iii) Estimate the variance of the H-T estimated total using the Sen-Yates-Grundy (SYG) estimator (4 marks).

(iv) Write down a nominal 95% confidence interval for the H-T estimated total based on the SYG estimator and a normal approximation (1 marks).

(v) Using Chebyshev's inequality, how many standard deviation units c would you need in order for the confidence interval to have at least 95% coverage? (3 marks).

(b) The Mathematics School also wants to survey the research interests and attitudes of individual staff members, with a series of face-to-face interviews. However, it does not have the resources to survey all staff. Instead, a sample of 40 staff members is to be chosen to be interviewed by 4 researchers. The school is keen to ensure that a representative sample of staff is chosen, and that the workload of each of the 4 researchers is manageable.

(vii) Explain briefly how stratified and multi-stage sampling might be used as part of the survey design for (b) (4 marks).

[20 marks total]

SOLUTION:

(a) (i) Define probabilities of selecting two different research groups i and k as p_i and p_k, where $i \neq k$.
$P(i \text{ chosen in first draw}) = p_i$
$P(k \text{ chosen in second draw}|i \text{ chosen in first draw}) = \frac{p_k}{(1-p_i)}$
$P(\text{ group } i \text{ chosen first, group } k \text{ chosen second.})$
$= P(i \text{ chosen in first draw}) \times P(k \text{ chosen in second draw}|i \text{ chosen in first draw})$
$= p_i \times \frac{p_k}{(1-p_i)}$
Conversely:
$P(\text{ group } k \text{ chosen first, group } i \text{ chosen second.})$
$= P(k \text{ chosen in first draw}) \times P(i \text{ chosen in second draw}|k \text{ chosen in first draw})$
$= p_k \times \frac{p_i}{(1-p_k)}$
Hence, a general expression for the probability that research groups i and k are both in a sample of size $n = 2$ is:
$= p_i \times \frac{p_k}{(1-p_i)} + p_k \times \frac{p_i}{(1-p_k)}$
(ii) Define number of papers published in research group i as t_i.

$$\begin{aligned}\hat{\tau}_{HT} &= \sum_{i \in S} \frac{t_i}{\pi_i} \\ &= \frac{18}{0.5929} + \frac{15}{0.3977} \\ &= 68.071\end{aligned}$$

(iii)

$$\begin{aligned}\hat{V}_{SYG}(\hat{\tau}_{HT}) &= \frac{1}{2}\sum_{i \in S}\sum_{k \in S, k \neq i}\left(\frac{\pi_i\pi_k - \pi_{ik}}{\pi_{ik}}\right)\left(\frac{t_i}{\pi_i} - \frac{t_k}{\pi_k}\right)^2 \\ &= 27.0040\end{aligned}$$

N.B: Since $\pi_{ik} = \pi_{ki}$, and we could select (i, k) or (k, i), the half in the expression above cancels out.
(iv)
The approximate s.e. of the total from $\sqrt{\hat{V}_{SYG}(\hat{\tau}_{HT})} =$ is: 5.1965 and the 95% critical points are -1.96 and 1.96. Hence the nominal 95% CI for the total is estimated as:

$$\begin{aligned}68.071 &\pm 1.96 \times 5.1965 \\ &= (57.9959, 78.25614).\end{aligned}$$

(v)
Chebyshev's inequality states that $\Pr(|X - E(X)| > a) \leq \frac{V(X)}{a^2}$, here $a = c\sqrt{V(\hat{\tau}_{HT}}$, where we approximate $V(\hat{\tau}_{HT}$ by $\hat{V}(\hat{\tau}_{HT} = 27$. Setting $\frac{V(X)}{(c\sqrt{V(\hat{\tau}_{HT})^2}} = \frac{1}{c^2} \leq 0.05$ and solving for c we get $c \geq 4.47$.
(b) (vi)
Stratify by gender, number of years working at the school, age.
Cluster by research group, or discipline areas within the maths school, choose 4 clusters at random of size n=10. Send one researcher each to talk to the individuals in each research group, thus evening out the workload and sending each research to only one place.

B3.

Assume that a variable Y_i is the the income of an individual in the north of England and that $Y_i \overset{\text{i.i.d}}{\sim} N(\mu, \sigma^2)$. If we were to take a sample, we know that the missing data generating mechanism is

$$\Pr(M_i = 1 | Y_i = y) = \frac{\exp^{\alpha+\beta y}}{1 + \exp^{\alpha+\beta y}} \qquad (1)$$

(a) If $\beta > 0$ will an available case analysis that estimates μ based on a sample overestimate or underestimate μ? (2 marks)
(b) Are data missing completely at random (MCAR) and, if not, under what conditions for the missing data generating mechanism are they? (4 marks)
(c) For $\mu = 0$, $\sigma^2 = 1$, $\beta = 1$, and $\alpha = -1.96$, find an upper bond of the proportion of missing. Express this in terms of the marginal probability $\Pr(M = 1)$, which is $\Pr(M = 1|Y = y)$ marginalised with respect to y and use monotonicity of (1) (5 marks)
(d) Given an example of how you can make the bound in (c) sharper by using the fact that the function $1/(1 + e^{f(y)})$ has an inflection point at $f(y) = 0$. You may need the result that if $Z \sim N(0, 1)$, then the expected value for Z truncated to the interval (a, b) is

$$E(Z|a < Z < b) = \frac{\frac{e^{-a^2/2}}{\sqrt{2\pi}} - \frac{e^{-b^2/2}}{\sqrt{2\pi}}}{\Phi(b) - \Phi(a)}$$

(9 marks)
[20 marks total]

SOLUTION:

(a) available, and complete case for that matter, analysis uses observations Y_i for which $M_i = 0$. If $\beta > 0$ the probability of missing is increasing in y, meaning that any estimate of μ only based on observed data will underestimate μ.
(b) If MCAR $f(M_i|Y_{\text{obs}}, Y_{\text{miss}}) = f(M_i)$. Here, if $M_i = 1$, $f(M_i|Y_{\text{obs}}, Y_{\text{miss}}) = \Pr(M_i = 1|Y_i = y_{i,\text{miss}})$. If $\beta = 0$ then $\Pr(M_i = 1|Y_i = y) = \Pr(M_i = 1) = \exp^{\alpha} / (1 + \exp^{\alpha})$.
(c)

$$\begin{aligned}
\Pr(M = 1) &= \int_{-\infty}^{\infty} \Pr(M = 1|Y = y)(2\pi)^{-1/2} e^{-y^2/2} dy \\
&= \int_{-\infty}^{0} \frac{e^{\alpha+y}}{1 + e^{\alpha+y}} (2\pi)^{-1/2} e^{-y^2/2} dy + \int_{0}^{\infty} \frac{e^{\alpha+y}}{1 + e^{\alpha+y}} (2\pi)^{-1/2} e^{-y^2/2} dy
\end{aligned}$$

where

$$\begin{aligned}
\int_{-\infty}^{0} \frac{e^{\alpha}}{1 + e^{\alpha}} (2\pi)^{-1/2} e^{-y^2/2} dy &< \frac{e^{\alpha}}{1 + e^{\alpha}} \int_{-\infty}^{0} (2\pi)^{-1/2} e^{-y^2/2} dy \\
&= \frac{e^{-1.96}}{1 + e^{-1.96}} \frac{1}{2} \\
&= \frac{0.123467}{2} = 0.0617
\end{aligned}$$

and, the second term in the sum, we observe that

$$\begin{aligned}
\int_0^\infty \frac{e^{\alpha+y}}{1+e^{\alpha+y}}(2\pi)^{-1/2}e^{-y^2/2}dy &= \int_0^{1.96} \frac{e^{\alpha+y}}{1+e^{\alpha+y}}(2\pi)^{-1/2}e^{-y^2/2}dy + \int_{1.96}^\infty \frac{e^{\alpha+y}}{1+e^{\alpha+y}}(2\pi)^{-1/2}e^{-y^2/2}dy \\
&< \frac{e^{-1.96+1.96}}{1+e^{-1.96+1.96}}\int_0^{1.96}(2\pi)^{-1/2}e^{-y^2/2}dy + \int_{1.96}^\infty (2\pi)^{-1/2}e^{-y^2/2}dy \\
&= \frac{1}{2}(\Phi(1.96)-0.5)+(1-\Phi(1.96)) \\
&= 3/4-\Phi(1.96)/2 = 0.2625
\end{aligned}$$

giving us an upper bound of $0.0617 + 0.2625 = 0.32$. **to here is sufficient for full marks**
We can also, for example, construct intervals $(y_{r-1}, y_r]$ and evaluate

$$\begin{aligned}
\int_{-\infty}^\infty \frac{e^{\alpha+y}}{1+e^{\alpha+y}}(2\pi)^{-1/2}e^{-y^2/2}dy &< \sum \int_{y_{r-1}}^{y_r} \frac{e^{\alpha+y}}{1+e^{\alpha+y}}(2\pi)^{-1/2}e^{-y^2/2}dy + 0.0194\Phi(-1.96)+(1-\Phi(1.96)) \\
&< \sum_{r:y_r\le 0}(0.053y_r+0.12)(\Phi(y_r)-\Phi(y_{r-1})) + \sum_{r:y_r>0}(0.19y_r+0.12)(\Phi(y_r)-\Phi(y_{r-1})) \\
&+ 0.02548285
\end{aligned}$$

According to this painstaking excercise we have an upper bound on the proportion of missing that is

Table 4: Simple interpolations

r	y_r	$0.053y_r+0.12$	$\Phi(y_r)-\Phi(y_{r-1})$	prod
1	-1	0.067	0.1336574	0.008955043
2	0	0.12	0.3413447	0.04096137
	y_r	$0.19y_r+0.12$	$\Phi(y_r)-\Phi(y_{r-1})$	prod
3	1	0.31	0.3413447	0.1058169
4	1.96	0.4924	0.1336574	0.0658128
				0.2215461

$0.2215461 + 0.02548285 = 0.247$
(d) As $\frac{e^{\alpha+y}}{1+e^{\alpha+y}}$ is convex on $y \in (-\infty, 0)$, this bound can be made sharper by linear interpolation, noting that $g(y) < \frac{g(a)-g(b)}{a-b}(y-b)+g(b)$, $g(y) = \frac{e^{\alpha+y}}{1+e^{\alpha+y}}$. Setting $b=-1.96$ and $a=0$, we get

$$\begin{aligned}
g(y) &\le \frac{\frac{e^{-1.96}}{1+e^{-1.96}} - \frac{e^{-3.92}}{1+e^{-3.92}}}{1.96}(y+1.96) + \frac{e^{-3.92}}{1+e^{-3.92}} \\
&= \frac{0.123467-0.019455}{1.96}(y+1.96)+0.019455 \\
&= \frac{0.104}{1.96}(y+1.96)+0.019455 \\
&= 0.053(y+1.96)+0.019455 \\
&= 0.053y+0.1233
\end{aligned}$$

