Two hours

THE UNIVERSITY OF MANCHESTER

LINEAR ANALYSIS

29 May 2015

09:45 – 11:45

Answer ALL 7 questions in Section A (25 marks in total). Answer 3 of the 4 questions in Section B (75 marks in total). If more than 3 questions from Section B are attempted, then credit will be given for the best 3 answers.

The use of electronic calculators is permitted provided they cannot store text.

SECTION A

Answer **ALL** of the seven questions

A1. State what is meant by the norm of a linear functional on a normed vector space.

[3 marks]

A2. State (without proof) Hölder's Inequality for vectors in $\mathbb{C}^n$.

[3 marks]

A3. Let $X = C[0,1]$ endowed with the uniform norm and let the operator $T : X \to X$ be defined as

$$Tf(x) = f(1-x^2).$$

Prove that T is linear and bounded.

[4 marks]

A4. State what is meant by an algebra in the space $C(X)$. State the Stone-Weierstrass Theorem.

[5 marks]

A5. State (without proof) the Parallelogram Law for Hilbert spaces.

[2 marks]

A6. State what is meant by a reflexive Banach space. Give an example of a non-reflexive space (without proof).

[3 marks]

A7. State what is meant by an inner product on a vector space over $\mathbb{R}$ or $\mathbb{C}$. State (without proof) the Cauchy-Schwarz Inequality for an inner product.

[5 marks]

SECTION B

Answer **THREE** of the four questions

B8.

(a) State what it means for two norms on a vector space to be equivalent.

(b) Prove that any two norms on $\mathbb{R}^n$ are equivalent.

(c) Prove that $\ell^2(\mathbb{R})$ is a proper subset of $\ell^\infty(\mathbb{R})$.

(d) Prove that the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are not equivalent on $\ell^2(\mathbb{R})$.

[25 marks]

B9.

(a) State what is meant by the Bernstein polynomial $B_n(f;x)$ of a continuous function $f:[0,1]\to\mathbb{R}$.

(b) State and prove the Weierstrass Approximation Theorem for $C[0,1]$. *(You may use the fact that $\sum_{k=0}^n \binom{n}{k} x^k(1-x)^{n-k}(k-nx)^2 = nx(1-x)$.)*

(c) Fix $n \in \mathbb{N}$ and let T be a linear operator on $C[0,1]$ acting by the formula

$$T(f(x)) = B_n(f;x).$$

Prove that T is bounded and compute its norm.

[25 marks]

B10.

(a) Let f denote a linear functional on a normed vector space. State what it means for f to be continuous.

(b) State what it means for f to be bounded.

(c) Prove that f is continuous if and only if it is bounded.

(d) Let a map $f:\ell^1(\mathbb{R})\to\mathbb{R}$ be defined as follows:

$$f(x_1,x_2,\dots) = \sum_{n=0}^{\infty} \frac{x_n}{n}.$$

Show that f is a bounded linear functional on $\ell^1(\mathbb{R})$ and compute its norm.

[25 marks]

B11. Let X be a Banach space and $T : X \to X$ be a linear operator.

(a) State what is meant by the spectrum of T.

Let $\mathrm{spec}(T)$ denote the spectrum of T.

(b) State what is meant by an eigenvalue of T. Show that any eigenvalue of T lies in $\mathrm{spec}(T)$.

(c) Let P be a polynomial. Prove that

$$\mathrm{spec}(P(T)) = \{P(\lambda) : \lambda \in \mathrm{spec}(T)\}.$$

Let $X = \ell^2(\mathbb{C})$ and put

$$T(x_1, x_2, x_3, x_4 \dots) = (x_1, -x_2, x_3, -x_4, \dots).$$

(d) Prove that $T^2 - I = 0$.

(e) Prove that both -1 and 1 are eigenvalues of T.

(f) Use (b), (c), (d) and (e) to compute $\mathrm{spec}(T)$.

[25 marks]

END OF EXAMINATION PAPER

Two hours

UNIVERSITY OF MANCHESTER

ANALYTIC NUMBER THEORY

26 May 2015
14.00–16.00

Answer **THREE** of the FOUR questions. (90 marks in total.) If more than THREE questions are answered, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

Throughout the paper you may assume that the Dirichlet Convolution of two multiplicative functions is multiplicative.

1.

i) Show, by estimating integrals or otherwise, that

$$\int_1^N \frac{du}{u^\sigma} + \frac{1}{N^\sigma} \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} < 1 + \int_1^N \frac{du}{u^\sigma},$$

for integer $N \geq 1$ and real $\sigma > 0$.

Deduce that the series defining $\zeta(\sigma)$ diverges for $\sigma \leq 1$, converges for $\sigma > 1$ and that $\zeta(\sigma)$ satisfies

$$\frac{1}{\sigma - 1} \leq \zeta(\sigma) \leq 1 + \frac{1}{\sigma - 1}$$

for $\sigma > 1$.

ii) Explain why

$$\sum_{n \in \mathcal{N}} \frac{1}{n^\sigma} = \prod_{p \leq N} \left(1 - \frac{1}{p^\sigma}\right)^{-1},$$

for $N \geq 1$, where $\mathcal{N} = \{n : p|n \text{ and } p \text{ prime } \Rightarrow p \leq N\}$ and the product is over primes.

iii) Prove that

$$\log \zeta(\sigma) \leq 1 + \sum_p \frac{1}{p^\sigma},$$

for $\sigma > 1$. The summation is over all primes p.

Deduce that there are infinitely many primes.

You may assume that $\sum_p \left(-\log\left(1 - 1/p^\sigma\right) - 1/p^\sigma\right) \leq 1$ *for* $\sigma \geq 1$.

[30 marks]

2.

i) By Partial Summation prove that for $s \in \mathbb{C}$, $s \neq 1$ we have

$$\sum_{1 \le n \le N} \frac{1}{n^s} = 1 + \frac{1}{s-1} + \frac{N^{1-s}}{1-s} - s \int_1^N \{u\} \frac{du}{u^{s+1}}$$

for any **integer** $N \geq 1$.

ii) Deduce that

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{1+s}} du, \tag{1}$$

for $\mathrm{Re} s > 1$.

Explain why (1) can be used to define $\zeta(s)$ for complex s with $\mathrm{Re} s > 0$, $s \neq 1$.

iii) Using parts i and ii prove that for all $s = \sigma + it$ with $\mathrm{Re} s > 0$, $s \neq 1$, and all integers $N \geq 1$,

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + O\left(\frac{|s|}{\sigma N^\sigma}\right).$$

Deduce that

$$\sum_{n=1}^\infty \frac{1}{n^{1+it}}$$

diverges for all real $t > 0$.

[30 marks]

3.

i) Write down the Euler product for the Riemann zeta function $\zeta(s)$.

ii) Let $\omega(n)$ denote the number of distinct prime factors of n. By looking at the Euler product of the Dirichlet Series on the left hand side of the identity below, prove that

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)},$$

for $\mathrm{Re}s > 1$.

You may assume that 2^ω *is multiplicative.*

iii) Let λ be Liouville's function, defined as $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime divisors of n counted with multiplicity. By looking at the Euler product of the Dirichlet Series on the left hand side of the identity below, prove that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)},$$

for $\mathrm{Re}s > 1$.

You may assume that λ *is multiplicative.*

iv) Explain why parts ii and iii suggest that

$$2^\omega * \lambda = 1.$$

Prove this by showing equality on prime powers.

[30 marks]

4.

i) State, without proof, Möbius Inversion, not forgetting to define all terms.

ii) a) Define Euler's phi function, ϕ.

b) Using Möbius Inversion or otherwise prove that $\phi = \mu * j$, i.e.

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d},$$

for all $n \geq 1$. Here j is the identity function, $j(n) = n$ for all n.

Deduce

1)
$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

for all $n \geq 1$.

2)
$$\sum_{d|n} \phi(d) = n,$$

for all $n \geq 1$.

iii) Prove that

$$\sum_{n \leq x} \frac{\phi(n)}{n} = \frac{1}{\zeta(2)} x + O(\log x).$$

You may assume that $\sum_{n>x} 1/n^2 = O(1/x)$ *and* $\sum_{n \leq x} 1/n = O(\log x)$.

[30 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

ALGEBRAIC TOPOLOGY

28 May 2015

09:45 — 11:45

Answer **ALL** FOUR questions in Section A (40 marks in total). Answer **THREE** of the FOUR questions in Section B (45 marks in total). If more than THREE questions from Section B are attempted then credit will be given for the best THREE answers.

Electronic calculators are permitted, provided they cannot store text.

SECTION A

Answer **ALL** FOUR questions.

A1. (a) Define what is meant by a *geometric simplicial surface.*

[The notions of *triangle* in $\mathbb{R}^n$, and its *vertices* and *edges* may be used without definition.]

(b) Define what is meant by the statement that a simplicial surface is *orientable.*

(c) Define what is meant by the statement that the orientability of a simplicial surface is a *topological invariant.*

[10 marks]

A2. (a) Define what it meant by a *geometric simplicial complex* K and its underlying space $|K|$.

[The notions of *geometric simplex* and *face* of a simplex may be used without definition.]

(b) An abstract simplicial complex has vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and simplices $\{v_1, v_2, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_5\}$ $\{v_4, v_5\}$, $\{v_4, v_6\}$ and $\{v_5, v_6\}$ and their faces. Draw a realization K of this simplicial complex as a geometric simplicial complex in $\mathbb{R}^2$.

(c) Define the *Euler characteristic* of a simplicial complex and calculate the Euler characteristic of the simplicial complex in part (b).

(d) Draw the first barycentric subdivision K' of the geometric simplicial complex K in part (b).

(e) Find the Euler characteristic of K'.

[10 marks]

A3. (a) Define what is meant by the *r-chain group* $C_r(K)$, the *r-cycle group* $Z_r(K)$, and the *r-boundary group* $B_r(K)$ of a simplicial complex K.

(b) Write down, without proof, generators for the groups $Z_1(K)$ and $B_1(K)$ of the simplicial complex K in Question A2(b). Hence find the first homology group $H_1(K)$.

[10 marks]

A4. (a) Let K be the simplicial complex $\bar{\Delta}^8$ consisting of all of the faces of the standard 8-simplex in $\mathbb{R}^9$ (including the 8-simplex itself). Write down the simplicial homology groups of K, justifying for your answer.

(b) Let L be the 3-skeleton of K. Calculate the Euler characteristic of L and hence, or otherwise, find its simplicial homology groups.

[10 marks]

SECTION B

Answer **THREE** of the FOUR questions.

If more than THREE questions are attempted then credit will be given for the best THREE answers.

B5. Let v_i be the ith standard basis vector in $\mathbb{R}^8$, $1 \leqslant i \leqslant 8$. Consider the set K of sixteen triangles with vertices v_i, v_j and v_k where ijk is one of

$$123,\ 126,\ 134,\ 148,\ 156,\ 157,\ 178,\ 237,\ 257,\ 258,\ 268,\ 346,\ 368,\ 378,\ 456,\ 458.$$

(a) Verify that K is a simplicial surface. [For the link condition, you need only check the vertices v_1 and v_8 to illustrate the method.]

(b) Represent the underlying space of $|K|$ as a polygon with edges identified in pairs, and hence represent $|K|$ by a symbol.

(c) Reduce the symbol to canonical form and hence determine the genus and orientability type of the surface $|K|$.

[15 marks]

B6. (a) Define what is meant by a *topological surface* and outline the definition of the *connected sum* $S_1 \# S_2$ of two surfaces S_1 and S_2.

(b) What is meant by a *triangulation* of a path-connected compact surface? Define the *Euler characteristic* $\chi(S)$ of a path-connected compact surface S. [You may assume that all such surfaces have a triangulation.]

(c) State and outline the proof of the relationship between $\chi(S_1)$, $\chi(S_2)$ and $\chi(S_1 \# S_2)$. Hence, calculate the Euler characteristic of the surfaces that arise in the classification theorem for compact path-connected surfaces. [You may assume the Euler characteristic of the 2-sphere, the torus and the projective plane without proof.]

(d) Explain the rôle of the Euler characteristic in proving the classification theorem.

[15 marks]

B7. (a) Describe a geometric simplicial complex K whose underlying space is homeomorphic to the *projective plane* and indicate why it has this property.

(b) Calculate the simplicial homology groups of K.

[15 marks]

B8. (a) Define what is meant by saying that two continuous functions $f_0 \colon X \to Y$ and $f_1 \colon X \to Y$ between topological spaces X and Y are *homotopic*. Prove that homotopy is an equivalence relation on the set of continuous functions from X to Y.

(b) Define what is meant by saying that two topological spaces X and Y are *homotopy equivalent*. Prove that, if X and Y are homotopy equivalent then X is path-connected if and only if Y is path-connected.

[15 marks]

END OF EXAMINATION PAPER

MATH 31082

Two hours

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY

21 May 2015
09:45—11:45

ANSWER THREE OF THE FOUR QUESTIONS
If four questions are answered credit will be given for the best three questions
All questions are worth 20 marks

Electronic calculators may not be used

P.T.O.

1.

(a) Explain what is meant by saying that G is a Riemannian metric on a manifold M.

Let $G = g_{ik}(x)dx^i dx^k$, $i, k = 1, 2, \ldots, n$ be a Riemannian metric on an n-dimensional manifold M.
Show that all diagonal components $g_{11}(x), g_{22}(x), \ldots, g_{nn}(x)$ are positive functions.

Let $G = cdu^2 + dudv + dv^2$ be Riemannian metric on 2-dimensional manifold M, where c is a real constant. Show that $c > \frac{1}{4}$.
(*Hint*: You may consider the length of a vector $\mathbf{X} = \partial_u + t\partial_v$ where t is an arbitrary real number.)

[8 marks]

(b) Explain what is meant by saying that a Riemannian manifold is locally Euclidean.

Show that the surface of the cylinder $x^2 + y^2 = 4$ in the Euclidean space $\mathbf{E}^3$ with the induced Riemannian metric is locally Euclidean.

[4 marks]

(c) Consider a Riemannian manifold M^n with a metric $G = g_{ik}(x)dx^i dx^k$.

Write down the formula for the volume element on M^n (area element for $n = 2$).

Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2} .$$

Calculate the area S_a of the domain $x^2 + y^2 \leq a^2$.

Find the limit S_a when $a \to \infty$.

Show that there is no isometry between the plane with this Riemannian metric and the Euclidean plane $\mathbf{E}^2$.

[8 marks]

P.T.O.

2.

(a) Explain what is meant by an affine connection on a manifold.

Give the definition of the canonical flat connection on the Euclidean space $\mathbf{E}^n$.

Calculate the Christoffel symbols Γ^r_{rr} and $\Gamma^r_{\varphi\varphi}$ of the canonical flat connection in the Euclidean space $\mathbf{E}^2$, where r, φ are polar coordinates ($x = r\cos\varphi, y = r\sin\varphi$).

[6 marks]

(b) Explain what is meant by the induced connection on a surface in Euclidean space.

Consider the surface of a saddle in the Euclidean space $\mathbf{E}^3$:

$$\mathbf{r}(u,v)\colon \quad \begin{cases} x = u \\ y = v \\ z = uv \end{cases} .$$

Calculate the induced connection at the point $u = v = 0$.

[6 marks]

(c) Write down the expression for the Christoffel symbols Γ^i_{km} of the Levi-Civita connection in terms of the Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

Suppose a symmetric connection ∇ is the Levi-Civita connection of a Riemannian manifold with metric $G = du^2 + u^2 dv^2$ in some local coordinates u, v.

Find Γ^u_{vv}.

Show that there exist other local coordinates u', v' such that in these coordinates all the Christoffel symbols of the connection ∇ identically vanish.

[8 marks]

P.T.O.

3.

(a) Define a geodesic on a Riemannian manifold as a parameterised curve.

Write down the differential equations for geodesics in terms of the Christoffel symbols.

What are the geodesics of the surface of a cylinder? Justify your answer.

[7 marks]

(b) Explain what is meant by the Lagrangian of a free particle on a Riemannian manifold.

Explain the relation between the Lagrangian of a free particle and the differential equations for geodesics.

Calculate the Christoffel symbols of the sphere of radius R in the Euclidean space $\mathbf{E}^3$ in spherical coordinates.
(You may use the Lagrangian of a free particle on this sphere $L = \frac{1}{2}\left(R^2\dot{\theta}^2 + R^2 \sin^2\theta\dot{\varphi}^2\right)$.)

Consider an arbitrary geodesic $\mathbf{r} = \mathbf{r}(t)$ on the sphere.
Show that the magnitude $I(t) = \sin^2\theta(t)\dot{\varphi}(t)$ is preserved along the geodesic.

[8 marks]

(c) Consider the Lobachevsky plane as an upper half-plane, $y > 0$ in $\mathbf{R}^2$ equipped with the metric $G = \frac{dx^2+dy^2}{y^2}$.

Consider a semicircle $C\colon x^2 + y^2 = 4x$ (i.e. $(x-2)^2 + y^2 = 4$), $y > 0$ in the Lobachevsky plane.
Consider two points $A = (2, 2)$ and $B = (3, \sqrt{3})$ on this semicircle.
Find the parallel transport of the vector $\mathbf{X} = \partial_x$ from the initial point A to the point B along the semicircle C.
(You may use the fact that the semicircle C is a geodesic in the Lobachevsky plane.)

[5 marks]

P.T.O.

4.

(a) Consider the conic surface in the Euclidean space $\mathbf{E}^3$

$$\mathbf{r}(h,\varphi)\colon \quad \begin{cases} x = 2h\cos\varphi \\ y = 2h\sin\varphi \\ z = h \end{cases} .$$

Let $\mathbf{e}, \mathbf{f}$ be the unit vectors in the directions of the vectors $\mathbf{r}_h = \frac{\partial \mathbf{r}}{\partial h}$ and $\mathbf{r}_\varphi = \frac{\partial \mathbf{r}}{\partial \varphi}$, and $\mathbf{n}$ be a unit normal vector to the cone.
Express these vectors explicitly.

For the obtained orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ calculate the 1-forms a, b and c in the derivation formula

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} .$$

Deduce from these calculations the mean curvature and the Gaussian curvature of the cone.

[8 marks]

(b) Give a definition of the curvature tensor for a manifold equipped with a connection.

Consider the surface of a cylinder in the Euclidean space $\mathbf{E}^3$ with the induced Riemannian metric. Explain why the Riemann curvature tensor of the Levi-Civita connection of this surface vanishes.

[6 marks]

(c) State the theorem about the result of parallel transport along a closed curve on a surface in the Euclidean space $\mathbf{E}^3$.

On the sphere $x^2 + y^2 + z^2 = R^2$ in the Euclidean space $\mathbf{E}^3$ consider a circle C which is the intersection of the sphere with the plane $z = R - h$, $0 < h < R$.
Let $\mathbf{X}$ be an arbitrary vector tangent to the sphere at a point of C.
Find the angle between $\mathbf{X}$ and the result of parallel transport of $\mathbf{X}$ along C.

[6 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

COMMUTATIVE ALGEBRA

14 May 2015

09:45 – 11:45

Answer ALL THREE questions in Section A (40 marks in total). Answer TWO of the THREE questions in Section B (40 marks in total). If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

Electronic calculators are permitted, provided they cannot store text.

SECTION A

Answer **ALL** questions in this section (40 marks in total)

A1. The polynomials $f = X^4 - Y + 1$, $g = Y + Z^2 + 1$, $h = YZ + Z$ generate the ideal I of $\mathbb{Q}[X, Y, Z]$.

(a) Find a Gröbner basis of I with respect to the lexicographic order Lex with $X \succ Y \succ Z$.

(b) Find the reduced Gröbner basis of I.

(c) Is the variety $\mathcal{V}(I) \subset \mathbb{Q}^3$ non-empty? Justify your answer.

[15 marks]

A2.

(a) Give a definition of a noetherian ring.

(b) Is it true that every subring of every noetherian ring is noetherian? Justify your answer briefly.

(c) Give a definition of a euclidean norm and a euclidean ring. Briefly state a reason why a euclidean ring is noetherian.

(d) Write down an example of a ring which is noetherian but not euclidean.

(e) State without proof Hilbert's Basis Theorem for polynomials over a field.

[10 marks]

A3. Let R be a commutative domain and let $a, b \in R$.

(a) What is meant by saying that a is irreducible in R?

(b) What is meant by saying that b is an associate of a?

(c) Prove: if a is irreducible and b is an associate of a, then b is irreducible.

Answer the following questions, giving reasons for your answer.

(d) Is $2X^3 + X^2 + X - 1$ irreducible in $\mathbb{Z}[X]$?

(e) Is $\frac{1}{20}X^5 + \frac{2}{15}X^3 + \frac{1}{5}X - \frac{3}{10}$ irreducible in $\mathbb{Q}[X]$?

(f) Is $X^8 + X + 1$ irreducible in $\mathbb{Z}_2[X]$?

[15 marks]

SECTION B

Answer **TWO** of the three questions in this section (40 marks in total).

If more than TWO questions from this section are attempted, then credit will be given for the best TWO answers.

B4. Let $M(X_1, \dots, X_n)$ denote the set of all monomials in $X_1, \dots, X_n$.

(a) Let S be a subset of $M(X_1, \dots, X_n)$. State Dickson's Lemma about minimal monomials of S.

(b) Prove Dickson's Lemma for $n = 2$.

(c) What is meant by saying that $\preccurlyeq$ is a monomial ordering on $M(X_1, \dots, X_n)$?

(d) Show: if $m, m' \in M(X_1, \dots, X_n)$, m divides m', and $\preccurlyeq$ is a monomial ordering, then $m \preccurlyeq m'$.

(e) Let $I \neq \{0\}$ be a monomial ideal of $\mathbb{Q}[X_1, \dots, X_n]$. Show that I contains a monomial, m, such that m is divisible by exactly 2015 other monomials contained in I.

[20 marks]

B5.

(a) What is meant by a prime in a commutative domain R?

(b) Show that a non-zero prime is irreducible.

(c) What is a unique factorisation domain (UFD)?

(d) Describe without proof all primes in the domain R, and state whether R is a UFD, if

i. $R = \mathbb{R}$, the field of real numbers;

ii. $R = \mathbb{R}[X]$, the ring of polynomials in X with real coefficients.

(e) Is $2 + 15i$ a prime in the UFD $\mathbb{Z}[i]$, the ring of Gaussian integers? Give reasons for your answer.

(f) Let a, b, c, d, e be non-zero elements of a commutative domain R (not necessarily a UFD) such that $ab = cde$ and $ac = bd$. Show that if c is a prime, then e is not a prime.

[20 marks]

B6.

(a) What is meant by saying that an ideal I of a commutative ring R is a maximal ideal?

(b) What is meant by the radical, $\sqrt{I}$, of an ideal I? What is a radical ideal?

(c) Show that a maximal ideal is a radical ideal. You may assume basic properties of the radical without particular comment.

(d) Describe all maximal ideals of the ring $\mathbb{C}[X, Y]$. Prove that the ideals in your list are maximal. (You do not have to prove that there are no other maximal ideals.)

(e) Give an example of an ideal $J \neq \{0\}$ of $\mathbb{C}[X, Y]$ such that J cannot be generated by two polynomials. Justify your example.

(f) For the ideal J from your example in part (e), find $\mathcal{V}(J)$ and $\sqrt{J}$.

[20 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO ALGEBRAIC GEOMETRY

Date: 18 May, 2015

Time: 09:45–11:45

Answer THREE of the FOUR questions.
If more than THREE questions are attempted, then credit will be given for the best THREE answers.

Electronic calculators are permitted, provided they cannot store text.

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1. In this question K denotes an arbitrary field and unless specified otherwise, varieties are defined over K.

(a) (i) Define the *affine algebraic variety* $\mathcal{V}(J) \subseteq K^n$ determined by an ideal $J \triangleleft K[x_1, x_2, \ldots, x_n]$.

(ii) Define the *ideal* $\mathcal{I}(X)$ of a set $X \subseteq K^n$.

(iii) Show that $\sqrt{J} \subseteq \mathcal{I}(\mathcal{V}(J))$ for any ideal $J \triangleleft K[x_1, x_2, \ldots, x_n]$ without any assumption on the field K.

(iv) State without proof the algebraic condition which K has to satisfy to ensure that $\sqrt{J} = \mathcal{I}(\mathcal{V}(J))$ for any integer $n \geq 1$ and any ideal $J \triangleleft K[x_1, x_2, \ldots, x_n]$.

(b) Let $J = \langle (x - y^2 + z^3)^2, yz^2 - z^3 + 2x^2 - y^2 + y \rangle \triangleleft \mathbb{C}[x, y, z]$.
Prove that $\left.\dfrac{\partial h}{\partial z}\right|_{(1,3,2)} = 0$ for every $h \in J$.
Show that $x - y^2 + z^3 \in \mathcal{I}(\mathcal{V}(J))$, but $x - y^2 + z^3 \notin J$.

(c) (i) Define what it means for an affine algebraic variety W to be *irreducible.*

(ii) Prove that if the affine algebraic variety W is irreducible then $\mathcal{I}(W)$ is a prime ideal.

(d) Find the irreducible components of the variety

$$W = \mathcal{V}(\langle 2xyz + x^2 + y^2 - 2xz + yz - 2x, yz(y - 1)\rangle) \subset \mathbb{A}^3(\mathbb{C}).$$

(You should either identify what kind of variety each component is or prove directly that it is irreducible.)

[20 marks]

2. (a) Let K be a field and let $V \subseteq \mathbb{A}^n(K)$ be an affine algebraic variety.

(i) Let $P \in V$ be a point and let $\mathbf{v} \in K^n$ be a non-zero vector. Let $\ell = \{P + t\mathbf{v} \mid t \in K\}$ be the line through P with direction vector $\mathbf{v}$. Explain what is meant by saying that ℓ is *tangent* to V at P and give the definition of the *tangent space* $T_P V$ to V at P.

(ii) Assume now that V is irreducible. Explain how the *dimension* of V and the *singular locus* of V can be defined in terms of the tangent spaces. (No proofs are required.)

(b) This part of the question is over $\mathbb{C}$. Let

$$I = \langle (x-1)^3 - y^2,\, x^2 + y^2 - 2x - z \rangle \lhd \mathbb{C}[x, y, z]$$

and let $W = \mathcal{V}(I) \subset \mathbb{C}^3$. (You may assume without proof that I is a prime ideal, therefore W is irreducible and $I = \mathcal{I}(W)$).

(i) Calculate the dimension of W and find the singular points of W or show that it has none.

(ii) Let t be the co-ordinate on the affine line $\mathbb{A}^1$. Show that

$$\phi(t) = (t^2 + 1, t^3, t^6 + t^4 - 1)$$

is a morphism $\phi : \mathbb{A}^1 \to W$.

(iii) Let $\psi : W \dashrightarrow \mathbb{A}^1$ be the rational map given by $\psi(x, y, z) = y/(x-1)$. Show that ψ is the inverse of ϕ considered as a rational map.

(iv) Show that W is a rational variety, but it is not isomorphic to $\mathbb{A}^1$.

[20 marks]

3. In this question K denotes an arbitrary field.

(a) Let $V \subseteq \mathbb{P}^m(K)$ and $W \subseteq \mathbb{P}^n(K)$ be projective algebraic varieties and assume that V is irreducible. Define the notions of a *rational map* $\Phi : V \dashrightarrow W$ and a *morphism* $\Phi : V \to W$.