Similarly, on $y \in (0, 1.96)$

$$\begin{aligned}
g(y) &\leq \frac{\frac{1}{2} - \frac{e^{-1.96}}{1+e^{-1.96}}}{1.96} y + \frac{e^{-1.96}}{1+e^{-1.96}} \\
&= \frac{0.5 - 0.123467}{1.96} y + 0.123467 \\
&= 0.1921087y + 0.123467
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{\alpha+y}}{1+e^{\alpha+y}} (2\pi)^{-1/2} e^{-y^2/2} dy &< \int_{-1.96}^{0} (0.053y + 0.12) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy + \int_{0}^{1.96} (0.19y + 0.12) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \\
&+ 0.0194\Phi(-1.96) + (1 - \Phi(1.96)) \\
&< 2 \times 0.475 \times 0.12 + 0.053 \int_{-1.96}^{0} \frac{ye^{-y^2/2}}{\sqrt{2\pi}} dy + 0.19 \int_{0}^{1.96} \frac{ye^{-y^2/2}}{\sqrt{2\pi}} dy + 0.0255 \\
&< 0.114 + 0.053 \int_{-1.96}^{0} \frac{ye^{-y^2/2}}{\sqrt{2\pi}} dy + 0.19 \int_{0}^{1.96} \frac{ye^{-y^2/2}}{\sqrt{2\pi}} dy + 0.0255
\end{aligned}$$

Now, we recognise the integrands as the expected values of truncated standard normal variates multiplied by their normalising constants.

$$\begin{aligned}
\int_{-1.96}^{0} \frac{ye^{-y^2/2}}{\sqrt{2\pi}} dy &= E(Z| - 1.96 < Z < 0) \left(\Phi(0) - \Phi(-1.96)\right) \\
&= \frac{\frac{e^{-1.96^2/2}}{\sqrt{2\pi}} - \frac{e^{-0^2/2}}{\sqrt{2\pi}}}{\Phi(0) - \Phi(-1.96)} \left(\Phi(0) - \Phi(-1.96)\right) \\
&= \frac{e^{-1.96^2/2} - 1}{\sqrt{2\pi}} = -0.34
\end{aligned}$$

and

$$\begin{aligned}
\int_{0}^{1.96} \frac{ye^{-y^2/2}}{\sqrt{2\pi}} dy &= E(Z|0 < Z < 1.96) \left(\Phi(1.96) - \Phi(0)\right) \\
&= \frac{\frac{e^{-1.96^2/2}}{\sqrt{2\pi}} - \frac{e^{-0^2/2}}{\sqrt{2\pi}}}{\Phi(1.96) - \Phi(0)} \left(\Phi(1.96) - \Phi(0)\right) \\
&= \frac{1 - e^{-1.96^2/2}}{\sqrt{2\pi}} = 0.34
\end{aligned}$$

and putting it together

$$\int_{-\infty}^{\infty} \frac{e^{\alpha+y}}{1+e^{\alpha+y}} (2\pi)^{-1/2} e^{-y^2/2} dy < 0.114 - 0.053 \times .34 + 0.19 \times .34 + 0.0255 = 0.193$$

END OF EXAMINATION PAPER

MATH39012 Mathematical Programming
Solutions 2015

Solution Q1

(a) Formulation:

$$\begin{aligned} \text{Maximize} \quad & 60W + 100C + 80S \\ \text{subject to} \quad & 6W + 8C + 10S \leq 5000 \\ & 100W + 150C + 120S \leq 60000 \\ & W + C + S \leq 500 \\ & W, C, S \geq 0 \end{aligned}$$

	W	C	S	
s_1	6	8	10	5000
s_2	100	$\boxed{150}$	120	60,000
s_3	1	1	1	500
	-60	-100	-80	0

	W	s_2	S	
s_1	$\frac{2}{3}$	$-\frac{8}{150}$	$\frac{540}{150}$	1800
C	$\frac{2}{3}$	$\frac{1}{150}$	$\frac{12}{15}$	400
s_3	$\frac{1}{3}$	$-\frac{1}{150}$	$\frac{30}{150}$	100
	$\frac{20}{3}$	$\frac{2}{3}$	0	$40,000$

Thus optimal to plant 400 acres corn only.

(b) Optimal dual solution $\boldsymbol{y}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$ where

$$B^{-1} = \begin{pmatrix} 1 & -\frac{4}{75} & 0 \\ 0 & \frac{1}{150} & 0 \\ 0 & -\frac{1}{150} & 1 \end{pmatrix}$$

and $\boldsymbol{c}_B^T = (0, 100, 0)$ so $\boldsymbol{y}^T = \left(0, \frac{2}{3}, 0\right)$.

Now $z = \boldsymbol{y}^T \boldsymbol{b}$ so $\delta z = \boldsymbol{y}^T \delta \boldsymbol{b}$ so for $\delta\ \boldsymbol{b} = (24, 0, 0)^T$ $\delta z = 0$. No value in additional man-hours.

$s_1 = 1800$ at optimum so man-hours are not a binding constraint.

(c) Add constraint $W \geq 100$ or $W - s_4 = 100$. Add $s_4 - W = -100$ to tableau

	W	s_2	S	
s_1	$\frac{2}{3}$	$-\frac{8}{150}$	$\frac{540}{150}$	1800
C	$\frac{2}{3}$	$\frac{1}{150}$	$\frac{12}{15}$	400
s_3	$\frac{1}{3}$	$-\frac{1}{150}$	$\frac{30}{150}$	100
s_4	$\boxed{-1}$	0	0	-100
	$\frac{20}{3}$	$\frac{2}{3}$	0	$40,000$

	s_4	s_2	S	
s_1	$\frac{2}{3}$	$-\frac{8}{150}$	$\frac{540}{150}$	$\frac{5200}{3}$
C	$\frac{2}{3}$	$\frac{1}{150}$	$\frac{12}{15}$	$\frac{1000}{3}$
s_3	$\frac{1}{3}$	$-\frac{1}{150}$	$\frac{30}{150}$	$\frac{200}{3}$
W	-1	0	0	100
	$\frac{20}{3}$	$\frac{2}{3}$	0	$\frac{118,000}{3}$

New optimal solution is $(W, C, S) = \left(100, \frac{1000}{3}, 0\right)$. Max profit $\$\frac{118000}{3}$

Solution Q2

(a) Write constraints as $\boldsymbol{Ax} - \boldsymbol{s} = \boldsymbol{b}$

$$\left[\begin{array}{c|c} \boldsymbol{A} & -\boldsymbol{I}_m \end{array}\right] \left[\begin{array}{c} \boldsymbol{x} \\ \hline \boldsymbol{s} \end{array}\right] = \boldsymbol{b}$$

then apply dual transformation to standard form

$$\begin{array}{lll} D: & \text{maximize} & \boldsymbol{y}^T\boldsymbol{b} \\ & \text{subject to} & \boldsymbol{A}\boldsymbol{y}^T \leq \boldsymbol{c}^T \\ & & -\boldsymbol{y}^T \leq \boldsymbol{0}^T, \quad \boldsymbol{y} \text{ unrestricted} \end{array}$$

Thus dual is

$$\begin{array}{lll} D: & \text{maximize} & \boldsymbol{y}^T\boldsymbol{b} \\ & \text{subject to} & \boldsymbol{A}\boldsymbol{y}^T \leq \boldsymbol{c}^T \\ & & \boldsymbol{y}^T \geq \boldsymbol{0}^T \end{array}$$

(b) Phase I problem:

$$\begin{array}{rc} \text{Minimize} & R_1 + R_2 \\ \text{subject to} & R_1 + x + 6y + 3z - s_1 = 2 \\ & R_2 + 2x - 5y + z - s_2 = 3 \\ & x, y, z, R_1, R_2, s_1, s_2 \geq 0 \end{array}$$

	x	y	z	s_1	s_2	
R_1	1	6	$\boxed{3}$	-1	0	2
R_2	2	-5	1	0	-1	3
	3	1	4	-1	-1	5

	x	y	R_1	s_1	s_2	
z	$\frac{1}{3}$	2		$-\frac{1}{3}$	0	$\frac{2}{3}$
R_2	$\boxed{\frac{5}{3}}$	-7		$\frac{1}{3}$	-1	$\frac{7}{3}$
	$\frac{5}{3}$	-7		$\frac{1}{3}$	-1	$\frac{7}{3}$

			15		0	0	
		R_2	y	R_1	s_1	s_2	
5	z		$\frac{17}{5}$		$-\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
2	x		$-\frac{21}{5}$		$\frac{1}{5}$	$-\frac{3}{5}$	$\frac{7}{5}$
			0		0	0	0
			$-\frac{32}{5}$		$-\frac{8}{5}$	$-\frac{1}{5}$	$\frac{19}{5}$

So tableau is immediately optimal for Phase II and $x^* = \frac{7}{5}$, $z^* = \frac{1}{5}$.

(b) Dual problem

$$\begin{aligned} \text{Maximize} \qquad & 2w_1 + 3w_2 \\ & w_1 + 2w_2 \leq 2 \\ & 6w_1 - 5w_2 \leq 15 \\ & 3w_1 + w_2 \leq 5, \quad w_1, w_2 \geq 0 \end{aligned}$$

Let dual slacks be $v_1, v_2, v_3 \geq 0$ then CS conditions $\Rightarrow v_1 = v_3 = 0$

$$w_1 + 2w_2 = 2$$

$$3w_1 + w_2 = 5$$

Hence $w_1^* = \frac{8}{5}$, $w_2^* = \frac{1}{5}$.

Duality theorem check that all variables are feasible for primal, dual and $2w_1^* + 3w_2^* = \frac{19}{5}$= minimum OF for primal.

Solution Q3

(a) The *incumbent* is the best solution found so far along any branch.

Fathoming a branch is concluding about the solution for the subproblem represented by that branch. For a max problem the LP relaxation produces an upper bound for the true integer solution for that subproblem.

Pseudocosts are used to evaluate alternative branching possibilities: choice of a branching variable and whether to set aside the "up" or "down" branch. They are the change in the value of the OF due to one iteration of the dual simplex procedure.

(b) Solution to LP relaxation

I	x_1	x_2	
s_1	-1	2	4
s_2	$\boxed{1}$	-1	1
s_3	4	1	12
	-5	-1	0

II	s_2	x_2	
s_1	1	1	5
x_1	1	-1	1
s_3	-4	$\boxed{5}$	8
	5	-6	5

III	s_2	s_3	
s_1	$\frac{9}{5}$	$-\frac{1}{5}$	$\frac{17}{5}$
x_1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{13}{5}$
x_2	$-\frac{4}{5}$	$\frac{1}{5}$	$\frac{8}{5}$
s_4	$\boxed{-\frac{1}{5}}$	$-\frac{1}{5}$	$-\frac{3}{5}$
	$\frac{1}{5}$	$\frac{6}{5}$	$\frac{73}{5}$

IV	s_4	s_3	
s_1	9	$\boxed{-2}$	-2
x_1	1	0	2
x_2	4	1	4
s_2	-5	1	3
	1	1	14

V	s_4	s_1	
s_3	9	-2	1
x_1	1	0	2
x_2	4	1	3
s_2	-5	1	2
	1	1	13

RHS≥ 0 in final tableau. End dual simplex iterations with $z^{LP} = 14\frac{3}{5}$.