(b) In this part of the question we identify $\mathbb{P}^1(K)$ with $K \cup \{\infty\}$ by identifying $(X_0 : X_1)$ with $X_0/X_1 \in K$ if $X_1 \neq 0$ and $(X_0 : 0)$ with ∞ as usual.
Recall that if $z_1, z_2, z_3, z_4 \in K \cup \{\infty\}$ with at least 3 of them pairwise distinct, their cross ratio is defined to be

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \in K \cup \{\infty\}$$

with appropriate rules for arithmetic involving ∞ and for division by 0.

(i) Let $z_1, z_2, z_3, z_4 \in K \cup \{\infty\}$ with at least 3 of them distinct. Show that $(z_1, z_2; z_3, z_4) = (\phi(z_1), \phi(z_2); \phi(z_3), \phi(z_4))$ for any projective transformation $\phi : \mathbb{P}^1 \to \mathbb{P}^1$.

(ii) Let $z_1, z_2, z_3, w_1, w_2, w_3$ be elements of $K \cup \{\infty\}$ such that z_1, z_2, z_3 are pairwise distinct and w_1, w_2, w_3 are also pairwise distinct. Prove that there exists a unique projective transformation $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\phi(z_i) = w_i$ for $i = 1, 2, 3$. (You may use standard properties of projective transformations without proof.)

(c) Find $a, b, c, d \in \mathbb{C}$ such that $\phi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, $\phi(z) = \dfrac{az + b}{cz + d}$ satisfies $\phi(3) = 4$, $\phi(2) = 3$ and $\phi(-1) = 6$.

[20 marks]

4. (a) Let K be a field. Let $E \subset \mathbb{P}^2(K)$ be a non-singular cubic curve and let O be a point of E. (You should not assume that the equation of E is in a special form.)
Describe without proof how one can define the addition operation on the points of E to obtain a commutative group in which O is the zero element.

(b) Let F be the elliptic curve given by the affine equation

$$y^2 = x^3 - x^2 - 3x$$

over $\mathbb{C}$ together with the point $O = (0 : 1 : 0)$ in homogeneous co-ordinates, which is taken to be the zero element of the group law as usual.

(i) Let $P = (-1, 1)$, $Q = (3, 3)$. Calculate $P + Q$ and $2P$.

(ii) Find the points of F which have order 2 in the group structure.

(c) The elliptic curve G is given by the affine equation

$$y^2 - 2xy - 2y = x^3 + 2x^2 + x - 1$$

over $\mathbb{C}$. Find p, $q \in \mathbb{C}$ such that the curve $y^2 = x^3 + px + q$ is isomorphic to G.

[20 marks]

END OF EXAMINATION PAPER

THE UNIVERSITY OF MANCHESTER

MATH 32112 Lie Algebras

Two hours

22 May 2015

14:00–16:00

Electronic calculators may be used, provided that they cannot store text.

Answer the **ONE** question in **SECTION A** (26 marks)
and
TWO of the **THREE** questions in **SECTION B** (27 marks each)

The total number of marks on the paper is 80. A further 20 marks are available from work during the semester making a total of 100.

SECTION A

Answer the **ONE** question.

A 1. Let $\Bbbk$ be a field.

(i) Let L be a vector space over $\Bbbk$ with a bilinear operation $[\,\cdot\,,\,\cdot\,]\colon L \times L \to L$. For $x, y, z \in L$ define $j(x, y, z) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]]$ and $j'(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]$.

When is L called a *Lie algebra*? Prove that if L is a Lie algebra then $[x, y] = -[y, x]$ and $j'(x, y, z) = 0$ for all $x, y, z \in L$.

[3 marks]

From now on assume that L is a Lie algebra over $\Bbbk$.

(ii) When is a subspace I of L called an *ideal* of L? When is a Lie algebra called *simple*? Prove that if I and J are ideals of L then so is $[I, J]$.

[4 marks]

(iii) Define the *lower central series* $L = L^1 \supseteq L^2 \supseteq \cdots \supseteq L^n \supseteq \cdots$ of L and show that all members of that series are ideals of L. When is L called *nilpotent*?

[3 marks]

(iv) Define the *derived series* $L = L^{(0)} \supseteq L^{(1)} \supseteq \cdots \supseteq L^{(n)} \supseteq \cdots$ of L and show that all members of that series are ideals of L. When is L called *solvable*? Prove that any nilpotent Lie algebra is solvable stating clearly all results that you use.

[6 marks]

(v) For any $x \in L$ define the *adjoint endomorphism* $\operatorname{ad} x$ of L and show that the map $\operatorname{ad}\colon L \to \mathfrak{gl}(L)$ is a representation of L. [6 marks]

(vi) Suppose A is an anti-commutative algebra over $\Bbbk$ with a $\Bbbk$-basis $\{u, v, w\}$ and the bracket operation $[\,\cdot\,,\cdot\,]$ such that $[u, v] = v$, $[u, w] = -w$ and $[v, w] = u$. Prove that if $\operatorname{char}(\Bbbk) \neq 2$ then A is a Lie algebra isomorphic to $\mathfrak{sl}(2, \Bbbk)$.

[4 marks]

SECTION B

Answer **TWO** of the **THREE** questions.

B2. Let $\Bbbk$ be an algebraically closed field and let L be a finite dimensional Lie algebra over $\Bbbk$.

(i) Let V be a vector space over $\Bbbk$. What is the *general linear Lie algebra* $\mathfrak{gl}(V)$? When is a linear map $\rho\colon L \to \mathfrak{gl}(V)$ called a *representation* of L? Define the *module structure* on V induced by a representation $\rho\colon L \to \mathfrak{gl}(V)$. When is a representation ρ called *irreducible*?

[4 marks]

(ii) When is an endomorphism A of V called *nilpotent*? Explain why 0 is the only eigenvalue of a nilpotent endomorphism A of V. Prove that if $A \in \mathfrak{gl}(V)$ is nilpotent then so is the endomorphism $\operatorname{ad} A$ of $\mathfrak{gl}(V)$.

[6 marks]

(iii) Explain what is meant by a *flag of subspaces* in V and state *Lie's theorem* for Lie subalgebras of $\mathfrak{gl}(V)$ in the form involving flags of subspaces in V. Show that Lie's theorem fails in positive characteristic by constructing an irreducible p-dimensional representation of the three-dimensional Heisenberg Lie algebra H_1 over a field $\Bbbk$ of characteristic $p > 0$.

[7 marks]

(iv) Define the *radical*, $\operatorname{rad} L$, of the Lie algebra L and explain what is meant by "L is semisimple". Use the canonical homomorphism $\beta\colon L \to L/\operatorname{rad} L$ to show that the factor algebra $L/\operatorname{rad} L$ is semisimple. State clearly all results you use.

[6 marks]

(v) Let L be a Lie algebra over $\Bbbk$ with basis $\{h, u, v, w\}$ such that

$$[h, u] = u,\ \ [h, v] = -v,\ \ [u, v] = w,\ \ [u, w] = [v, w] = 0.$$

Determine the derived subalgebra and the centre of the Lie algebra L.

[4 marks]

B3. Let L be a finite dimensional Lie algebra over an algebraically closed field $\Bbbk$.

(i) When is a symmetric bilinear form $\gamma\colon L \times L \to \Bbbk$ called *L-invariant*? Define the radical $\operatorname{Rad}\gamma$ of γ and show that $\operatorname{Rad}\gamma$ is an ideal of L. Define the *Killing form* κ of L and prove that if L is simple and κ is non-zero, then κ is non-degenerate. (You may assume that the Killing form κ is symmetric and L-invariant.)

[5 marks]

(ii) Suppose that the Lie algebra L is nilpotent. Show that for all $x, y \in L$ the linear operator $(\operatorname{ad} x) \circ (\operatorname{ad} y)$ is nilpotent and use this to prove that the Killing form of L is zero.

[4 marks]

(iii) Let I be an ideal of L. Show that the Killing form κ_I of the Lie algebra I coincides with the restriction of the Killing form of L to $I \times I$.

[5 marks]

(iv) Show that any abelian ideal A of L is contained in $\operatorname{Rad}\kappa$. Use this to prove that if the Killing form κ of L is non-degenerate then L is semisimple.

[5 marks]

(v) Suppose $L = I_1 \oplus \cdots \oplus I_m$ is a direct sum of its simple ideals $I_1, \ldots, I_m$. Use the canonical projections $\pi_k\colon L \to I_k$, where $1 \le k \le m$, to prove that L is semisimple.

[5 marks]

(vi) Let L be a 2-dimensional Lie algebra with basis $\{u, v\}$ such that $[u, v] = v$. Compute the radical of the Killing form of L.

[3 marks]

B4. Let L be a finite dimensional Lie algebra over an algebraically closed field $\mathbb{k}$.

(i) Let V be a finite dimensional vector space over $\mathbb{k}$ and let $x \in \mathfrak{gl}(V)$. Define a *Jordan decomposition* $x = x_s + x_n$ of x and explain why such a decomposition is unique. When is a Lie subalgebra L of $\mathfrak{gl}(V)$ called *separating*?

[6 marks]

(ii) Let V be a finite dimensional vector space over K. When is V called an L-*module*? When is an L-module called *completely reducible*? State *Weyl's theorem* on finite dimensional representations of semisimple Lie algebras.

[5 marks]

(iii) Suppose L is a semisimple Lie algebra and $\mathrm{char}(\mathbb{k}) = 0$. By applying Weyl's theorem to the adjoint representation of L prove that there exist simple ideals $I_1, \ldots, I_m$ of L such that $L = I_1 \oplus \cdots \oplus I_m$.

[4 marks]

(iv) Let A be a finite dimensional algebra over $\mathbb{k}$ (not necessarily associative or Lie). When is an endomorphism D of A called a *derivation* of A? Prove that the set $\mathrm{Der}(A)$ of all derivations of A is a Lie subalgebra of $\mathfrak{gl}(A)$.

[5 marks]

(v) Let $L = \mathfrak{sl}(2, \mathbb{k})$ and suppose $\mathrm{char}(\mathbb{k}) \neq 2$. By computing the determinant of the Gram matrix of the Killing form κ of L with respect to the standard basis $S = \{e, h, f\}$ or otherwise prove that κ is non-degenerate.

[4 marks]

(vi) Let L be as in part (v) and let $x = e + 2f$. Determine the semisimple and nilpotent parts of the Jordan decomposition of x.

[3 marks]

END OF EXAM PAPER

Two hours

UNIVERSITY OF MANCHESTER

GREEN'S FUNCTIONS, INTEGRAL EQUATIONS AND APPLICATIONS

20 May 2015

14:00 - 16:00

Answer ALL six questions (100 marks in total)

Electronic calculators may be used, provided that they cannot store text.

Given equations:

- For functions f and g defined in either two or three dimensions, and a domain D with smooth boundary in either two or three dimensions

$$\int_D \left(f(\mathbf{x})\nabla^2 g(\mathbf{x}) - g(\mathbf{x})\nabla^2 f(\mathbf{x})\right)\, d\mathbf{x} = \int_{\partial D} \left(f(\mathbf{x})\nabla g(\mathbf{x}) - g(\mathbf{x})\nabla f(\mathbf{x})\right)\cdot \mathbf{n}(\mathbf{x})\, ds \tag{1}$$

where the boundary ∂D is oriented by the outward pointing unit normal $\mathbf{n}(\mathbf{x})$ at each point $\mathbf{x}$ in ∂D.

1. For this question, suppose that we have a one dimensional boundary value problem

$$\begin{cases} \mathcal{L}u(x) = p(x)u''(x) + r(x)u'(x) + q(x)u(x) = f(x), \quad x \in (a,b) \\ \mathcal{B} \end{cases} \tag{2}$$

where $\mathcal{B}$ are a pair of homogeneous boundary conditions on $u(x)$. **On this question you do not need to prove that your answers are correct**.

(a) Answer whether the following statement is true or false: There is always a Green's function for the boundary value problem (2).

[3 marks]

(b) Suppose that $G(x, x_0)$ is a Green's function for (2). Write a formula for the solution $u(x)$ of (2) using $G(x, x_0)$.

[3 marks]

(c) As in part (b), suppose that $G(x, x_0)$ is a Green's function for (2). What does it mean for $G(x, x_0)$ to have reciprocity? What property of (2) is equivalent to reciprocity for the Green's function $G(x, x_0)$?

[6 marks]

(d) Suppose that p, r, and q are real-valued functions. What conditions on p, r, and q must be satisfied for the operator $\mathcal{L}$ to be formally self-adjoint?

[3 marks]

2. Find the Green's function for the boundary value problem

$$\begin{cases} u''(x) + u(x) = f(x), \quad x \in (0, \pi/2), \\ u'(0) = 0, \ u'(\pi/2) = 0. \end{cases}$$

[10 marks]

3. This question concerns the boundary value problem

$$\begin{cases} u''(x) - u'(x) - 6u = f(x), \quad x \in (0, L), \\ u(0) = 0, \ u'(L) - 4u(L) = 0. \end{cases} \tag{3}$$

(a) Calculate the adjoint operator and boundary conditions for this boundary value problem.

[6 marks]

(b) For what values of L is there a unique solution to the boundary value problem (3) for any continuous function f?

[6 marks]

(c) For values of L for which there is not a unique solution of (3) for any f, state what condition f must satisfy for there to be a solution of (3). When f satisfies this condition, how many different solutions are there? (**NOTE: You do not have to find a formula for the solutions.**)

[5 marks]

4. Consider forced vibrations of an infinite inhomogeneous string. In appropriate units the displacement $U(x,t)$ at point x along the string and time t is governed by the linear wave equation

$$\frac{\partial^2 U}{\partial x^2}(x,t) = \frac{1}{c(x)^2}\frac{\partial^2 U}{\partial t^2}(x,t) + F(x,t), \quad x,\ t \in \mathbb{R}.$$

Assume that $c(x) = c_0$ is constant for $|x|$ sufficiently large.

(a) Assume that $F(x,t)$ and $U(x,t)$ are time harmonic and so

$$F(x,t) = e^{-i\omega t} f(x), \qquad U(x,t) = e^{-i\omega t} u(x)$$

for some functions $u(x)$ and $f(x)$. What equation is satisfied by $u(x)$? Use the usual notation $k(x) = \omega/c(x)$ for the wave number.

[3 marks]

(b) State the radiation conditions on $u(x)$ which would be applied to ensure there are no waves coming from infinity using the notation $k_0 = \omega/c_0$. Explain why these are the correct conditions.

[6 marks]

(c) Find the Green's function $G(x, x_0)$ which satisfies for every x_0

$$\frac{\mathrm{d}^2 G}{\mathrm{d}x^2}(x,x_0) + k_0^2 G(x,x_0) = \delta(x - x_0),$$

and the radiation conditions from part (b) in the variable x.

[7 marks]

(d) Suppose that the string is homogeneous except for one small weight attached at the position $x = x_0$, and that this is modeled by setting

$$\frac{1}{c(x)^2} = \frac{1}{c_0^2}\big(1 + M\delta(x - x_0)\big)$$

where M is a constant. Also suppose that $f(x) = 0$. Decompose $u(x)$ into the sum of an incident field, and scattered field

$$u(x) = u_{in}(x) + u_{sc}(x)$$

where the incident field is $u_{in}(x) = e^{ik_0 x}$ with $k_0 = \omega/c_0$, and the scattered field satisfies the radiation conditions. Use the Green's function from part (c) to find $u_{sc}(x)$.

[9 marks]

5. This question concerns the Fredholm integral equation

$$u(x) = \lambda \int_0^1 x e^{y^2} u(y) \mathrm{d}y + f(x). \tag{4}$$

(a) For what values of $\lambda \in \mathbb{C}$ is there a unique solution $u(x)$ to (4) for any continuous function f?

[3 marks]

(b) For the values of λ found in part (a), find a formula for the solution $u(x)$ of (4).

[6 marks]

(c) When λ is not one of the values found in part (a) find a condition on f so that there is a solution for (4). Also write a general solution for (4) in this case.

[3 marks]

6. In this problem $\mathcal{B}_1 = \{\mathbf{x} \in \mathbb{R}^3 \ : \ |\mathbf{x}| < 1\}$ will be the unit ball in three dimensions, and $G(\mathbf{x}, \mathbf{x}_0)$ will be the Dirichlet Green's function for the Laplacian ∇^2 on $\mathcal{B}_1$.
(a) Write down the boundary value problem satisfied by $G(\mathbf{x}, \mathbf{x}_0)$.

[5 marks]

(b) Show that the solution $u(\mathbf{x})$ of the Dirichlet problem

$$\begin{cases} \nabla^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{B}_1 \\ u|_{\partial \mathcal{B}_1} = h \end{cases} \tag{5}$$

is given in terms of $G(\mathbf{x}, \mathbf{x}_0)$ by

$$u(\mathbf{x}) = \int_{\partial \mathcal{B}_1} \mathbf{x}_0 \cdot \nabla_{\mathbf{x}_0} G\Big(\mathbf{x}, \mathbf{x}_0\Big) \ h\Big(\mathbf{x}_0\Big) \ \mathrm{d}s(\mathbf{x}_0).$$

You may use the fact that $G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}_0, \mathbf{x})$.

[6 marks]

(c) Use the method of images and the free-space Green's function

$$G_{3\infty}(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} = -\frac{1}{4\pi} \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}.$$

to find $G(\mathbf{x}, \mathbf{x}_0)$. *Hint: To apply the method of images here find for every* $\mathbf{x}_0 \in \mathcal{B}_1$ *an image point* $\widetilde{\mathbf{x}}_0(\mathbf{x}_0)$ *outside of* $\mathcal{B}_1$ *and function* $a(\mathbf{x}_0)$ *such that*

$$G_{3\infty}(\mathbf{x}, \mathbf{x}_0) + a(\mathbf{x}_0) G_{3\infty}(\mathbf{x}, \widetilde{\mathbf{x}}_0) = 0$$

for all $\mathbf{x} \in \partial \mathcal{B}_1$.

[10 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

WAVE MOTION

15 May 2015

09:45 – 11:45

Answer **ALL** six questions.

Electronic calculators are permitted, provided they cannot store text.

1. A uniform elastic string is attached to a fixed point at $x = 0$ and constrained so that the other end moves along the line $x = l$. The boundary condition at $x = l$ is $\partial y/\partial x = 0$ where $y(x, t)$ denotes the transverse displacement from the equilibrium position.

(i) Write down the boundary condition to be satisfied by $y(x, t)$ at $x = 0$ and the appropriate equation of motion.

(ii) Look for separable solutions of the form $y(x, t) = X(x)T(t)$. Hence show that the general solution for a string released from a state of rest has the form

$$y(x,t) = \sum_{n=1,3,..}^{\infty} A_n \sin\left(\frac{n\pi x}{2l}\right) \cos\left(\frac{n\pi ct}{2l}\right),$$

where $\{A_n\}$ are unknown coefficients and c is the wave speed for the elastic string.

[10 marks]

2. Small-amplitude water waves exist on an inviscid, incompressible fluid layer of depth h. The waves (of amplitude A) have a velocity potential

$$\phi(x, z, t) = \frac{gA}{\omega} \frac{\cosh(\hat{K}[z-h])}{\cosh \hat{K}h} \cos(\hat{K}x - \omega t),$$

where $\hat{K} \tanh(\hat{K}h) = \omega^2/g$, with g being the acceleration due to gravity. The coordinates (x, z) have z measured downwards from the mean location of the free surface.

In the motion resulting from the potential ϕ above, write down the equations of motion for a particle with position $x(t), z(t)$. Obtain approximations for $x(t)$ and $z(t)$, and sketch the shape of the particle paths.

[10 marks]

3. You are given that a physical system supports plane waves of frequency ω with a dispersion relation of

$$\omega^2 = \frac{A^2 m^2}{|\underline{K}|^2},$$

where A is a constant, $\underline{K} = k\underline{i} + l\underline{j} + m\underline{k}$ is a wavenumber vector and $\{\underline{i}, \underline{j}, \underline{k}\}$ are the usual Cartesian basis vectors.

(i) Explain why propagating waves in this system must have a period greater than a critical value (explicitly state what the critical value is).

(ii) Show that the group velocity for these waves must be perpendicular to the wavenumber vector. What does this tell us about energy propagation in this system?

[10 marks]

4. Consider small disturbances in an inviscid, compressible fluid of sound speed c that is otherwise uniform and at rest. If the velocity potential, $\phi(r,t)$, is spherically symmetric about the origin O, show that the three-dimensional wave equation reduces to the one-dimensional wave equation for the quantity $r\phi(r,t)$.

State D'Alembert's solution, and hence show that a solution with only outwardly propagating waves exists in the form

$$\phi(r,t) = -\frac{1}{4\pi r} m(t - r/c)$$

for any smooth function m.

For this form of velocity potential, compute the volume flux $Q(t)$ through a sphere of radius R centred at O. Show that as $R \to 0$, $Q(t) \to m(t)$.

You may use:
In a spherical polar coordinate system,

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right),$$

when ϕ is a spherically symmetric function of position.

[10 marks]

5. A wave motion exists in an incompressible, inviscid, fluid (occupying $z > 0$), such that the linearised conditions at the mean position of the free surface are

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}, \quad \text{and} \ -\rho\frac{\partial \phi}{\partial t} = -\rho g \eta + T\frac{\partial^2 \eta}{\partial x^2} \quad \text{on } z = 0\,.$$

Here T is a constant surface tension, ρ is the fluid density, g is the local gravitational acceleration, ϕ is the velocity potential of the wave and η is the free surface deflection caused by the wave.

(i) By combining these conditions show that the free surface condition including the effects of surface tension is

$$\frac{\partial^2 \phi}{\partial t^2} = g\frac{\partial \phi}{\partial z} - \frac{T}{\rho}\frac{\partial^3 \phi}{\partial^2 x \partial z} \quad \text{on } z = 0\,.$$

(ii) Consider a travelling wave in a fluid layer of constant (mean) depth h above a fixed impermeable boundary, with a velocity potential of the form

$$\phi(x, z, t) = \Phi(z)\cos(Kx - \omega t)\,,$$

where K and ω are positive constants.

Determine $\Phi(z)$, and hence obtain the dispersion relation for these waves including surface tension effects.

(iii) By assuming that $\gamma = Kh$ is small, express the phase speed c in the form

$$c = c_0 + c_1\gamma^2 + O(\gamma^4)\,,$$

where $c_{0,1}$ are to be determined.

(iv) State the definition of the group velocity, c_g, and show that

$$c_g = c + \gamma\frac{\partial c}{\partial \gamma}\,.$$

Hence, for $\gamma \ll 1$, interpret the effect of choosing a fluid depth of

$$h = \sqrt{\frac{3T}{g\rho}},$$

on the energy propagation speed in this system.

[20 marks]

6. Consider a compressible fluid, for which the governing equations for small amplitude sound waves are

$$\frac{\partial p_1}{\partial t} + K_1 \frac{\partial u_1}{\partial x} = 0\,, \tag{1}$$

and

$$\frac{\partial u_1}{\partial t} + \frac{1}{\rho_1}\frac{\partial p_1}{\partial x} = 0\,. \tag{2}$$

Here K_1 and ρ_1 are positive constants, p_1, u_1 are the pressure and velocity associated with the sound field, x is a spatial coordinate and t is time.

(i) Show that these equations can be reduced to the one dimensional wave equation for the pressure p_1, with a speed of sound of c_1. Give c_1 in terms of the constants ρ_1 and K_1. Hence show that a general solution is

$$p_1 = f(t - x/c_1) + g(t + x/c_1)\,,$$

and find the corresponding expression for u_1.

(ii) Suppose that the fluid has an interface at $x = 0$, with the sound field governed by (1) and (2) in a region $x < 0$ and an analogous system for $x > 0$ but with $\{p_1, u_1, K_1, \rho_1, c_1\}$ replaced by $\{p_2, u_2, K_2, \rho_2, c_2\}$. Given a solution in $x > 0$ of

$$p_2 = h(t - x/c_2)\,,$$

give the corresponding solution for u_2.

You are given that the boundary conditions to be applied at $x = 0$ are that $p_1 = p_2$ and $u_1 = u_2$. Hence determine g and h in terms of f.

(iii) In the case $K_1 = K_2 = 9$, $\rho_1 = 1$ and $\rho_2 = 9$, for a pressure wave of unit amplitude hitting the interface at $x = 0$, find the amplitudes of the transmitted and reflected waves.

[20 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL BIOLOGY

3 June 2015

14:00–16:00

Answer **ALL THREE** questions in Section A (30 marks in total).
Answer **TWO** of the THREE questions in Section B (50 marks in total).
If all THREE questions from Section B are attempted,
credit will be given for the TWO best answers.

Electronic calculators are permitted, providing they cannot store text.

SECTION A

Answer **ALL** 3 questions

A1. The following differential equation models a fish population with self-limiting growth that is subject to harvesting:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \frac{hN}{A+N}. \tag{A1.1}$$

Here the parameters r, K, h and A are all positive real numbers.

(a) Explain briefly what each of the parameters means.

(b) Show that by a suitable choice of dimensionless variables one can reduced the system (A1.1) to the form

$$\frac{dx}{d\tau} = x\,(1-x) - \frac{\gamma x}{a+x}. \tag{A1.2}$$

Your answer should include expressions for x, τ, γ and a in terms of N, t and the parameters r, K, h and A.

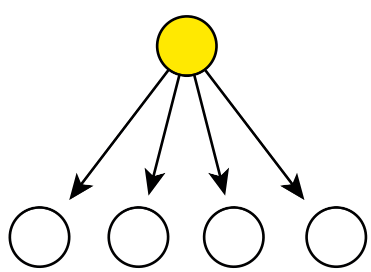
(c) Set $a = 1$ in (A1.2) and determine which values of γ give rise to a sustainable fishery. That is, if $a = 1$, for which values of γ does (A1.2) have a stable, non-zero steady-state population?

[10 marks]

A2. Explain what is meant by the following terms:

- a *motif* in a genetic regulatory network;
- a *coherent 3-node feed-forward loop.* Your answer should include an example of such a motif;
- a *graphlet* in a genetic regulatory network.

In a randomly-assembled regulatory network with $N = 8$ genes and $E = 9$ interactions, what is the probability of finding one or more copies of the graphlet illustrated below?



You may give either a numerical answer or an expression in terms of binomial coefficients, but you should explain your reasoning thoroughly.

[10 marks]

A3. During the development of an embryo, certain cell-fate decisions are controlled by a morphogen M that diffuses through the tissue and is degraded, so that its concentration $M(x,t)$ evolves according to the PDE

$$\frac{\partial M}{\partial t} = D\frac{\partial^2 M}{\partial x^2} - \alpha M^3, \tag{A3.1}$$

where D and α are positive real numbers. The morphogen is produced at one end of the embryo and is degraded so strongly that it is sufficient to study (A3.1) on the domain $x \geq 0$, $t \geq 0$ with the boundary conditions

$$M(0,t) = M_0 \qquad \text{and} \qquad \lim_{x\to\infty} M(x,t) = 0 \qquad \forall t \geq 0$$

(a) Show that (A3.1) has a steady-state concentration profile of the form

$$M(x) = \frac{\gamma}{(x+\epsilon)^\nu}$$

and find expressions for γ, ϵ and ν in terms of M_0, D and α.

Cells at position x develop into one of three types—A, B or C—according to the steady-state concentration of the morphogen $M(x)$:

$$\text{cell type at } x = \begin{cases} A & \text{if } M(x) \geq \theta_1 \\ B & \text{if } \theta_1 > M(x) \geq \theta_2 \\ C & \text{if } \theta_2 > M(x) \end{cases}$$

where $\theta_1 > \theta_2 > 0$.