Table of pseudocosts

	u	v	f	uf	$v(1-f)$
x_1	1	–	$\frac{3}{5}$	$\frac{3}{5}$	–
x_2	6	$\frac{1}{4}$	$\frac{3}{5}$	$\frac{18}{5}$	$\frac{1}{4} \cdot \frac{2}{5} = \frac{1}{10}$

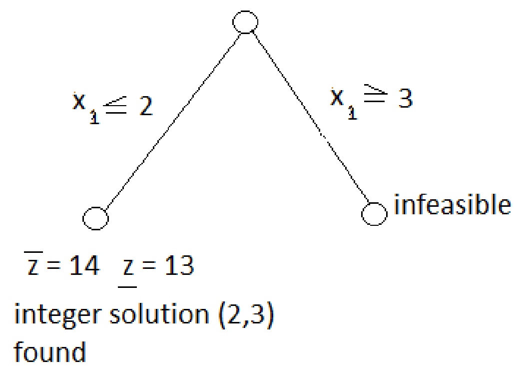
Up branch on x_1 ($x_1 \geq 3$) is infeasible, so consider down branch ($x_1 \leq 2$)

Add in $x_1 + s_4 = 2$ and eliminate x_1 from $x_1 + \frac{1}{5}s_2 + \frac{1}{5}s_3 = \frac{13}{5}$

Add $s_4 - \frac{1}{5}s_2 - \frac{1}{5}s_3 = -\frac{3}{5} \Rightarrow$ optimal solution

$$(x_1^*, x_2^*) = (2, 3) \qquad z^* = 13$$

Solution tree:



Solution Q4.

(a) Expected payoff (B/W)

$$E(\boldsymbol{r}, \boldsymbol{c}) = \boldsymbol{r}^T\,\boldsymbol{A}\boldsymbol{c} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} r_i c_j$$

Fundamental Theorem of Matrix Games (B/W):

$\exists$ strategies $\boldsymbol{r}'$, $\boldsymbol{c}'$ s.t.

$$\begin{aligned} E(\boldsymbol{r}', \boldsymbol{c}) &\geq v \qquad \text{for all column strategies } \boldsymbol{c} \\ E(\boldsymbol{r}, \boldsymbol{c}') &\leq v \qquad \text{for all row strategies } \boldsymbol{r} \end{aligned}$$

v is the value of the game.

(b) (i) Colin's problem to determine $\boldsymbol{c}' = \boldsymbol{x}^*$ optimal for

$$\min_x \max_{i=1}^{m} \left\{\boldsymbol{a}_i^T \boldsymbol{x}\right\}, \left(\boldsymbol{a}_i^T = \text{row } i \text{ of } \boldsymbol{A}\right)$$

i.e.

$$\min v$$

such that

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &\leq v\boldsymbol{1} \\ \boldsymbol{1}^{\boldsymbol{T}}\boldsymbol{x} &= 1 \\ \boldsymbol{x} &\geq \boldsymbol{0} \end{aligned}$$

(ii) Let $\boldsymbol{x}' = \frac{1}{v}\boldsymbol{x}$, then $\boldsymbol{1}^{\boldsymbol{T}}\boldsymbol{x}' = \frac{1}{v}$, so problem transforms to

$$\max\ \boldsymbol{1}^{\boldsymbol{T}}\boldsymbol{x}' = \frac{1}{v}$$

such that

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x}' &\leq \boldsymbol{1} \\ \boldsymbol{x}' &\geq \boldsymbol{0} \end{aligned}$$

Colin's LP:

$$
\begin{array}{ll}
\text{Maximize} & x_1 + x_2 + x_3 \\
\text{subject to} & 4x_1 + 3x_2 + x_3 \leq 1 \\
& x_2 + 2x_3 \leq 1 \\
& x_1, x_2, x_3 \geq 0
\end{array}
$$

Max	x_1	x_2	x_3	
s_1	$\boxed{4}$	3	1	1
s_2	0	1	2	1
	-1	-1	-1	0

Max	s_1	x_2	x_3	
x_1	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
s_2	0	1	$\boxed{2}$	1
	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{3}{4}$	$\frac{1}{4}$

Max	s_1	x_2	x_3	
x_1				$\frac{1}{8}$
x_3				$\frac{1}{2}$
	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{3}{8}$	$\frac{5}{8}$

so $\frac{5}{8} = \frac{1}{v}$ hence Colin's optimal strategy is

$$
\begin{aligned}
\boldsymbol{x}' &= \tfrac{1}{v}\, \boldsymbol{x} = \tfrac{8}{5}\left(\tfrac{1}{8}, 0, \tfrac{4}{8}\right) \\
&= \left(\tfrac{1}{5}, 0, \tfrac{4}{5}\right)
\end{aligned}
$$

(iii) Rose's problem (dual to Colin's)

$$
\begin{array}{ll}
\text{Min} & y_1 + y_2 \\
\text{s.t.} & 4y_1 \geq 1 \\
& 3y_1 + y_2 \geq 1 \\
& y_1 + 2y_2 \geq 1 \\
& y_1, y_2 \geq 0
\end{array}
$$

By CS conditions $v_1 = v_3 = 0$ so $y_1 = \frac{1}{4} = \frac{2}{8}$, $y_2 = \frac{3}{8}$. Hence Rose's optimal strategy

$$
\begin{aligned}
\boldsymbol{y}' &= v\boldsymbol{y}^* = \tfrac{8}{5}\left(\tfrac{2}{8}, \tfrac{3}{8}\right) \\
&= \left(\tfrac{2}{5}, \tfrac{3}{5}\right)
\end{aligned}
$$

(iv) Game favours Rose as $v = \frac{5}{8} > 0$.
Rose should pay $\frac{5}{8}$ to make game fair.

A1/

$$V = S^2 f_1(t) + f_2(t)$$ 4

$$S^2 f_1' + f_2' + \tfrac{1}{2}\sigma^2 S^2 \cdot 2 f_1$$
$$+ 2r S^2 f_1 - r[S^2 f_1 + f_2] = 0$$ 2

$O(S^0)$ $\quad f_2' - r f_2 = 0$ 2

$O(S^2)$ $\quad f_1' + (\sigma^2 + r) f_1 = 0$ 2

$$f_2 = A_2 e^{rt}$$

$$f_1 = A_1 e^{-(\sigma^2 + r)t}$$ 2

$f_2(T) = -w^2$
$$\Rightarrow f_2 = -w^2 e^{-r(T-t)}$$

$f_1(T) = 1$
$$\Rightarrow f_1 = e^{(\sigma^2 + r)(T-t)}$$

$$V = S^2 e^{(\sigma^2 + r)(T-t)} - K^2 e^{-r(T-t)}$$

4

$$\Delta = \frac{\partial V}{\partial S} = 2S e^{(\sigma^2 + r)(T-t)}$$

Δ

$$\Pi = V - \Delta S$$

3

see sixth

15

A2 (i) $B_c(S,T) = H(S-X)$

$B_p(S,T) = H(X-S)$

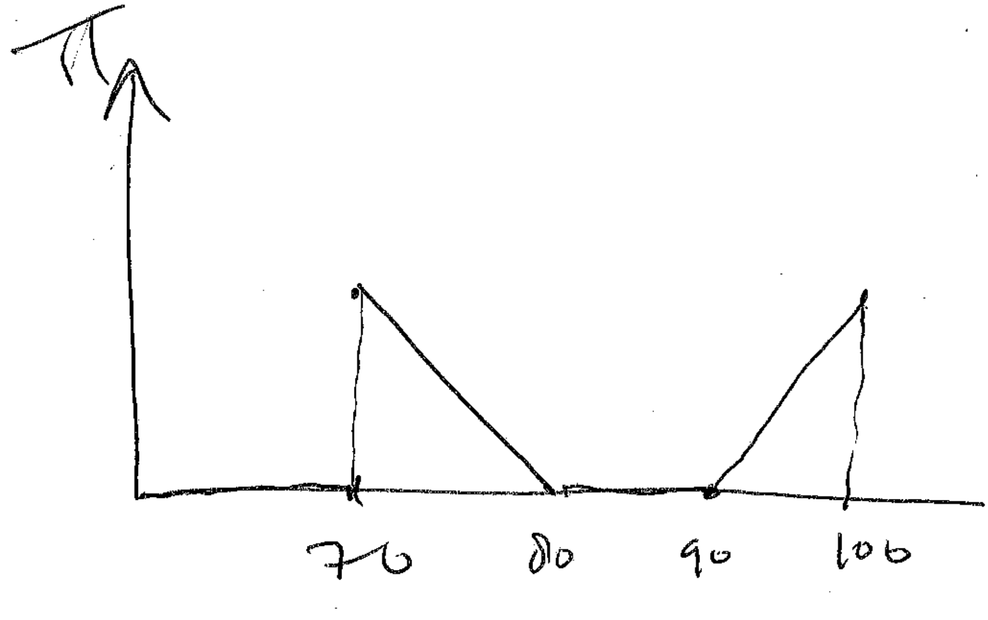
2

(ii) $B_c(S,T) + B_p(S,T) = 1$

So $B_c(S,t) + B_p(S,t) = e^{-r(T-t)}$

2

(iii)



For $S < 70$, $\Pi = 10B_c(70)$

For $70 < S < 80$, $\Pi = 10B_c(70)$

$- C(80)$

For $80 < S < 90$ (2)

$$\Pi = 10B_c(70) - C(70) + C(80)$$

For $90 < S < 100$

$$\Pi = 10B_c(70) - C(70) + C(80) + C(90)$$

For $S > 100$

$$\Pi = 10B_c(70) - C(70) + C(80) + C(90) - C(100) - 10B_c(100)$$

Seem similar (but not intuitively though)

15

A7 $T-t = 0.5$ (1

$S = 22$

$X = 21$

$\sigma = 0.15$

$r = 0.04$ 3

$d_1 = 0.680$, $d_2 = 0.574$ 2

$X e^{-r(T-t)} = 20.584$ 2

$C = 1.7797$ 2

$P = 0.3679$ 2

To break even (call)

$1.7797 = S - 21 \Rightarrow \cancel{\ldots}$ $S = 22.779$

$\Rightarrow \Delta S = 0.779$ 2

To break even (put

$0.367 = 21 - S \Rightarrow S = 20.637$ 2

$\Rightarrow \Delta S = -1.367$

see s[illegible]