(b) Assume that $M_0 > \theta_1$, so that cells near $x = 0$ develop into type A. Find an expression for the position $x_1^\star$ of the boundary between cells of types A and B.

(c) Define $x_2^\star$ to be the position of the boundary between cells of types B and C. Show that $\Delta = x_2^\star - x_1^\star$ is independent of M_0 and discuss what this implies about the robustness of developmental patterning.

[10 marks]

SECTION B

Answer **2** of the 3 questions

B4. Consider the following system of differential equations, which models the interaction between two species,

$$\frac{dx_1}{d\tau} = x_1\left[1 - \frac{x_1}{1+\beta_2 x_2}\right] \qquad \text{and} \qquad \frac{dx_2}{d\tau} = x_2\left[1 - \frac{x_2}{1+\beta_1 x_1}\right], \tag{B4.1}$$

where the parameters β_1 and β_2 are positive real numbers.

(a) Explain briefly the nature of the interaction and describe the behaviour of the population of each species when left to develop in isolation.

(b) Sketch the null clines of (B4.1) in the cases

(i) $\beta_1\beta_2 < 1$

(ii) $\beta_1\beta_2 = 1$

(iii) $\beta_1\beta_2 > 1$

(c) Find the community matrix A for the system (B4.1) and show that, at an equilibrium point $(x_1^\star, x_2^\star)$ with $x_1^\star, x_2^\star > 0$, it reduces to the form

$$A = \begin{bmatrix} -1 & \beta_2 \\ \beta_1 & -1 \end{bmatrix}.$$

(d) In the case $\beta_1\beta_2 < 1$, find all equilibrium points of (B4.1) and classify their stability.

(e) In the cases $\beta_1\beta_2 \geq 1$, discuss the long-term behaviour of this system: do the two populations converge to some equilibrium values? Justify your answer.

[25 marks]

B5. This problem concerns a non-fatal infectious disease such as the common cold, which confers immunity on survivors. For such a disease we can divide the whole population among the following three categories:

Susceptible people who have never had the disease

Infectious people who currently have the disease and so can infect susceptibles

Recovered people who have been infected, but have recovered and so are now immune.

If we use the symbols S, I and R to denote, respectively, susceptible, infected and recovered persons, then a standard model for the spread of the disease is equivalent to the following system of chemical reactions:

$$\begin{aligned} S + I &\xrightarrow{\beta} I + I \\ I &\xrightarrow{\gamma} R \end{aligned} \tag{B5.1}$$

Note that there are no reactions correponding to birth or death: on the timescale of interest here we can ignore these processes.

(a) Write down the stoichiometric matrix N for the system of reactions (B5.1).

(b) By performing Gaussian elimination on N, or by any other means, find a conserved quantity for the system (B5.1).

(c) Find the rank r of the stoichiometric matrix N and decompose N as a product $N = LN_R$ where N_R, the reduced stoichiometry matrix, has full rank and L is of the form

$$L = \begin{bmatrix} I_r \\ L_0 \end{bmatrix},$$

where I_r is the $r \times r$ identity matrix. Your answer should include expressions for L, L_0 and N_R.

If we restrict attention to a single household we can model the spread of infection with the Gillespie algorithm.

(d) Consider a household that initially has one infected person, one susceptible person and no recovered persons. List the set of states the system can subsequently occupy.

(e) Draw a directed graph whose nodes are the states listed in part (d) and whose edges represent transitions allowed by the reactions (B5.1).

(f) Using the notation $\pi_{jk}(t)$ to indicate the probability that the household is, at time t, in a state with j infected persons and k who have recovered, write down a system of linear differential equations for the evolution of the $\pi_{jk}(t)$ associated with the states found in part (d).

(g) Using the same notation as in part (f), show that $\lim_{t\to\infty} \pi_{01}(t) + \pi_{02}(t) = 1$.

[25 marks]

B6. The main example in Turing's famous paper on morphogenesis involves a ring of N cells, each containing two morphogens, X and Y. In the j-th cell these have concentrations X_j and Y_j which evolve according to the equations

$$\begin{aligned} \frac{dX_j}{dt} &= f(X_j, Y_j) + \mu\,(X_{j+1} - 2X_j + X_{j-1}) \\ \frac{dY_j}{dt} &= g(X_j, Y_j) + \nu\,(Y_{j+1} - 2Y_j + Y_{j-1})\,. \end{aligned} \tag{B6.1}$$

Here the index j is defined periodically so, for example, $X_0 \equiv X_N$ and $X_{N+1} \equiv X_1$.

(a) Assume there are numbers $X^\star$ and $Y^\star$ such that

$$f(X^\star, Y^\star) = g(X^\star, Y^\star) = 0.$$

Show that there is an equilibrium for (B6.1) with $X_j = X^\star$ and $Y_j = Y^\star\ \forall j$ and explain why one might refer to this as a *spatially uniform equilibrium.*

Now restrict attention to the case $N = 3$ and consider perturbations $x_j = X_j - X^\star$, $y_j = Y_j - Y^\star$.

(b) By linearizing (B6.1), obtain equations of the form

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \left(\begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} + \begin{bmatrix} \mu\mathcal{D}_3 & 0 \\ 0 & \nu\mathcal{D}_3 \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \tag{B6.2}$$

where I is the 3×3 identity matrix and $\mathcal{D}_3$ and $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^3$ are

$$\mathcal{D}_3 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{x} = [x_1,\, x_2,\, x_3]^T, \qquad \boldsymbol{y} = [y_1,\, y_2,\, y_3]^T.$$

Explain how to evaluate the parameters a, b, c and d appearing in (B6.2).

(c) Show that $\boldsymbol{u} = [2, -1, -1]^T$ is a right eigenvector of $\mathcal{D}_3$ and find its eigenvalue λ_u. Also, find a second eigenvector.

(d) Show that there are solutions to (B6.2) of the form

$$\boldsymbol{x}(t) = \theta(t)\boldsymbol{u} \qquad \text{and} \qquad \boldsymbol{y}(t) = \eta(t)\boldsymbol{u}, \tag{B6.3}$$

where the functions $\theta(t)$ and $\eta(t)$ obey differential equations of the form

$$\begin{aligned} \frac{d\theta}{dt} &= (a + \mu\lambda_u)\theta + b\eta \\ \frac{d\eta}{dt} &= c\theta + (d + \nu\lambda_u)\eta. \end{aligned}$$

(e) Find numerical values for a b, c and d that guarantee that the spatially uniform solution is unstable against perturbations of the form (B6.3).

[25 marks]

END OF EXAMINATION PAPER

Two Hours

THE UNIVERSITY OF MANCHESTER

NUMERICAL ANALYSIS 2

14th May 2015

14:00-16:00

Answer **ALL** five questions in **SECTION A** (40 marks in total)
and
Answer **TWO** of the three questions in **SECTION B** (20 marks each)
If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** five questions

A1. Let $f \in C[a,b]$ and let $w(x)$ be a continuous non-negative weight function on $[a,b]$.

(a) The coefficients of the best weighted L_2 approximation of the form $g(x) = \sum_{i=0}^{n} c_i \phi_i(x)$ to $f(x)$ can be found by solving the so-called normal equations $A\underline{c} = \underline{f}$. Define the entries of the matrix A and the vector $\underline{f}$.

(b) Prove that if the functions ϕ_i are linearly independent then the matrix A is non-singular and hence the normal equations are uniquely solvable.

(c) State a good choice for the functions ϕ_i and provide a brief explanation to justify your answer.

[10 marks]

A2. Calculate the [2/1] Padé approximant $r_{21}(x)$ of $f(x) = \exp(2x)$ and give two reasons why in general Padé approximants offer more flexibility than single polynomials for approximating functions.

[8 marks]

A3. Suppose that we want to approximate the integral $I(f) = \int_0^1 f(x)dx$ using the three-point closed Newton–Cotes rule

$$\frac{1}{6}\left(f(0) + 4f(1/2) + f(1)\right)$$

and the three-point open Newton–Cotes rule

$$\frac{1}{3}\left(2f(1/4) - 4f(1/2) + 2f(3/4)\right).$$

Suppose that errors are made in evaluating f at the chosen quadrature points x_i. Let ϵ_i be the error in $f(x_i)$ and suppose $|\epsilon_i| \leq 10^{-3}$. Analyse the effect of this data error for both quadrature schemes. Why are negative weights not desirable?

[6 marks]

A4. Let $f(x, y)$ be continuous for $x \in [0, 1]$ and for all $y \in \mathbb{R}$. Define the "Lipschitz condition" on f that guarantees existence and uniqueness of a solution to the following initial value problem:

$$y'(x) = f(x, y), \qquad 0 \le x \le 1, \quad y(0) = y_0.$$

Determine which (if any) of the following two functions satisfy the condition.

(1) $f(x, y) = 3x + 2y^2$,

(2) $f(x, y) = e^{-x}\sin(y)$.

[7 marks]

A5. The ℓ-step Adams linear multistep methods for the initial value problem $y' = f(x, y)$, $x \ge x_0$, $y(x_0) = y_0$ have the form

$$y_{n+1} = \sum_{i=1}^{\ell} a_i y_{n+1-i} + h \sum_{i=0}^{\ell} b_i f_{n+1-i},$$

where a_i, b_i are given constants, h is the step size, and $f_n = f(x_n, y_n)$.

(a) Explain how we can derive the ℓ-step Adams–Bashforth method and the ℓ-step Adams–Moulton method by replacing the integrand in

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))\, dx$$

with a suitable polynomial. Briefly comment on the accuracy of the two methods and the computational cost of implementing them.

(b) Briefly describe how to combine the ℓ-step Adams–Bashforth method and the ℓ-step Adams–Moulton method to form a predictor-corrector method. Why is it beneficial to combine them as a predictor-corrector pair?

[9 marks]

SECTION B

Answer **TWO** of the three questions

B6. Let $T_n(x) = \cos(n \cos^{-1}(x))$ denote the Chebyshev polynomial of degree n.

(a) State the leading term (the term with the highest power of x) of $T_{n+1}(x)$ and list all the values of x in $[-1, 1]$ such that $T_{n+1}(x) = \pm 1$.

[3 marks]

(b) The Chebyshev equioscillation theorem states that the polynomial $p_n(x)$ of degree $\leq n$ is a best L_∞ approximation to $f \in C[a, b]$ from the set of polynomials of degree $\leq n$ if and only if there exists an alternant. Explain fully what is meant by an "alternant".

[3 marks]

(c) Write down the best L_∞ approximation $p_0(x) = c$ to $f \in C[a, b]$ from the set of constants (i.e., the set of polynomials of degree zero). Justify your answer using the result in part (b).

[4 marks]

(d) Using any of your previous answers, prove that $q_n(x) = x^{n+1} - 2^{-n} T_{n+1}(x)$ is a best L_∞ approximation to $f(x) = x^{n+1}$ on $[-1, 1]$ from the set of polynomials of degree $\leq n$.

[4 marks]

(e) Assume that $f \in C[-1, 1]$ has $n + 1$ continuous derivatives on $[-1, 1]$ and let $p_n(x)$ be the polynomial of degree at most n that interpolates f at the $n + 1$ distinct points $\{x_i\}_{i=0}^n$ in $[-1, 1]$. Recall that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i),$$

for some (unknown) $\xi_x \in [-1, 1]$, where $f^{(n+1)}$ is the $(n + 1)$st derivative of f. Explain why Chebyshev points are a good choice for $\{x_i\}_{i=0}^n$. What can go wrong if we use uniformly spaced points as $n \to \infty$?

[6 marks]

B7. Consider the integral

$$I(f) = \int_a^b w(x) f(x)\, dx,$$

where $w(x)$ is a continuous and non-negative weight function on $[a, b]$ and let $\{\phi_i(x)\}$ be the system of polynomials that are orthogonal with respect to $w(x)$.

(a) Using the idea of interpolatory quadrature, derive an expression for the weights w_i in the n-point Gauss rule

$$G_n(f) = \sum_{i=1}^{n} w_i f(x_i),$$

that ensures the rule is exact for all polynomials of degree $\leq n - 1$.

[4 marks]

(b) If the weights w_i are chosen as in part (a) and the nodes $x_1, \ldots, x_n$ are chosen as the zeros of the polynomial $\phi_n(x)$ of degree n, prove that the Gauss rule $G_n(f)$ is exact for polynomials of degree $\leq 2n - 1$.

[8 marks]

(c) Derive the two point Gauss–Legendre rule

$$\int_{-1}^{1} f(x)\, dx \approx w_1 f(x_1) + w_2 f(x_2),$$

using the method of undetermined coefficients. That is, find the appropriate nodes x_1, x_2 and weights w_1, w_2.

HINT: You may find it useful to define a quadratic polynomial $\Phi(x) = x^2 + ax + b$ whose roots coincide with x_1 and x_2, and find a and b.

[8 marks]

B8. Consider the initial value problem $y'(x) = f(x, y(x))$, $x \geq x_0$, $y(x_0) = y_0$. Recall that a two-stage Runge–Kutta scheme has the general form

$$\begin{aligned} k_1 &= f\left(x_n, y_n\right) \\ k_2 &= f\left(x_n + c_2 h, y_n + h a_{21} k_1\right) \\ y_{n+1} &= y_n + h\left(b_1 k_1 + b_2 k_2\right) \end{aligned}$$

where b_1, b_2, c_2 and a_{21} are constants, h denotes the step size and y_n is the approximation to $y(x_n)$ at the grid point x_n.

(a) Define the "truncation error" (denoted $\tau(h)$) and the "order" of a numerical method for the initial value problem.

[3 marks]

(b) Show that for two-stage Runge–Kutta schemes, the truncation error satisfies

$$\tau(h) = y(x_{n+1}) - y(x_n) - hb_1 y'(x_n) - hb_2 f(x_n + c_2 h, y(x_n) + ha_{21} y'(x_n)).$$

[3 marks]

(c) By examining the terms in h^0, h^1 and h^2 in the truncation error $\tau(h)$ show that a two-stage Runge–Kutta method has order two if the following conditions are satisfied:

$$b_1 + b_2 = 1, \qquad b_2 c_2 = \frac{1}{2}, \qquad b_2 a_{21} = \frac{1}{2}.$$

HINT: Recall that if $y'(x) = f(x, y(x))$ then

$$y''(x) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f.$$

[10 marks]

(d) By examining the test problem $y'(x) = \lambda y(x)$ where $\lambda \in \mathbb{C}$, derive a quadratic polynomial $p(\lambda h)$ such that the condition $|p(\lambda h)| < 1$ provides the region of absolute stability for a two-stage Runge–Kutta method of order two.

[4 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

MARKOV PROCESSES

21 May 2015

14:00-16:00

Answer 3 of the 4 questions. If more than 3 questions are attempted, credit will be given for the best 3 answers. Each question is worth 20 marks.

Electronic calculators may be used, provided that they cannot store text.

Answer 3 questions

1.
a) Urns 1 and 2, between them, contain four balls. At each step, one of the four balls is chosen at random and moved from the urn which it occupies, into the other urn. Consider the Markov chain $\{X_n, n \geq 0\}$ defined as the number of balls in urn 1 immediately following the nth step.

(i) Draw the transition diagram for the Markov chain, indicating the paths of positive probability.

[2 marks]

(ii) Derive the transition probabilities and write down the transition probability matrix.

[3 marks]

(iii) Determine the stationary distribution of the Markov chain.

[3 marks]

(iv) Determine the period of each state of the Markov Chain.

[2 marks]

(v) Given the Markov chain starts in state zero, determine the expected number of steps taken until state two is first visited.

[3 marks]

b) Let $\{X_n, n \geq 0\}$ be a Markov chain with state space S. Prove each of the following identities:

$$\text{(i) For } n \geq 1,\ p_{ij}(n) = \sum_{k=1}^{n} f_{ij}(k) p_{jj}(n-k);$$

[4 marks]

$$\text{(ii) For } n, m \geq 0,\ p_{ij}(n+m) = \sum_{k \in S} p_{ik}(m) p_{kj}(n).$$

[3 marks]

where

$p_{ij}(n) = P(X_n = j \mid X_0 = i)$ for $n \geq 1$, $p_{ii}(0) = 1$ and $p_{ij}(0) = 0$ for $i \neq j$,
$f_{ij}(n) = P(X_k \neq j, 1 \leq k < n, X_n = j \mid X_0 = i)$.

2. a) $\{X_n, n \geq 0\}$ denotes a Markov Chain whose state space is the set of non-negative integers. The transition probabilities are given by

$$\begin{array}{rl} P_{0,0} = 1-\alpha & P_{0,1} = \alpha,\ 0 < \alpha < 1 \\ P_{i,i+1} = \alpha & \qquad i \geq 1 \\ P_{i,0} = 1-\alpha & \qquad i \geq 1. \end{array}$$

i) Carefully stating any results to which you appeal, show that state zero is recurrent, and hence all the states are recurrent.

[4 marks]

ii) Define the term "positive recurrent". Show that zero is a positive recurrent state.

[4 marks]

iii) Show that the stationary distribution $(\pi_n)_{n\geq 0}$ satisfies the system of equations

$$\begin{array}{rl} \pi_0 = (1-\alpha)(\pi_0 + \pi_1 + \pi_2 + ...) & \\ \alpha\pi_i = \pi_{i+1} & \quad i \geq 0. \end{array}$$

Hence calculate the stationary distribution.

[4 marks]

b) The Markov Chain in a) is now modified so that

$$\begin{array}{rl} P_{0,0} = 1/2 & P_{0,1} = 1/2 \\ P_{i,i+1} = \frac{i+1}{i+2} & \quad i \geq 1 \\ P_{i,0} = \frac{1}{i+2} & \quad i \geq 1. \end{array}$$

Carefully stating any results to which you appeal, determine whether or not a stationary distribution exists.

[4 marks]

c) Prove that in an irreducible recurrent Markov chain, either all states are positive recurrent, or all are null recurrent.

[4 marks]

3. At time 0 a population comprises M individuals. Lifetimes of individuals are independent and each exponentially distributed with parameter μ. Let $\{X_t, t \geq 0\}$ denote the number of individuals alive in the population at time t.

i) Show that the forward equations for the process $\{X_t, t \geq 0\}$, include

$$p'_{M,i}(t) = \mu(i+1)p_{M,i+1}(t) - \mu i p_{M,i}(t) \qquad 0 \leq i < M;$$
$$p'_{M,M}(t) = -\mu M p_{M,M}(t).$$

[5 marks]

Hence, or otherwise, show that, for $t \geq 0$,

ii) $p_{M,M}(t) = \exp(-\mu M t)$;

iii) $p_{M,M-1}(t) = M\{\exp(-\mu t)\}^{M-1}\{1 - \exp(-\mu t)\}$.

[6 marks]

iv) By considering the probability of any given individual from the initial M living beyond time t, or otherwise, derive an expression for $p_{M,i}(t)$ for each $0 \leq i \leq M$. Hence write down the expected number of individuals alive at time t.

[6 marks]

v) Find the expected time until the population becomes extinct.

[3 marks]

4. Consider the following model. A factory contains $m \geq 1$ machines and employs r mechanics, where $1 \leq r \leq m$. If, at some time, a machine is found to be working, the time until it next breaks down is exponentially distributed with parameter λ. If, at some time, a machine is found to be undergoing repair by one of the mechanics, the time until repair completion is exponentially distributed with parameter μ. No mechanic is unoccupied while there are broken machines unattended. Assume all repair time and time to breakdown random variables are independent. Let $\{X(t) : t \geq 0\}$ denote the number of machines out of service at time t.

(i) Stating carefully any results to which you appeal, write down the transition rates.

(ii) Show that the limiting distribution $(P_n)_{n \geq o}$ of the number of machines out of service is given by

$$
\begin{aligned}
P_n &= \binom{m}{n}\left(\frac{\lambda}{\mu}\right)^n P_0 && 0 \leq n \leq r \\
P_n &= \frac{m!}{(m-n)!r!r^{n-r}}\left(\frac{\lambda}{\mu}\right)^n P_0 && r < n \leq m.
\end{aligned}
$$

[10 marks]

(b)

i) A garage has the capacity to service 5 cars at any one time. Cars arrive for service in a poisson process at rate 2 per hour. The times taken to service cars are independent and each exponentially distributed, with mean 27 minutes. Arriving cars which find the garage full, leave and do not return. Show that the proportion of cars which leave without being serviced is at least

$$0.9^5 \frac{e^{-0.9}}{5!}.$$

ii) Reducing the capacity to 2 would save the garage *£20* per hour in running costs. Given that each car represents an income of *£40*, determine whether this option is cost-effective.

[10 marks]

END OF EXAMINATION PAPER

Statistical Tables to be provided

Two hours

THE UNIVERSITY OF MANCHESTER

Time Series Analysis

19 May 2015
09:45 – 11:45

Electronic calculators are permitted, provided they cannot store text.

Answer **ALL** Four questions in Section A (32 marks in all)

and

Two of the Three questions in Section B (24 marks each).

If more than Two questions from Section B are attempted, then credit will be given for the best Two answers.

SECTION A
Answer ALL four questions

A1. Suppose that a time series $\{X_t\}$ follows the model

$$X_t = a + bt + ct^2 + \varepsilon_t,$$

where a, b and c are constants, $c \neq 0$ and $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$.

a) Determine whether $\{X_t\}$ is stationary or not.

b) Show that the time series $\{Z_t\}$ obtained by differencing $\{X_t\}$ twice, is a moving average process of order 2 (MA(2)).

c) Determine if the MA(2) model for $\{Z_t\}$ obtained above is invertible.

[8 marks]

A2. A time series, $\{X_t\}$, follows a seasonal ARIMA (SARIMA) model of order $(1, 0, 0) \times (0, 1, 1)_{12}$.

a) Write out the model in operator form (i.e. using the $\mathbf{B}$ operator).

b) Write down the model in difference equation form (i.e. in terms of current and past values of X_t and ε_t).

[6 marks]

A3. Consider the following ARIMA(1,1,0) model for a time series $\{X_t\}$:

$$(1 - \mathbf{B})(1 - 0.2\mathbf{B})X_t = \varepsilon_t,$$

where $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$ with $\sigma^2 = 4$.

When best linear preditors are used, calculate the variances of the prediction errors for horizons from one to three.

[8 marks]

A4. Let $\{X_t\}$ be a stationary time series with mean μ and autocovariances γ_k, $k = 0, \pm 1, \pm 2, \ldots$. Let $Y_t = (1 - \mathbf{B})X_t$.

a) Show that $\mathrm{Cov}(Y_t, Y_{t-k}) = 2\gamma_k - (\gamma_{k-1} + \gamma_{k+1})$.

b) Show that $\{Y_t\}$ is stationary.

c) Suppose further that $\mu = 0$ and $\{X_t\}$ is an ARMA process with representation $\phi(\mathbf{B})X_t = \theta(\mathbf{B})\varepsilon_t$. Find a model for $\{Y_t\}$.

[10 marks]

SECTION B
Answer 2 of the 3 questions

B5.

a) Let $\{X_t, t = \ldots, -2, -1, 0, 1, 2, \ldots\}$, be a stationary process with mean $\mathrm{E}\, X_t = \mu \neq 0$.
   i) Give the definition of an autoregressive model of order $p > 0$ (AR(p) model) for $\{X_t\}$.
   ii) Write down the characteristic polynomial of the AR(p) model and state any restrictions imposed on it.
   iii) What do we mean when we say that an AR(p) model is causal?
   iv) What can be said about the partial autocorrelations of an AR(p) process for lags greater than p?
   v) Prove your claim in iv) *(You may use without proof any properties of linear predictors studied in this course.)*
   vi) Explain how result iv) may be used in model building.

b) Consider the following MA(1) process:

$$X_t = \varepsilon_t + \theta\varepsilon_{t-1},$$

where $\{\varepsilon_t\}$ is a white noise with $\mathrm{E}\,\varepsilon_t^2 = \sigma^2$. Denote by β_k the partial autocorrelation of $\{X_t\}$ for lag k.

Find β_1 and β_2. Give your answers in terms of the parameters of the MA(1) model.

[24 marks]

B6. Let $x_1, \ldots, x_T$ be an observed trajectory of a time series, $\{X_t\}$, driven by the following ARIMA$(0, 2, 2)$ model:

$$(1 - \mathbf{B})^2 X_t = (1 - 0.81\mathbf{B} + 0.38\mathbf{B}^2)\varepsilon_t,$$

where $\{\varepsilon_t\}$ is white noise with variance 1.

a) Write down the model in difference equation form.

b) Identify $\{X_t\}$ as an $I(d)$ process. Explain your answer.

c) Give a recursive equation for the computation of forecasts $\hat{X}_{T+k|T,\ldots,1}$ for $k \geq 2$, from forecasts $\hat{X}_{T+k-1|T,\ldots,1}, \hat{X}_{T+k-2|T,\ldots,1}, \ldots$, for smaller horizons. Justify your equation. By solving it, obtain a general formula for $\hat{X}_{T+k|T,\ldots,1}$ in terms of $\hat{X}_{T+1|T,\ldots,1}$ and $\hat{X}_{T+2|T,\ldots,1}$. (While working on this, you may wish to denote $\hat{X}_{T+1|T,\ldots,1}$ and $\hat{X}_{T+2|T,\ldots,1}$ by single letters.)

d) What is the form of the curve on which all forecasts lie for $k \geq 2$?

e) Suppose now that $T = 95$, $x_{91} = 15.1$, $x_{92} = 15.8$, $x_{93} = 15.9$, $x_{94} = 15.2$, $x_{95} = 15.9$. You are also given that $\varepsilon_{94} = -1.286$ and $\varepsilon_{95} = 0.586$.
Calculate numerically the forecasts $\hat{X}_{95+k|95,\ldots,1}$, for $k = 1, 2, 3$ and the variances of the corresponding prediction errors.

[24 marks]

B7.

The exchange rates for British pound sterling to New Zealand dollar for the period from January 1991 to September 2000 are given in the table below. The data are mean values taken over quarterly periods of three months, with the first quarter being January to March and the last quarter being October to December.

	Qtr1	Qtr2	Qtr3	Qtr4
1991	2.9243	2.9422	3.1719	3.2542
1992	3.3479	3.5066	3.0027	2.8440
1993	2.8378	2.7301	2.7008	2.6138
1994	2.5874	2.5787	2.5470	2.4701
1995	2.3895	2.3750	2.3859	2.2766
1996	2.2351	2.2450	2.3208	2.3390
1997	2.3687	2.5120	2.6917	2.8435
1998	3.0922	3.2528	3.1852	3.0340
1999	2.9593	3.0498	3.1869	3.2286
2000	3.1925	3.3522	3.5310	

a) Figures 7.1–7.2 show the original and differenced time series, as well as the corresponding sample autocorrelation functions. Explain what features of the time series can be observed in these plots and identify one or two models to be fitted.

b) Several ARIMA models were fitted to this time series with the **R** function `arima`. Some results and diagnostics are provided in Figures 7.3–7.6.
   i) Study the provided numerical and graphical information. Comment on the quality of the fit of the ARIMA(0,2,2) model. Your commentary should not include any comparisons with alternative models at this point.
   ii) From the models for which output is provided in Figure 7.3 to Figure 7.6, which would be your preferred choice for predicting subsequent exchange rates? Justify your choice.
   iii) Explain the deficiences, if any, in the model chosen by you, and indicate how you would go about trying to eliminate them.

c) Assuming the ARIMA(0,2,0) model to be the correct one, give a point prediction and a 95% prediction interval of the exchange rate for the last quarter of year 2000. (Note that the output from fitting this model can be found in Figure 7.4.)

d) The following model was also fitted to the exchange rate data.

$$X_t = X_{t-1} + b_{t-1} + \varepsilon_t \qquad (1)$$
$$b_{t-1} = 0.167(X_{t-1} - X_{t-2}) + 0.833 b_{t-2}, \qquad (2)$$

where $\{b_t\}$ is an unobserved time series and $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$.
   i) Write down Equations (1)–(2) in operator form.
   ii) Show that this model implies that

$$(1 - \mathbf{B})^2 X_t = (1 - 0.833\mathbf{B})\varepsilon_t.$$

(Hint: use the operator form of the equations to eliminate b_{t-1}.)