A4

(i) $\pi(t=T) = 2S - 3P(x)$ ✓

$= 2S - 3\max(x-S, 0)$

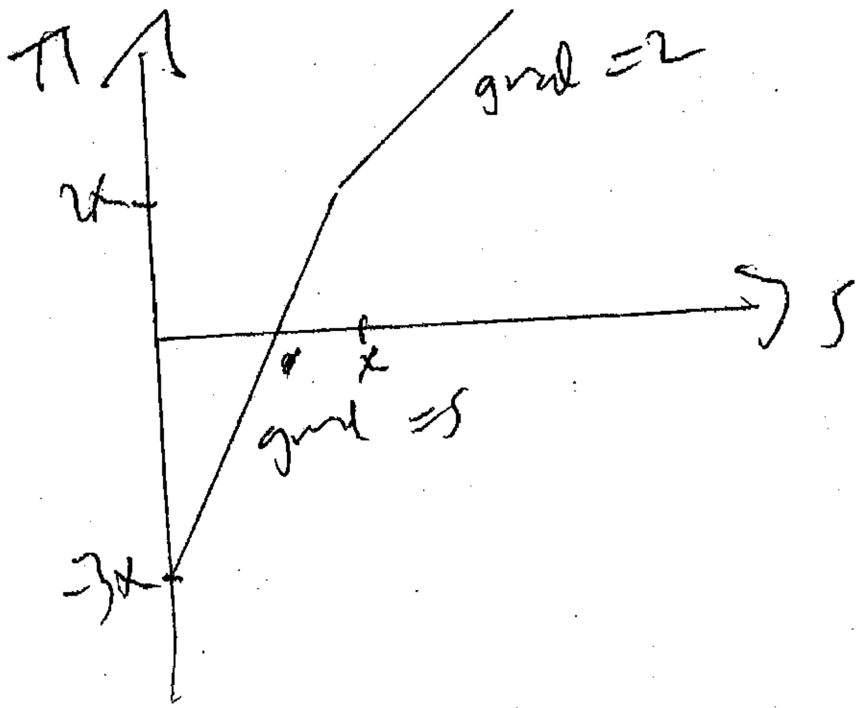
$0 < S < x$

$\pi(t=T) = 2S - 3(x-S)$

$= 5S - 3x$

$S > x$

$\pi = 2S$



3

(iv) $\pi(t=T) = 2\,c(x) - 3\,p(x)$

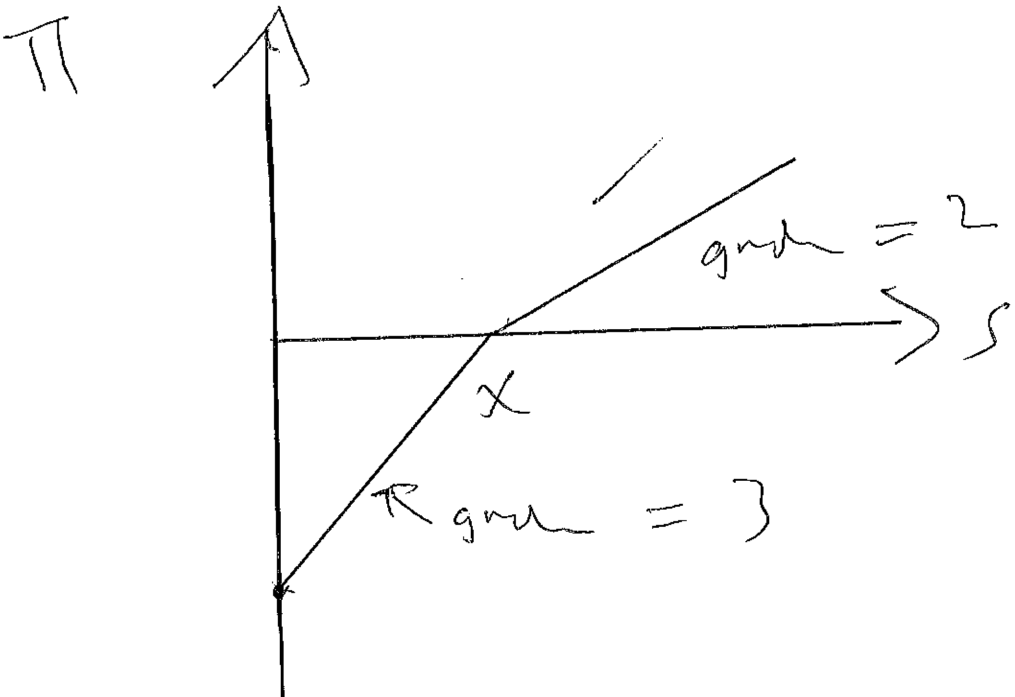
$0 < S < X$

$\pi(t=T) = 0 - 3(X-S)$

$= 3S - 3X$

$S > X$

$\pi(t=T) = 2(S-X)$



3

(iii) $\pi = 2p(X_1)$ (J)

$+ 2c(X_2)$

$\pi(t=T) = 2\max(X_1 - S, 0)$

$+ 2\max(S - X_2, 0)$

If $X_2 < X_1$

If $0 < S < X_2$

$\pi(t=T) = -2(S - X_4) = 2(X_1 - S)$

If $X_2 < S < X_1$

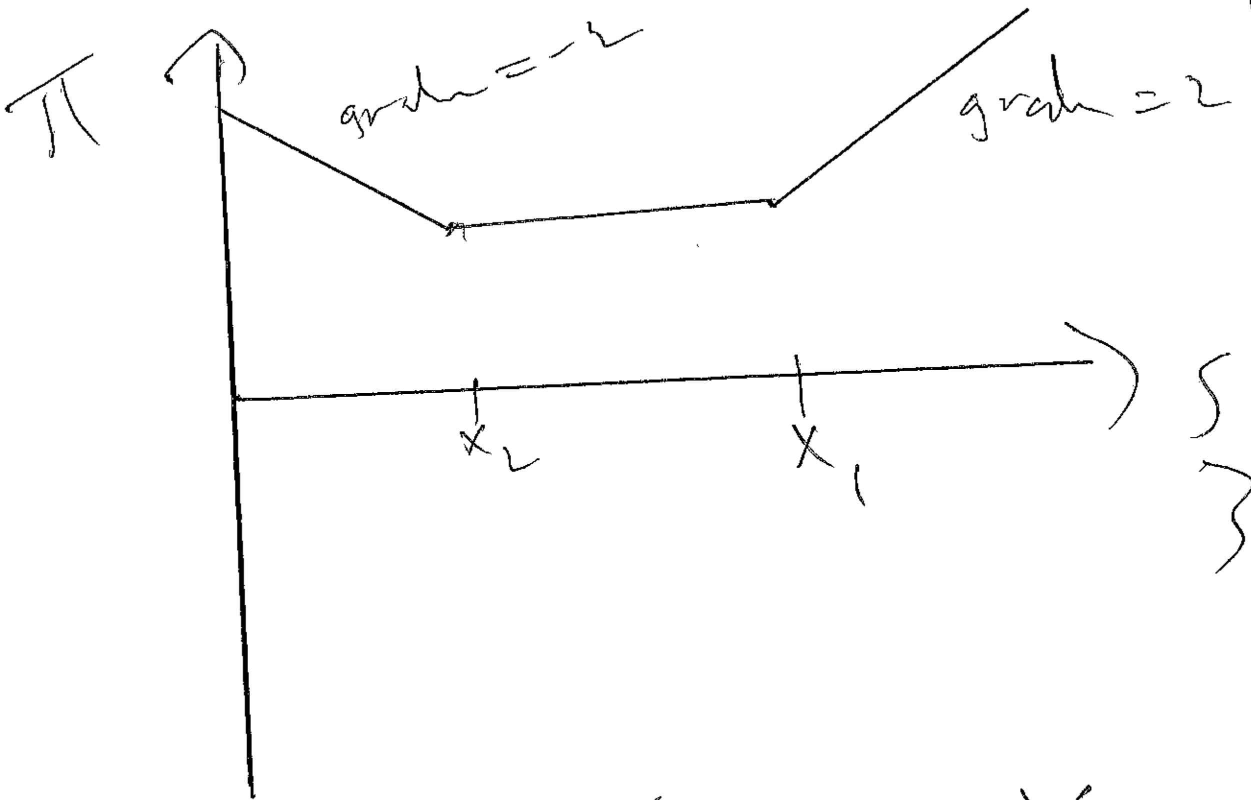
$\pi(t=T) = 2(X_1 - S) + 2(S - X_2)$

$= 2(X_1 - X_2) > 0$

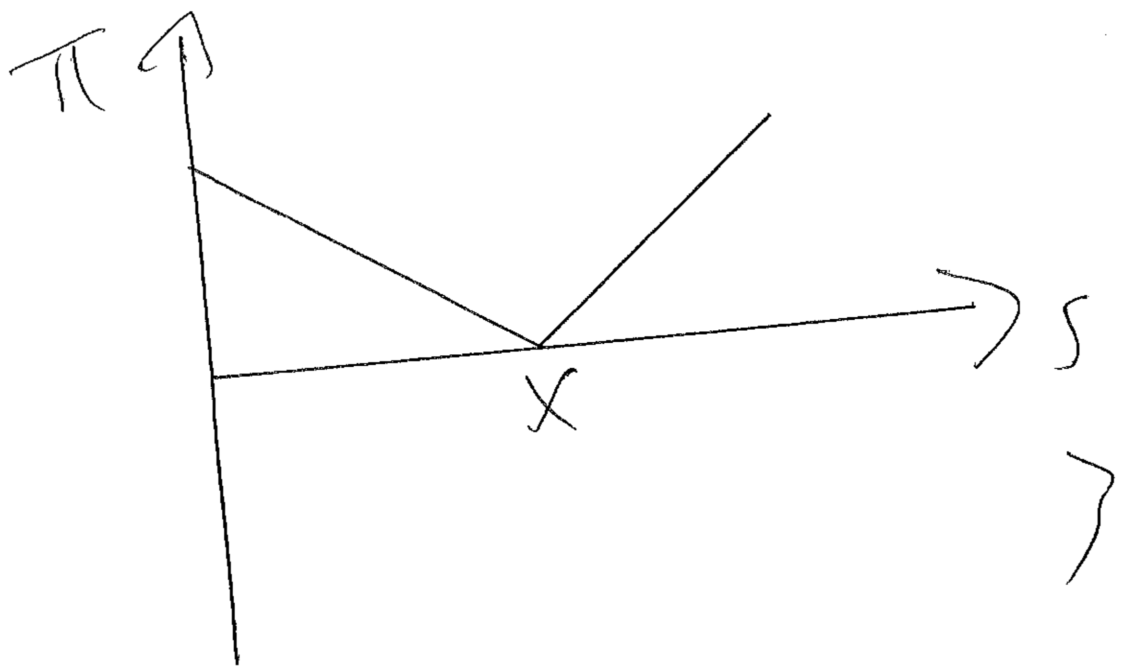
If $S > X_1$

$\pi(t=T) = 2(S - X_2)$

(4)



If $x_1 = x_2 = X$



If $X_2 > X_1$ (5)