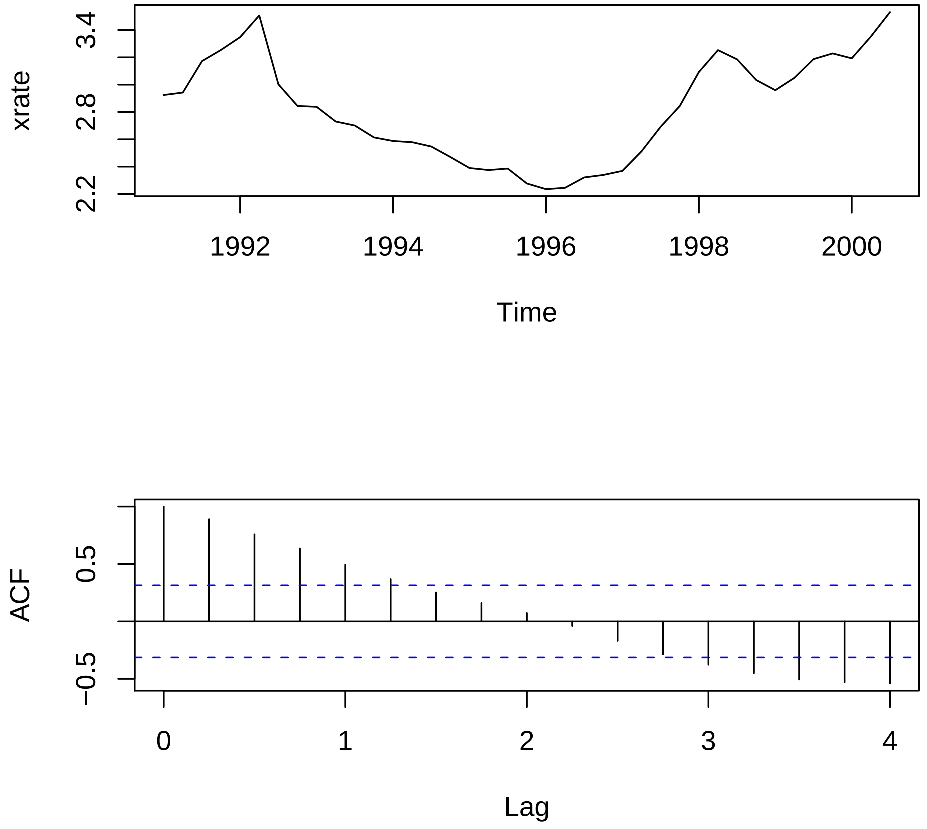


Figure 7.1: exchange rate data: the time series and the sample autocorrelation function.

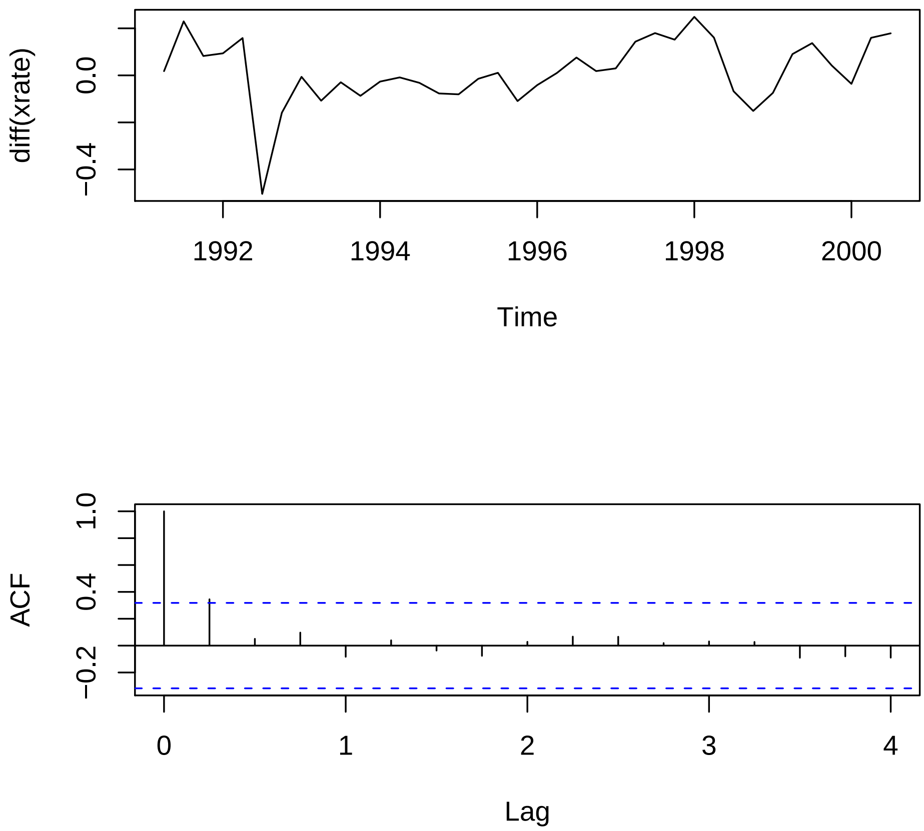


Figure 7.2: Differenced exchange rate data: the time series and the sample autocorrelation function.

```
Call:
arima(x = tsuknz, order = c(0, 1, 1))

Coefficients:
         ma1
      0.4771
s.e.  0.1973

sigma^2 estimated as 0.01587:  log likelihood = 24.67,  aic = -45.35
```

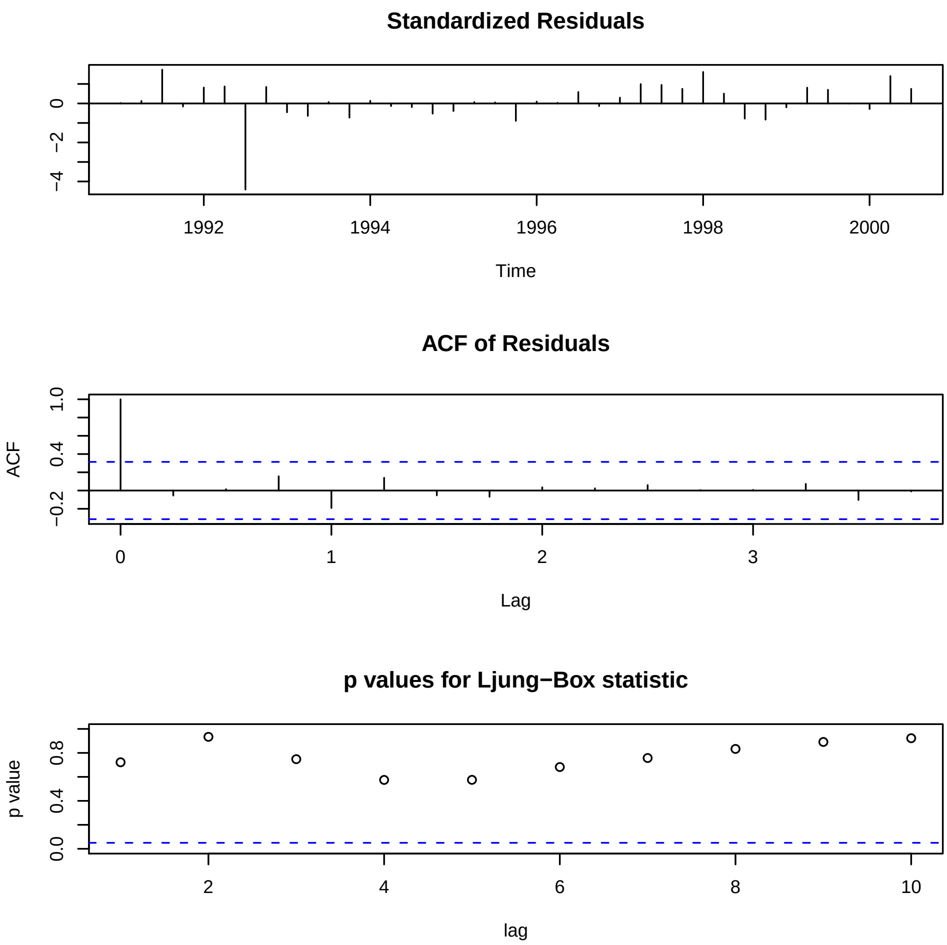


Figure 7.3: Model 1.

```
Call:
arima(x = tsuknz, order = c(0, 2, 0))


sigma^2 estimated as 0.02415:  log likelihood = 16.38,  aic = -30.76
```

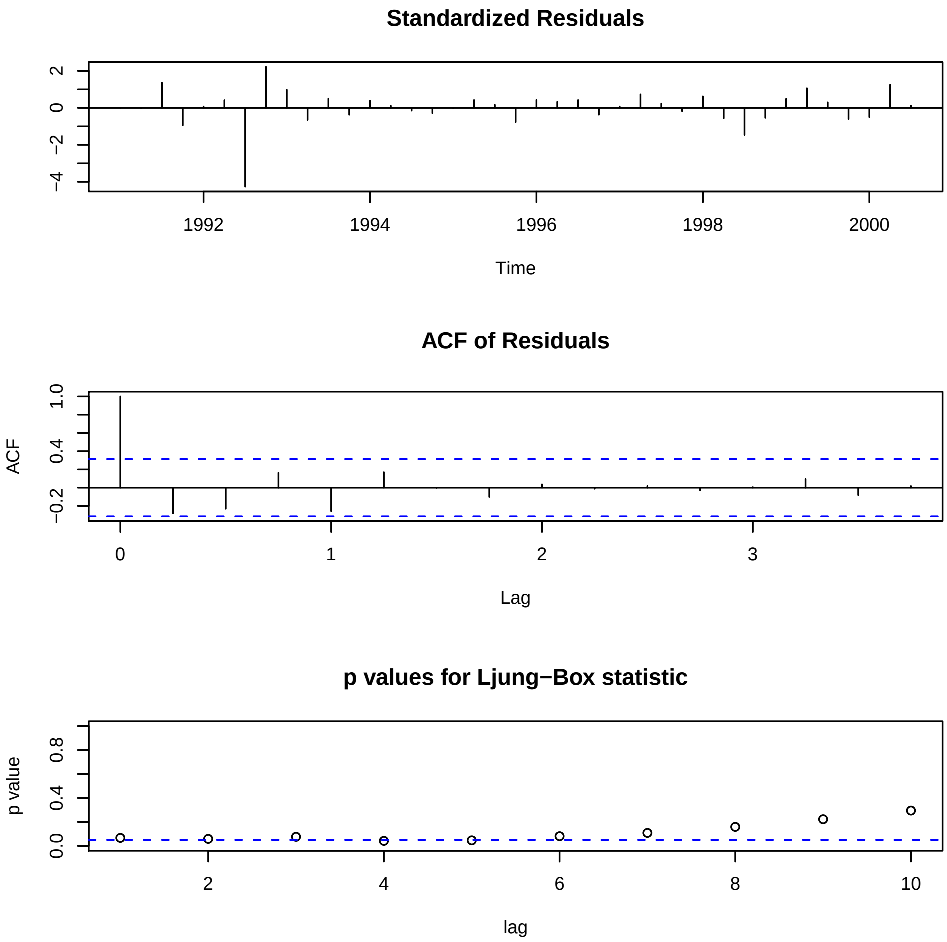


Figure 7.4: Model 2.

```
Call:
arima(x = tsuknz, order = c(0, 2, 1))

Coefficients:
          ma1
      -0.7162
s.e.   0.2465

sigma^2 estimated as 0.01919:  log likelihood = 20.28,  aic = -36.56
```

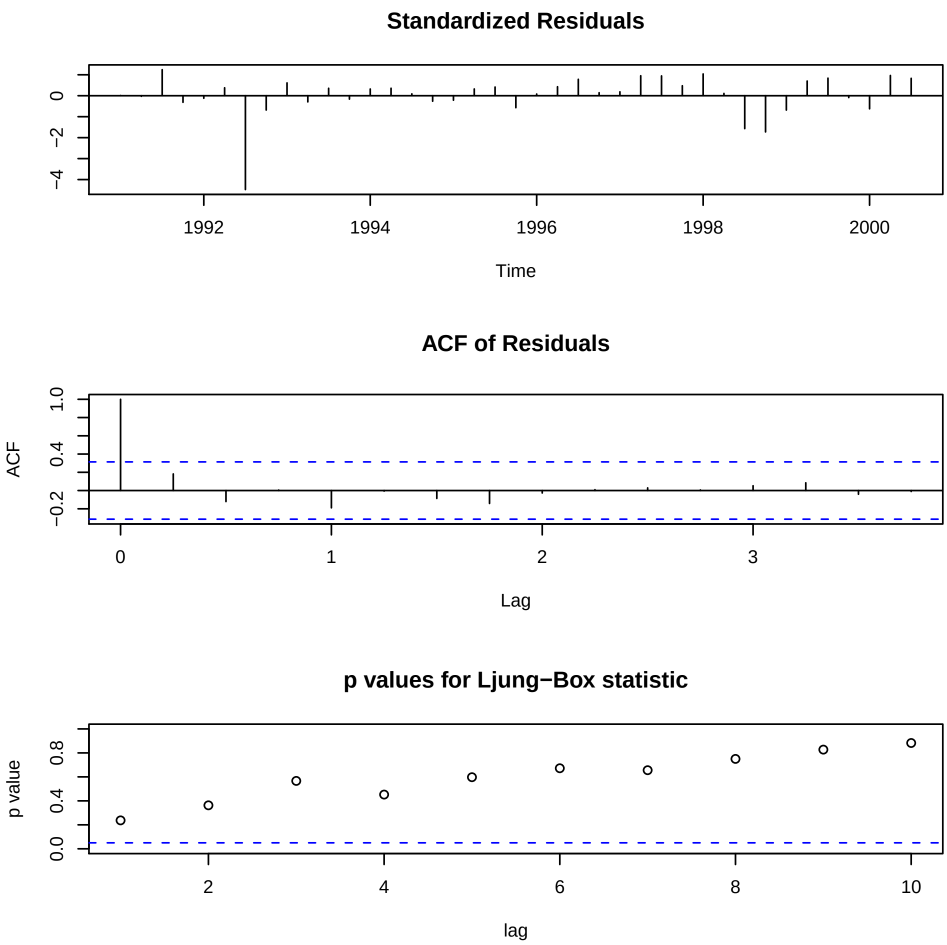


Figure 7.5: Model 3.

```
Call:
arima(x = tsuknz, order = c(0, 2, 2))

Coefficients:
          ma1      ma2
      -0.4696  -0.4260
s.e.   0.2039   0.2171

sigma^2 estimated as 0.01678:  log likelihood = 22.37,  aic = -38.73
```