If $0 < S < X_1$

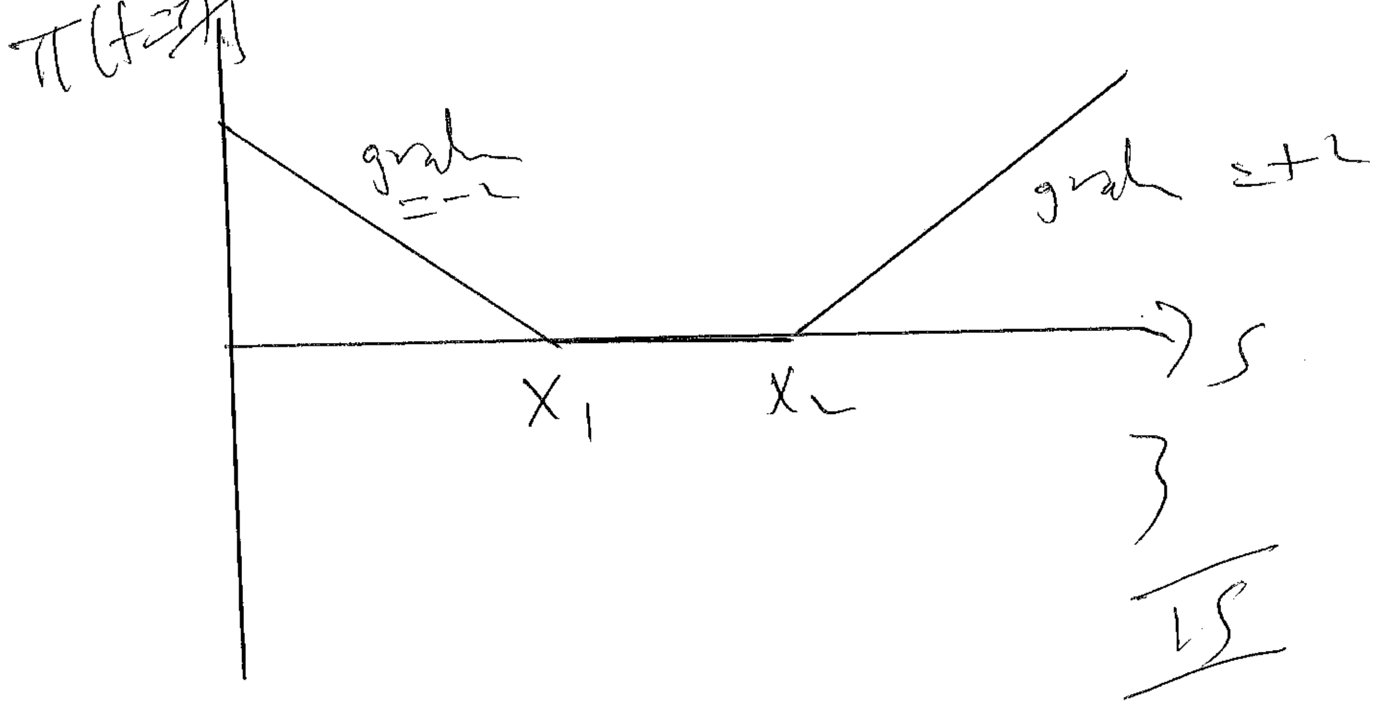
$\pi(t=T) = 2(X_1 - S)$

If $X_1 < S < X_2$

$\pi(t=T) = 0$

If $S > X_2$

$\pi(t=T) = 2(S - X_2)$



3

15

See student

B5 If $V = S^{1-2r/\sigma^2} V_1(\xi, t)$

$$S^{1-2r/\sigma^2}\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2\Big[S^{-1-2r/\sigma^2} V_1\left(1-\frac{2r}{\sigma^2}\right)\left(\frac{2r}{\sigma^2}\right.$$

$$+2\left(1-\frac{2r}{\sigma^2}\right) S^{-2r/\sigma^2}\frac{\partial V_1}{\partial S} + S^{1-2r/\sigma^2}\frac{\partial^2 V_1}{\partial S^2}\Big]$$

$$+rS\left[\left(1-\frac{2r}{\sigma^2}\right)V_1 S^{-2r/\sigma^2} + S^{1-2r/\sigma^2}\frac{\partial V_1}{\partial S}\right]$$

$$-r S^{1-2r/\sigma^2} V_1 = 0 \qquad \boxed{2}$$

$$\frac{\partial V_1}{\partial t} + \Big[-r\left(1-\cancel{\frac{2r}{\sigma^2}}\right)V_1 + (\sigma^2 - r) S\frac{\partial V_1}{\partial S}$$

$$+\frac{1}{2}S^2\sigma^2\frac{\partial^2 V_1}{\partial S^2}\Big] \cancel{-\frac{2r^2}{\sigma^2}} V_1 = 0 \qquad \boxed{2}$$

?

(ii) $\xi = S_d^2/S$

$$\frac{\partial V_1}{\partial S} = -\frac{S_d^2}{S^2}\frac{\partial V_2}{\partial \xi}$$

$$\frac{\partial^2 v_1}{\partial S^2} = \frac{2 S_d^2}{S^3} \frac{\partial v_2}{\partial \xi} + \frac{S_d^4}{S^4} \frac{\partial^2 v_2}{\partial \xi^2} \qquad 2$$

$$\boxed{2}$$

$$\frac{\partial v_2}{\partial t} + \Big(-r\left(1 - \frac{2r}{\sigma^2}\right) v_2 - (\sigma^2 - r) \frac{S_d}{S} \frac{\partial v_2}{\partial \xi} + \frac{S_d^2}{S} \sigma^2 \frac{\partial v_2}{\partial \xi} + \frac{1}{2} \sigma^2 \frac{S_d^4}{S^2} \frac{\partial^2 v_2}{\partial \xi^2} \Big) - \frac{2r^2}{\sigma^2} v_2 = 0$$

$$\frac{\partial v_2}{\partial t} + r\xi \frac{\partial v_2}{\partial \xi} + \frac{1}{2} \sigma^2 \xi^2 \frac{\partial^2 v_2}{\partial \xi^2} - r v_2 = 0$$

$$\boxed{2}$$

(iii) C_{D0} is a linear combination of two solutions of the BSE (which is linear) hence C_{D0} is a solution of the BSE

$$\boxed{2}$$

(iv) require

(a) $C_{D_0} \sim S$ as $S \to \infty$

(b) $C_{D_0} = \max(S - X, 0)$ at $t = T$

(c) $C_{D_0} = 0$ on $S = S_d$

$\boxed{3}$

$C_{D_0} \sim A\, V(S, t) \sim S \sim S$ as $S \to \infty$

$\Rightarrow A = 1$

$\boxed{2}$

(c) requiring

$0 = A\, V(S_d, t) + B\, V(S_d, t)$

$\Rightarrow B = -A = -1$ $\boxed{2}$

$$C_{D_0} = V(S, t) - \left(\frac{S}{S_d}\right)^{1 - \frac{2r}{\sigma^2}} V\left(\frac{S_d^2}{S}, t\right)$$

At $t = T$

$$C_{D_0} = \max(S - X, 0) - \left(\frac{S}{S_d}\right)^{1 - 2r/\sigma^2} \max\left(\frac{S_d^2}{S} - X, 0\right)$$

If $S > X$, $\frac{S_d^2}{S} < X$ (4)

$\Rightarrow C_{D0} = S - X$

If $S_d < S < X$,

$\max(S - X, 0) = 0$

and $\max\left(\frac{S_d^2}{S} - X, 0\right) = 0$

$\therefore$ Final condition satisfied $\boxed{7}$

$\boxed{\text{Not seen before}}$

B6 If stock S before dividend, value $S - d_y S$ after, otherwise arbitrage possibilities

Holder of put option receives no dividend, ∴ value of option across dividend date unchanged

$$p(t_d^-, S^-) = p(t_d^+, S^+)$$

$$\Rightarrow p(t_d^-, S) = p(t_d^+, S(1-d_y))$$

For $t_d^+ \leq t < T$

$$p_d(S, t) = p(S, t; X)$$

$$\cancel{p(S(1-d_y); t; X)}$$

$$p_d(S, t_d^-) = p(t_d^+ S(1-d_y); X)$$

$$P(S(1-d_y), T; X) = \max(X - S(1-d), 0)$$

$$= (1-d)\max\left(\frac{X}{1-d_y} - S, 0\right)$$

$$= (1-d_y)\, P\left(S, T, \frac{X}{1-d_y}\right)$$

Basic result for $(1-d_y)$ puts with exercise price $\frac{X}{1-d_y}$ 2

similar to last week.

If $d_1, d_2, \ldots, d_n$ dividends,

If $t_{k-1} < t < t_k$

$$P(t, S) = (1-d_n)(1-d_{n-1})\ldots(1-d_k) \cdot P\left(S, T, \frac{X}{(1-d_n)(1-d_{n-1})\ldots(1-d_k)}\right)$$

(unseen) 6

Pt (i) $dS_i = \mu_i S_i \, dt + \sigma_i S_i \, dW_i$

$\Rightarrow$ Geometric Brownian motion

$\Rightarrow$ changes in S_i linked to magnitude of S_i

$\Rightarrow$ S_i cannot go negative

(iii) $d\Pi = dV - \Delta_1 dS_1 - \Delta_2 dS_2$

$$= \left[\frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2 + \frac{1}{2} \frac{\partial^2 V}{\partial S_1^2} dS_1^2 + \frac{1}{2} \frac{\partial^2 V}{\partial S_2^2} dS_2^2 + \frac{\partial^2 V}{\partial S_1 \partial S_2} dS_1 dS_2 \right]$$

$$- \Delta_1 [\mu_1 S_1 \, dt + \sigma_1 S_1 \, dW_1]$$

$$- \Delta_2 [\mu_2 S_2 \, dt + \sigma_2 S_2 \, dW_2]$$

$$= \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial S_1}\left[\mu_1 S_1\,dt + \sigma_1 S_1\,dW_1\right]\Big($$

$$+ \frac{\partial V}{\partial S_2}\left[\mu_2 S_2\,dt + \sigma_2 S_2\,dW_2\right] + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2}dt$$

$$+ \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}dt + \sigma_1\sigma_2\rho S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2}dt$$

$$- \Delta_1\left[\mu_1 S_1\,dt + \sigma_1 S_1\,dW_1\right]$$

$$- \Delta_2\left[\mu_2 S_2\,dt + \sigma_2 S_2\,dW_2\right]$$

$$+ o(dt)$$

$\Delta_i = \frac{\partial V}{\partial S_i}$ removes dW_i $\Big)$

(iii) $$d\Pi = \Big[\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S_1^2}S_1^2\sigma_1^2$$

$$+ \frac{1}{2}\frac{\partial^2 V}{\partial S_2^2}S_2^2\sigma_2^2 + \frac{\partial^2 V}{\partial S_1 \partial S_2}S_1 S_2 \sigma_1\sigma_2\Big]dt$$

$$= r\Pi\,dt$$

$$dt\, r\left[V - S_1\frac{\partial V}{\partial S_1} - S_2\frac{\partial V}{\partial S_2}\right]$$

$$\Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2\frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2\frac{\partial^2 V}{\partial S_1\partial S_2}$$

$$+ rS_1\frac{\partial V}{\partial S_1} + rS_2\frac{\partial V}{\partial S_2} - rV = 0\Big)$$

see Sinsla

(iii) If $\frac{\partial}{\partial t} = 0$, $\rho = 0$ B

$$\tfrac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \tfrac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}$$

$$+ r S_1 \frac{\partial V_1}{\partial S_1} + r S_2 \frac{\partial V_2}{\partial S_2} - rV = 0$$

If $V = A_1 S_1^{\lambda_1} + A_2 S_2^{\lambda_2}$

unseen

$O(S_i^{\lambda_i})$:-

$$\tfrac{1}{2}\sigma_i^2 (\lambda_i)(\lambda_i - 1) + r\lambda_i - r = 0$$

$$\lambda_i = -\frac{2r}{\sigma_i^2} < 0 \quad C$$

[$\lambda_i = 1$ unacceptable for a put]

On S_1^*, S_2^K

V, $\frac{\partial V}{\partial S_i}$ continuous

(iv) $A_1 S_1^{\kappa\lambda_1} + A_2 S_2^{\kappa\lambda_2} = X - S_1^{\kappa} - S_2^{\kappa}$

$$\lambda_1 A_1 S_1^{\kappa\lambda_1 - 1} = -1$$
$$\lambda_2 A_2 S_2^{\kappa\lambda_2 - 1} = -1$$

$$\Rightarrow A_1 S_1^{\kappa\lambda_1} = -\frac{S_1^{\kappa}}{\lambda_1}$$
$$A_2 S_2^{\kappa\lambda_2} = -\frac{S_2^{\kappa}}{\lambda_2}$$

$$-\frac{S_1^{\kappa}}{\lambda_1} - \frac{S_2^{\kappa}}{\lambda_2} = X - S_1^{\kappa} - S_2^{\kappa}$$

$$S_1^{\kappa}\left(1 - \frac{1}{\lambda_1}\right) + S_2^{\kappa}\left(1 - \frac{1}{\lambda_2}\right) = X$$ 3

If S_1 or $S_2 \to 0$ 1

$V \to \infty$ unseen

20

Table For $N(x)$ When $x \leq 0$

This table shows values of $N(x)$ for $x \leq 0$. The table should be used with interpolation. For example

$$N(-0.1234) = N(-0.12) - 0.34[N(-0.12) - N(-0.13)]$$
$$= 0.4522 - 0.34 \times (0.4522 - 0.4483)$$
$$= 0.4509$$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−3.0	0.0014	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.5	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
−3.6	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.7	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.8	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−4.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Solutions 39522

① $A_{\overset{1}{55:50}\,\overline{10|}} = A_{\overset{1}{55}:\overline{10|}} + A_{\overset{1}{50}:\overline{10|}} - A_{\overset{1}{55:50}\,\overline{10|}}$ [2]

$= \left(A_{55} - v^{10}\frac{l_{65}}{l_{55}}A_{65}\right) + \left(A_{50} - v^{10}\frac{l_{60}}{l_{50}}A_{60}\right) - \left(A_{55:50} - v^{10}\frac{l_{65}}{l_{60}}\frac{l_{60}}{l_{50}}A_{65:60}\right)$

[2]

$v^{10} = 0.67556$ $\quad l_{65}/l_{60} = 9647.8/9904.8 = .97405$ (male)

$d = 0.038462$ $\quad l_{60}/l_{50} = 9848.4/9952.7 = .98952$ (female)

[1]

1st term: $A_{55} = 1 - 0.038462 \times 17.364 = 0.33215$

$A_{65} = 1 - 0.038462 \times 13.666 = 0.47438$

first term $= 0.01999$. [1]

2nd term: $A_{50} = 1 - .038462 \times 19.539 = 0.24849$.

$A_{60} = 1 - .038462 \times 16.652 = 0.35953$.

2nd term $= 0.008151$. [1]

3rd term: $A_{55:50} = 1 - .038462 \times 16.602 = 0.36145$

$A_{65:60} = 1 - .038462 \times 12.682 = 0.51222$

3rd term $= 0.02792$ [2]

$\therefore$ answer $= 0.01999 + 0.008151 - 0.02792 = \underline{\underline{0.00021}}$

[1]

Total: 10

(2) a) either $\int_5^\infty v^t \, {}_tp_y \, dt - \int_5^\infty v^t \, {}_tp_{xy} \, dt$

or $\int_5^\infty v^t \, {}_tp_y \, {}_tq_x \, dt$ [3]

b) if both alive: $\int_0^\infty v^s \, {}_sp_{y+t} \, {}_sq_{y+t} \, ds$ [1]

if x is dead: $\int_0^\infty v^s \, {}_sp_{y+t} \, ds$ [1]

if y is dead : NIL [1]

c) for A there is a step change on the death of x giving an increase in the reserve or, if y dies, a reduction in reserve to zero. [1]

For B, on the death of x the reserve for the annuity to x falls but the reserve for the reversionary annuity increases – compensating effects. For B, on the death of y while the reserve for the reversionary annuity falls to zero the reserve for the annuity to x remains unchanged. Hence combined there is a smaller percentage change to the reserves for B. [2]

The level of reserves for A is potentially more volatile. [1]

Total = 10

(3) a) $_tp_x^{HS}$ = probability that healthy life at x is sick at x+t.

$$EPV = \int_0^\infty e^{-\delta t} \, _tp_x^{HS} \, dt \, . \quad \delta = \text{force of interest} \quad [3]$$

b) $_tp_x^{SS}$ = probability that a sick life at x is sick at x+t

$$EPV = \int_0^\infty e^{-\delta t} \, _tp_x^{SS} \, dt \, . \quad [2]$$

c) $_tp_x^{\overline{SS}}$ = probability of being continuously sick between x and x+t.

$$EPV = \int_0^\infty e^{-\delta(t+1)} \, _tp_x^{HS} \, _1p_{x+t}^{\overline{SS}} \, dt \, . \quad [4]$$

d) The valuation must change [1] because $_tp_x^{SS}$ allows a payment to be made at time t if x is sick without condition [1] and therefore does not require x to have been continuously sick for the 12 months prior to t. [1]

[3]

total = 12

---

(4) Dependent probabilities ! [redacted] [3]

$$(aq)^r_{60} = \frac{0.05}{0.051}\left(1 - e^{-0.051}\right) = \frac{.05}{.051} \times .04972 = 0.4875 \quad [1]$$

$$\therefore (aq)^d_{60} = 0.00097 \quad [1]$$

$$aq^r_{61} = \frac{.06}{.0612}\left(1 - e^{-0.0612}\right) = \frac{.06}{.0612} \times .05936 = 0.05820 \quad [1]$$

$$\therefore (aq)^d_{61} = \; = 0.00116 \quad [1]$$

$$(aq)^r_{62} = \frac{.07}{.0714}\left(1 - e^{-0.0714}\right) = \frac{.07}{.0714} \times .06891 = 0.06756 \quad [1]$$

$$\therefore (aq)^d_{62} = 0.00135 \quad [1]$$

Table:

Age	$(al)_x$	$(ad)^d_x$	$(ad)^r_x$
60	100,000	97	4875
61	95028	110	5530
62	89388	121	6039
63	83228		

[3]

Total = 12

---

5) a)

$$\underbrace{\sum_{t=0}^{24}}_{[1]} \underbrace{\frac{10}{60} \times 30{,}000}_{[1]} \times \underbrace{\frac{z_{40+t+1/2}}{s_{39}}}_{[2]} \times \underbrace{v^{t+1/2}}_{[1]} \times \underbrace{\frac{r_{40+t}}{l_{40}}}_{[1]} \times \underbrace{\bar{a}_{40+t+1/2}}_{[1]}$$

$$+ \frac{10}{60} \times 30{,}000 \times \frac{z_{65}}{s_{39}} \times v^{25} \times \frac{r_{65}}{l_{40}} \times \bar{a}_{65} \quad [2]$$

marks a) = [9]

b) $\frac{10}{60} \times 30{,}000 \times \underbrace{\frac{s_{40}}{s_{39}}}_{[1]} \times \underbrace{{}^zM^{ra}_{40} / {}^sD_{40}}_{[3]} = 5000 \times \frac{7.814}{7.623} \times \frac{128026}{25059}$ [1]

$= £26185$

marks for b) = [5]

c) At age 65, the 40 year old is expected to have a final salary of $30{,}000 \times z_{65}/s_{39} = 30{,}000 \times 11.157/7.623 = 43{,}910$ which is below the limit of £50,000, and will be for earlier retirement. [2]

d) Final salary at age 60, the earliest age of retirement, the expected final average salary is $40{,}000 \times 10.350/7.623 = 54309 > 50{,}000$

Final average salary will be higher at later ages and therefore as the limit applies past service pension $= \frac{10}{60} \times 50{,}000 = £\underline{\underline{8{,}333}}$ [3]

Total = 19

6) Use $CF_t + {}_{t-1}V(1+i) - p_{x+t-1} \cdot {}_tV = (PRO)_t$

where CF_t is cash flow as given and $(PRO)_t$ is the element in the profit vector ~~at~~ for year t. Define $CF'_t = CF_t - p_{x+t-1} \cdot {}_tV$ [2]

As cash flow is positive in years 6 and 7 ${}_5V = 0$

$\therefore$ year 5: $CF'_5 = (50)$ $\therefore (PRO)_5 = 0$ ${}_4V = \frac{50}{1.05} = 47.62$ [2]

year 4: $CF'_4 = (100) - \underbrace{p_{43}}_{.9988} \cdot 47.62 = -147.56$

Hence $(PRO)_4 = 0$ ${}_3V = \frac{147.56}{1.05} = 140.53$ [2]

year 3: $CF'_3 = 150 - \underbrace{p_{42}}_{.9989} \times 140.53 = 9.63$ [2]

Hence $(PRO)_3 = 9.63$ and ${}_2V = 0$

year 2 – ignore as positive cash flow and no closing reserve [1]

year 1 – ${}_0V$ = nil and ${}_1V$ = nil hence cash flow unaffected [1]

Hence the profit vector is $-200, 0, 9.63, 0, 0, 200, 300$

total = 10

7) **Unit fund** :

	Year 1	Year 2
Units at start of year	0	537.97
Allocation	550	1025
Less b/o spread	(27.50)	(51.25)
	522.50	1511.72
add interest at 4% pa	543.40	1572.19
Less management charge	(5.43)	(15.72)
Units at end of year	537.97	1556.47

[6]

Non Unit fund

	Year 1	Year 2
Unallocated premium	450	(25)
plus B/O spread	27.50	51.25
Less expenses	(250)	(30)
	227.50	(3.75)
plus interest at 2% pa	232.05	(3.83)
plus management charge	5.43	15.72
Less cost of guarantee	(7.70)	–
	229.78	11.89 [6]

Hence profit vector = 229.78 , 11.89 [1]

Profit signature : 229.78 , 11.89 × .992 = 229.78 , 11.79 [1]

Net Present value : 229.78 + 11.79/1.06 = £ 240.91 [2]

EPV of premiums : $1000 + 1000 \times \frac{.992}{1.06}$ = £ 1935.85 [2]

Profit margin : $\frac{240.91}{1935.85}$ = 12.44% [1]

total = 19

8a) 4 possible reasons are:

Within the respective areas there may be differences in the mix of occupations, housing, climate, education, nutrition (4 out of 5) [4]

b) There may be some changes but essentially it is the same person (or group of people) in a new location – simply moving does not alter mortality expectations immediately.

Education will not change and nor is nutrition likely to change.

The office change might mean new occupations but it is more likely that the people will be doing similar jobs.

Climate could be different and housing may also change. [4]

Total = 8

See over for further comments

Marking schedule

Q1 - Variation of course notes example

Q2 - a) ~~and~~ bookwork
b) and c) new

Q3 - a) Bookwork, examples, Tutorials
b) ——— " ———
c) Hard tutorial question
d) new

Q4 - bookwork and tutorial

Q5 a) - tutorial question based on bookwork
b)
c) and d) new

Q6 Tutorial

Q7 Worked example in course notes with small variations (±15 minutes to do calculations)

Q8 a) bookwork
b) new

Solutions to exam MATH39542 Risk Theory 2015

General remarks The syllabus of MATH39542 Risk Theory consists of (i) ruin theory, (ii) premium principles and risk measures, (iii) Bayesian statistics and (iv) credibility theory. The four questions cover these topics in exactly this order.

Most of the questions are similar (though the degree of similarity can vary) to questions on the example sheets, though none are copy-pasted from the example sheets; other models (not just other parameters) are used instead and/or the question is phrased differently. The exceptions are:

- Questions 1(a) and (b) are close to material seen in the lecture notes.
- Question 1(c) can be considered as a 'new' question.
- Questions 2(a) and (b) are bookwork.
- Question 3(c) can be considered as a 'new' question.

Answer to 1

(a) Let X_1 denote the size of the first claim. The cdf of X_1 is given by $F_{X_1}(x) = 1-\mathrm{e}^{-\alpha x}$. Therefore using a formula in the notes for the Laplace exponent $\xi(\theta)$, we get

$$\xi(\theta) = c\theta - \theta\lambda \int_0^\infty \mathrm{e}^{-\theta x}(1-F_{X_1}(x))\mathrm{d}x = c\theta - \theta\lambda \int_0^\infty \mathrm{e}^{-(\theta+\alpha)}\mathrm{d}x = c\theta - \lambda\frac{\theta}{\theta+\alpha}.$$

(b) Noting that $\mathbb{E}[X_1] = 1/\alpha$, the Laplace transform of the ruin probability is given by

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\theta u}\phi(u)\mathrm{d}u =& \frac{\xi(\theta) - \theta(c-\lambda\mathbb{E}[X_1])}{\theta\xi(\theta)}\\ =&\lambda\frac{1/\alpha - 1/(\theta+\alpha)}{c\theta - \lambda\theta/(\theta+\alpha)}\cdot\frac{\theta+\alpha}{\theta+\alpha}\\ =&\lambda\frac{\theta/\alpha}{\theta(c(\theta+\alpha)-\lambda)}\\ =&\frac{\lambda}{\alpha c}\cdot\frac{1}{\theta+\alpha-\lambda/c}\\ =&\frac{\lambda}{\alpha c}\int_0^\infty \mathrm{e}^{-\theta u}\mathrm{e}^{-(\alpha-\lambda/c)u}\mathrm{d}u,\end{aligned}$$

where the first equality follows by a formula from the notes. By uniqueness of the Laplace transform it follows that $\phi(u) = \frac{\lambda}{\alpha c}\mathrm{e}^{-(\alpha-\lambda/c)u}$ for $u \ge 0$.

(c) (i) Denote by $\phi_1(u)$, respectively $\phi_2(u)$, the ruin probability of the first respectively second lob at initial capital u. Since both lobs have exponentially distributed claims, it follows by the stated formula in part (b) that

$$\phi_1(u_1) = \frac{2}{1*4}\mathrm{e}^{1-\frac{2}{4}u_1} = \tfrac{1}{2}\mathrm{e}^{-\frac{1}{2}u_1}, \quad \phi_2(u_2) = \frac{1}{0.7*5}\mathrm{e}^{0.7-\frac{1}{5}u_2} = \tfrac{2}{7}\mathrm{e}^{-\frac{1}{2}u_2}.$$

Let T_1, respectively T_2, denote the ruin time of lob 1 respectively lob 2. The probability that at least one lob gets ruined is given by

$$\begin{aligned}\mathbb{P}(T_1<\infty \text{ or } T_2<\infty) =& \mathbb{P}(T_1<\infty) + \mathbb{P}(T_2<\infty) - \mathbb{P}(T_1<\infty \text{ and } T_2<\infty)\\ =&\mathbb{P}(T_1<\infty) + \mathbb{P}(T_2<\infty) - \mathbb{P}(T_1<\infty)\mathbb{P}(T_2<\infty)\\ =&\phi_1(u_1)+\phi_2(u_2)-\phi_1(u_1)\phi_2(u_2)\\ =&\tfrac{1}{2}\mathrm{e}^{-\frac{1}{2}u_1} + \tfrac{2}{7}\mathrm{e}^{-\frac{1}{2}u_2} - \tfrac{1}{2}\mathrm{e}^{-\frac{1}{2}u_1} * \tfrac{2}{7}\mathrm{e}^{-\frac{1}{2}u_2},\end{aligned}$$

(ii) We want to minimise the function

$$f(u_1,u_2) := \mathbb{P}(T_1<\infty \text{ or } T_2<\infty)$$

subject to the constraint $u_1+u_2 = 3$. Substituting the constraint $u_1 = 3-u_2$ into f leads us to look at the function $g(u_2) := f(3-u_2, u_2)$ and we need to minimise this function over the interval $[0,3]$. We have

$$\begin{aligned}g'(u_2) =& \frac{\mathrm{d}}{\mathrm{d}u_2}\left(\tfrac{1}{2}\mathrm{e}^{-\frac{3}{2}}\mathrm{e}^{\frac{1}{2}u_2} + \tfrac{2}{7}\mathrm{e}^{-\frac{1}{2}u_2} - \tfrac{11}{14}\mathrm{e}^{-\frac{3}{2}}\right)\\ =& \tfrac{1}{4}\mathrm{e}^{-\frac{3}{2}}\mathrm{e}^{\frac{1}{2}u_2} - \tfrac{1}{7}\mathrm{e}^{-\frac{1}{2}u_2}.\end{aligned}$$

We have $g'(u_2) = 0$ if

$$u_2 = \log(\tfrac{4}{7}\mathrm{e}^{\frac{3}{2}}) = \tfrac{3}{2} + \log\tfrac{4}{7} = 0.9404.$$

Since $g''(u_2) = \frac{1}{8}\mathrm{e}^{-\frac{3}{2}}\mathrm{e}^{\frac{1}{2}u_2} + \frac{1}{14}\mathrm{e}^{-\frac{1}{2}u_2} > 0$ for all $u \in [0,3]$, the point $u_2 = 0.9404$ is the minimum of $g(\cdot)$ over the interval $[0,3]$. So the optimal allocation is $u_2 = 0.9404$ and $u_1 = 3 - 0.9404 = 2.0596$.

(d) From the lecture notes we know that the ruin probability at 0 initial capital is given by

$$\frac{\widetilde{\lambda}\mathbb{E}[Y_1]}{\widetilde{c}},$$

where $\widetilde{\lambda}$ is the claim intensity, Y_1 the first claim amount and $\widetilde{c}$ the premium rate of the first lob under the reinsurance scheme. We have $\widetilde{\lambda} = 2$, $\widetilde{c} = 4 - 2.5 = 1.5$ and $Y_1 = \min\{X_1, 0.8\}$, where X_1 is exponentially distributed with parameter 1. We have with $M = 0.8$,

$$\begin{aligned}\mathbb{E}[Y_1] =& \int_0^M x\mathrm{e}^{-x}\mathrm{d}x + \int_M^\infty M\mathrm{e}^{-x}\mathrm{d}x\\ =& 1 - M\mathrm{e}^{-M} - \mathrm{e}^{-M} + M\mathrm{e}^{-M}\\ =& 1 - \mathrm{e}^{-M}\\ =& 1 - \mathrm{e}^{-0.8} = 0.5506.\end{aligned}$$

Hence the ruin probability at 0 initial capital is $\frac{2*0.5506}{1.5} = 0.734$.

Answer to 2

Let X be a risk, i.e. a positive random variable and let $\pi(X)$ be the corresponding premium.

(a) The exponential premium principle means that the premium is given by

$$\pi(X) = \frac{1}{\beta}\log\mathbb{E}\left[\mathrm{e}^{\beta X}\right],$$

where $\beta > 0$.

(b) A premium principle satisfies the no rip-off property if for any risk X which is bounded from above by a constant C, i.e. $X \leq C$, we have that the premium is also bounded by C, i.e. $\pi(X) \leq C$.

(c) With the Esscher premium principle the premium for a risk X is given by

$$\pi(X) = \frac{\mathbb{E}\left[X\mathrm{e}^{\beta X}\right]}{\mathbb{E}\left[\mathrm{e}^{\beta X}\right]},$$

where $\beta > 0$. Let X and Y be independent, positive random variables. Note that for independent X and Y and functions f and g,

$$\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)].$$

For the additivity property, we need to show $\pi(X+Y) = \pi(X) + \pi(Y)$. By using linearity of expectation and the independence of X and Y, we have

$$\begin{aligned}
\pi(X+Y) =& \frac{\mathbb{E}\left[(X+Y)\mathrm{e}^{\beta(X+Y)}\right]}{\mathbb{E}\left[\mathrm{e}^{\beta(X+Y)}\right]} \\
=& \frac{\mathbb{E}\left[X\mathrm{e}^{\beta X}\mathrm{e}^{\beta Y}\right] + \mathbb{E}\left[Y\mathrm{e}^{\beta X}\mathrm{e}^{\beta Y}\right]}{\mathbb{E}\left[\mathrm{e}^{\beta X}\mathrm{e}^{\beta Y}\right]} \\
=& \frac{\mathbb{E}\left[X\mathrm{e}^{\beta X}\right]\mathbb{E}\left[\mathrm{e}^{\beta Y}\right] + \mathbb{E}\left[\mathrm{e}^{\beta X}\right]\mathbb{E}\left[Y\mathrm{e}^{\beta Y}\right]}{\mathbb{E}\left[\mathrm{e}^{\beta X}\right]\mathbb{E}\left[\mathrm{e}^{\beta Y}\right]} \\
=& \frac{\mathbb{E}\left[X\mathrm{e}^{\beta X}\right]}{\mathbb{E}\left[\mathrm{e}^{\beta X}\right]} + \frac{\mathbb{E}\left[Y\mathrm{e}^{\beta Y}\right]}{\mathbb{E}\left[\mathrm{e}^{\beta Y}\right]} \\
=& \pi(X) + \pi(Y).
\end{aligned}$$

We conclude that the Esscher premium principle is additive.

(d) The cdf of X is given by

$$F_X(x) = \mathbb{P}(X \leq x) = \begin{cases} 0 & x < 200, \\ 0.45 & 200 \leq x < 300, \\ 0.80 & 300 \leq x < 400, \\ 0.92 & 400 \leq x < 500, \\ 1 & x \geq 500. \end{cases}$$

Hence the Value-at-Risk with confidence level 0.90 is given by

$$\mathrm{VaR}(X;p) = \inf\{x \geq 0 : F_X(x) \geq 0.90\} = 400.$$

For the TVaR, note that $\mathrm{VaR}(X;t) = 400$ for $0.90 \leq t \leq 0.92$ and $\mathrm{VaR}(X;t) = 500$ for $0.092 < t \leq 1$. Therefore the Tail-Value-at-Risk with confidence level 0.90 is given by

$$\begin{aligned}
\mathrm{TVaR}(X;p) =& \frac{1}{1-0.90}\int_{0.90}^{1} \mathrm{VaR}(X;t)\mathrm{d}t \\
=& 10 * \left(\int_{0.90}^{0.92} 400\mathrm{d}t + \int_{0.92}^{1} 500\mathrm{d}t\right) \\
=& 480.
\end{aligned}$$

Answer to 3

(a) By Bayes' Theorem we have

$$f_{\Theta|X}(\theta|x) = c(x) f_{X|\Theta}(x|\theta) f_\Theta(\theta),$$

where f denotes the pdf/pmf of the respective random variable and c is a constant depending on x. Hence for any $x \in \mathbb{R}$

$$f_{\Theta|X}(\theta|x) = \begin{cases} \frac{c(x)}{4\sqrt{2\pi}} e^{-(x+1)^2/2} & \text{if } \theta = -1 \\ \frac{c(x)}{2\sqrt{2\pi}} e^{-x^2/2} & \text{if } \theta = 0 \\ \frac{c(x)}{4\sqrt{2\pi}} e^{-(x-1)^2/2} & \text{if } \theta = 1. \end{cases}$$

As

$$\sum_{\theta \in \{-1,0,1\}} f_{\Theta|X}(\theta|x) = 1$$

it follows that

$$\frac{c(x)}{\sqrt{2\pi}} = \left(\frac{1}{4} e^{-(x+1)^2/2} + \frac{1}{2} e^{-x^2/2} + \frac{1}{4} e^{-(x-1)^2/2} \right)^{-1}.$$

Plugging this back in and simplifying a bit we get

$$f_{\Theta|X}(\theta|x) = \begin{cases} \left(1 + 2e^{x+1/2} + e^{2x}\right)^{-1} & \text{if } \theta = -1 \\ 2\left(e^{-x-1/2} + 2 + e^{x-1/2}\right)^{-1} & \text{if } \theta = 0 \\ \left(e^{-2x} + 2e^{-x+1/2} + 1\right)^{-1} & \text{if } \theta = 1. \end{cases}$$

(b) We know from the notes that the Bayesian estimate under the squared error loss function is equal to $\mathbb{E}[\Theta|X = x]$ and hence

$$\begin{aligned} &\hat{\theta}_B(x) = \mathbb{E}[\Theta|X = x] \\ &= -1\cdot\left(1 + 2e^{x+1/2} + e^{2x}\right)^{-1} + 0\cdot 2\left(e^{-x-1/2} + 2 + e^{x-1/2}\right)^{-1} + 1\cdot\left(e^{-2x} + 2e^{-x+1/2} + 1\right)^{-1} \\ &= \frac{2\left(e^{x+1/2} - e^{-x+1/2}\right) + e^{2x} - e^{-2x}}{\left(1 + 2e^{x+1/2} + e^{2x}\right)\left(e^{-2x} + 2e^{-x+1/2} + 1\right)}. \end{aligned}$$

(c) We know from the notes that the Bayesian estimate is the decision function that minimises the posterior risk, i.e. the decision function d^* that attains the minimum in

$$\min_d \mathbb{E}[l(\Theta, d(x))|X = x].$$

Note that equivalently we may fix x and minimise over $d(x) \in \mathbb{R}$. In this case we are given that $x = 0$ which yields

$$\begin{aligned}\mathbb{E}[l(\Theta, d(0))|X = 0] &= |-1-d(0)| \cdot f_{\Theta|X}(-1|0) + |d(0)| \cdot f_{\Theta|X}(0|0) + |1-d(0)| \cdot f_{\Theta|X}(1|0) \\ &= |-1-d(0)| \cdot (2+2e^{1/2})^{-1} + |d(0)| \cdot (1+e^{-1/2})^{-1} + |1-d(0)| \cdot (2+2e^{1/2})^{-1}.\end{aligned}$$

Note that the continuous function

$$f(z) = |-1-z| \cdot (2+2e^{1/2})^{-1} + |z| \cdot (1+e^{-1/2})^{-1} + |1-z| \cdot (2+2e^{1/2})^{-1}$$

is linear on each of the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$, and it satisfies $f(\pm\infty) = \infty$. Hence it attains its minimum in either $z = \pm 1$ or $z = 0$. As

$$f(-1) = f(1) = (1+e^{-1/2})^{-1} + (2+2e^{1/2})^{-1} \approx 0.8 \quad \text{and} \quad f(0) = 2(2+2e^{1/2})^{-1} \approx 0.4$$

it follows that f attains its minimum in $z = 0$. Therefore $d^*(0) = 0$ i.e. the Bayes estimate is 0. (Which is of course not very surprising given the symmetry of the problem).

Answer to 4

(a) With $x = 0.1$ denoting the observed claim amount in Year 1, the posterior distribution of Θ is

$$\begin{aligned} f_{\Theta|X}(\theta|x) &\propto f_{X|\Theta}(x|\theta) f_\Theta(\theta) \\ &\propto \frac{2x}{\theta^2} \mathbf{1}_{\{\theta > x\}} * 4\theta^3 \\ &\propto 8x\theta \mathbf{1}_{\{\theta > x\}}, \end{aligned}$$

where the symbol $\propto$ stands for 'proportional to'. In order to determine the constant of proportionality, which we denote by c, we must have

$$1 = \int_0^1 f_{\Theta|X}(\theta|x) \mathrm{d}\theta = 8cx \int_x^1 \theta \mathrm{d}\theta = 8cx\tfrac{1}{2}(1 - x^2),$$

which implies $c = \frac{1}{4x(1-x^2)}$ and thus $f_{\Theta|X}(\theta|x) = \frac{2\theta}{1-x^2}$, $0 < x < \theta < 1$.
We also need to compute

$$\mu(\theta) := \mathbb{E}[X|\Theta = \theta] = \int_0^\infty x f_{X|\Theta}(x|\theta) \mathrm{d}x = \frac{1}{\theta^2} \int_0^\theta 2x^2 \mathrm{d}x = \tfrac{2}{3}\theta.$$

By a theorem in the notes, the Bayesian credibility estimate (which is defined as the Bayesian estimate of $\mu(\Theta)$ under squared error loss) is equal to the expectation with respect to the posterior distribution, of $\mu(\Theta)$. Hence

$$\hat{\mu}_B(x) = \mathbb{E}[\mu(\Theta)|X = x] = \int_0^\infty \mu(\theta) f_{\Theta|X}(\theta|x) \mathrm{d}\theta = \frac{4}{3(1 - x^2)} \int_x^1 \theta^2 \mathrm{d}\theta = \frac{4}{9} \frac{1 - x^3}{1 - x^2}.$$

Since we have $x = 0.1$, the Bayesian credibility estimate is given by $\hat{\mu}_B(0.1) = \frac{74}{165} = 0.448$.

(b) We have $\mu(\theta) = \frac{2}{3}\theta$ and

$$\begin{aligned} \nu(\theta) :=& \mathrm{Var}(X|\Theta = \theta) \\ =& \mathbb{E}[X^2|\Theta = \theta] - \mu(\theta)^2 \\ =& \int_0^\theta \frac{2x^3}{\theta^2} \mathrm{d}x - \tfrac{4}{9}\theta^2 \\ =& \left(\tfrac{1}{2} - \tfrac{4}{9}\right)\theta^2 = \tfrac{1}{18}\theta^2. \end{aligned}$$

This implies

$$\begin{aligned} \mu :=& \mathbb{E}[\mu(\Theta)] = \tfrac{2}{3}\mathbb{E}[\Theta] = \tfrac{2}{3} \int_0^1 4\theta^4 \mathrm{d}\theta = \tfrac{8}{15}, \\ \kappa :=& \mathrm{Var}(\mu(\Theta)) = \tfrac{4}{9}\mathrm{Var}(\Theta) = \tfrac{4}{9} \int_0^1 4\theta^5 \mathrm{d}\theta - \left(\tfrac{8}{15}\right)^2 = \tfrac{4}{9}\tfrac{2}{3} - \tfrac{64}{225} = \tfrac{8}{675}, \\ \nu :=& \mathbb{E}[\nu(\Theta)] = \tfrac{1}{18}\mathbb{E}[\Theta^2] = \tfrac{1}{18}\tfrac{2}{3} = \tfrac{1}{27}. \end{aligned}$$

From the lecture notes we know that the Bühlmann credibility estimate based on the observed value of $X = x$ is given by

$$(1 - w)\mu + wx,$$

where the credibility factor w is given by $w = \frac{\kappa}{\nu+\kappa}$. In our case $w = \frac{\frac{8}{675}}{\frac{1}{27}+\frac{8}{675}} = \frac{8}{33}$ and so Bühlmann's credibility estimate is given by

$$\hat{\mu}_{BM} = w * x + (1 - w) * \mu = \tfrac{8}{33} * 0.1 + \tfrac{25}{33} * \tfrac{8}{15} = \tfrac{212}{495} = 0.428.$$

(c) The size of the portfolio in year i is denoted by m_i. We have $m_1 = 250$ and $m_2 = 300$. We let X_i be the average claim amount of the group per policyholder in year i. Note that we observe

$$X_1 = 135/250 = \tfrac{27}{50}.$$

From the lecture notes we know that the Bühlmann credibility estimate of X_2 given the observation of X_1 is given by

$$\hat{\mu}_{BM} = (1 - \widetilde{w})\mu + \widetilde{w}X_1,$$

where the credibility factor $\widetilde{w}$ is given by $\widetilde{w} = \frac{\kappa m_1}{\nu+\kappa m_1} = \frac{250\kappa}{\nu+250\kappa}$ with μ, κ, ν as in part (b). Therefore

$$\hat{\mu}_{BM} = (1 - \tfrac{80}{81}) * \tfrac{8}{15} + \tfrac{80}{81} * \tfrac{27}{50} = \tfrac{656}{1215}.$$

Hence the Bühlmann credibility estimate of the total claim amounts of the portfolio in year 2 is

$$\tfrac{656}{1215} * m_2 = \tfrac{13120}{81} = 161.975.$$