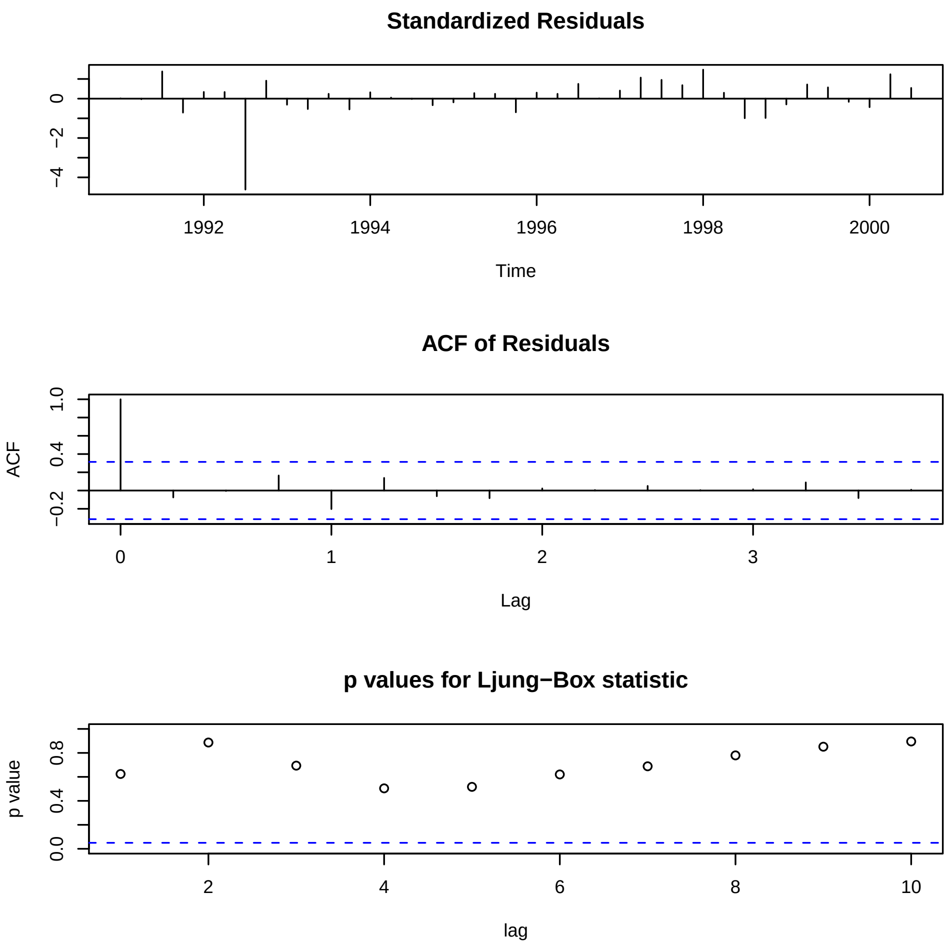


Figure 7.6: Model 4.

[24 marks]

END OF EXAMINATION PAPER

Two Hours

Mathematical formula books and statistical tables are to be provided

THE UNIVERSITY OF MANCHESTER

GENERALISED LINEAR MODELS

28 May 2015

9:45 – 11:45

Answer ALL TWO questions in Section A.
Answer TWO of the THREE questions in Section B.

If more than TWO questions from section B are attempted
then credit will be given to the best TWO answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** of the two questions

A1.

(a) Show that the negative binomial distribution $\mathrm{NB}(2, p)$ with probability mass function

$$P(Y = y) = (y + 1)p^2(1 - p)^y, \ y = 0, 1, 2, \ldots$$

belongs to the exponential family, where $p \in [0, 1]$ is an unknown parameter.

[3 marks]

(b) Derive the mean and variance of the negative binomial distribution in (a) using Property 1 of the exponential family.

[4 marks]

(c) Explain the role of the link function $g(\mu)$ in a generalised linear model and define the canonical link.

[4 marks]

(d) Derive the canonical link for a generalised linear model when the response variable follows the negative binomial distribution in (a).

[4 marks]

(e) Show that when the canonical link is used in a generalised linear model, the observed Fisher information is equal to the expected Fisher information. You may use the fact that the Fisher scores are

$$\frac{\partial \ell}{\partial \beta_j} = \frac{1}{a(\phi)} \sum_{i=1}^{n} \frac{x_{ij}}{V(\mu_i)g'(\mu_i)}(y_i - \mu_i), \ j = 1, \ldots, p,$$

where $V(\mu)$ is the variance function.

[5 marks]

[20 marks in total for this question.]

A2.

(a) Write down a generalised linear model with canonical link for Poisson responses $y_i \sim Pois(\lambda_i)$ on a single explanatory variable x with values x_i, $i = 1, \dots, n$, and state any further assumptions on $y_1, \dots, y_n$. An intercept or constant term should be included in the model.

[3 marks]

(b) Write down the data matrix X and work out the weight matrix W for the model in (a). These matrices are used in the iteratively re-weighted least squares algorithm.

[3 marks]

(c) Following on from (b), let

$$\underset{\sim}{\xi} = X\underset{\sim}{\beta} + W^{-1}(\underset{\sim}{y} - \underset{\sim}{\mu}),$$

where $\underset{\sim}{y} = (y_1, \dots, y_n)'$, $\underset{\sim}{\mu}$ is the corresponding mean vector, and $\underset{\sim}{\beta} = (\beta_0, \beta_1)'$ is the vector of model parameters. By differentiating the Poisson log-likelihood function and setting the partial derivatives to zero, show that the maximum likelihood estimator $\hat{\underset{\sim}{\beta}}$ of $\underset{\sim}{\beta}$ satisfies

$$X'WX\hat{\underset{\sim}{\beta}} = X'W\underset{\sim}{\xi}.$$

[6 marks]

(d) Differentiate the log-likelihood in (c) again to show that the expected/observed Fisher information matrix $I(\underset{\sim}{\beta})$ based on $y_1, \dots, y_n$ in (a) is given by

$$I(\underset{\sim}{\beta}) = X'WX,$$

where X and W are as in (b) and (c).

[4 marks]

(e) Derive the deviance of the model in (a) when fitted to data and simplify it.

[4 marks]

[20 marks in total for this question.]

SECTION B

Answer **TWO** of the three questions

B1. The Gamma distribution $\Gamma(2, \mu/2)$ with probability density function (pdf)

$$f(y;\mu) = \frac{4y}{\mu^2} \exp\left(-\frac{2y}{\mu}\right), \; y > 0,$$

belongs to the exponential family, where $\mu > 0$ is an unknown parameter. The dispersion parameter $\phi = 2$ is known.

(a) Express the pdf in exponential family form with $a(\phi) = 1/\phi$ on the denominator. State what the canonical parameter θ is and what the functions $b(\theta)$ and $c(y, \phi)$ are.

[3 marks]

(b) Show that the mean response is μ and find the variance function $V(\mu)$.

[4 marks]

(c) A generalised linear model with $\Gamma(2, \mu/2)$ response, linear predictor $\eta = \beta_0 + \beta_1 x$ and reciprocal link $g(\mu) = 1/\mu$ was fitted to $n = 25$ observations on x = pre-heat temperature and y = tensile strength. The MLE's were $\hat{\beta}_0 = 0.1676$ and $\hat{\beta}_1 = -0.000364$ with standard errors $se(\hat{\beta}_0) = 0.0500$, $se(\hat{\beta}_1) = 0.0036$ and correlation -0.8678. The deviance was 15.36 for the fitted model.

(i) Does the model provide adequate fit to the data? Use the scaled deviance to answer this question.

[4 marks]

(ii) Calculate the linear predictor and the fitted tensile strength at temperature $x = 15°C$.

[4 marks]

(iii) Find an approximate 95% confidence interval for the mean tensile strength at $x = 15°C$.

[5 marks]

[20 marks in total for this question.]

B2. A car insurance company conducted a survey of its policy holders. Data on the following four variables were recorded.

- y is the number of policy holders who made a claim during a three-month period,
- n is the number of policy holders surveyed (and responded)

for each combination of

- age with categories '<25', '25-29', '30-35', '>35' in years, and
- car with categories '<1.0L', '1.0-1.5L', '1.5-2.0L', '>2.0L' for the engine size.

Logistic regression was used to analyse how the probability of making a claim depends on the age group of the policy holder and the engine size of the car.

(a) The following full and additive effects models were fitted.

```
> fit1 <- glm(cbind(y, n-y) ~ age * car, family = binomial)
> fit0 <- glm(cbind(y, n-y) ~ age + car, family = binomial)
```

What is the difference between the two models? [2 marks]

(b) The following analysis of deviance table was obtained.

```
> anova(fit0, fit1, test="Chisq")
Analysis of Deviance Table

Model 1: cbind(y, n - y) ~ age + car
Model 2: cbind(y, n - y) ~ age * car
  Resid. Df Resid. Dev Df Deviance P(>|Chi|)
1         9     12.193
2         0      0.000  9   12.193    0.2027
```

What conclusion can you make from the above about the difference between the two models? Give your reason and significance level used.

[4 marks]

(c) Is the effect of car significant in the presence of age? Use the following output to answer this question and give the significance level used.

```
> anova(fit0, test="Chisq")
Analysis of Deviance Table
Terms added sequentially (first to last)

     Df Deviance Resid. Df Resid. Dev P(>|Chi|)
NULL                    15    212.175
age   3   94.698        12    117.477 < 2.2e-16 ***
car   3  105.284         9     12.193 < 2.2e-16 ***
```

[4 marks]

(d) Is the additive model adequate for the data? Can it be simplified further by dropping one or more terms? Use the following output to answer these questions. Given your reasons and significance levels used.

```
> summary(fit)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.59528    0.08308 -19.202  < 2e-16 ***
age>35      -0.62887    0.07796  -8.067 7.21e-16 ***
age25-29    -0.22863    0.09237  -2.475 0.013316 *
age30-35    -0.40827    0.09031  -4.521 6.16e-06 ***
car>2.0      0.66970    0.07915   8.461  < 2e-16 ***
car1.0-1.5   0.18539    0.05379   3.447 0.000568 ***
car1.5-2.0   0.45760    0.05908   7.745 9.56e-15 ***

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 212.175  on 15  degrees of freedom
Residual deviance:  12.193  on  9  degrees of freedom

> 1-pchisq(12.193, 9)
[1] 0.2026463
```

[4 marks]

(e) Estimate the probability of a claim from an older policy holder (>35 years) driving a medium engine sized car (1.5-2.0L) and give a 95% confidence interval using the following variances and covariances of estimators.

```
> vcov(fit0)
              (Intercept)          age>35       car1.5-2.0
(Intercept)  0.006902304   -0.0053429877    -0.0018941565
age>35      -0.005342988    0.0060772730    -0.0002045798
car1.5-2.0  -0.001894156   -0.0002045798     0.0034908773
```

[6 marks]

[20 marks in total for this question.]

B3.

(a) Let y_{ij} be the number of observations in row i column j of a contingency table, where $i = 1, \ldots, I$, $j = 1, \ldots, J$. What is the joint distribution of the random variables behind these observations?

[3 marks]

(b) Show that the distribution in (a) can be obtained from independent Poisson random variables by conditioning on their sum.

[5 marks]

(c) A random sample of 100 motor vehicles stopped at roadside were classified according to type and mechanical condition, with the following results:

Mechanical	Vehicle Type		
Condition	Car	Lorry	Van
Good	28	13	11
Satisfactory	30	27	8
Poor	19	6	17

(i) Use the following results to test independence between the row and column classifications at the 5% significance level.

```
> summary(fit)
Call: glm(formula = y ~ type + condition, family = poisson)

Coefficients:
                      Estimate Std. Error z value Pr(>|z|)
(Intercept)             3.2261     0.1610  20.035  < 2e-16 ***
typelorry              -0.5152     0.1863  -2.765 0.005701 **
typevan                -0.7603     0.2019  -3.766 0.000166 ***
conditionpoor          -0.2136     0.2075  -1.029 0.303259
conditionsatisfactory   0.2231     0.1861   1.199 0.230387

Null deviance: 37.889  on 8  degrees of freedom
Residual deviance: 16.236  on 4  degrees of freedom
```

[4 marks]

(ii) Can vehicle condition be removed from the additive model? Give reasons for your answer.

[4 marks]

(iii) When the mean age, x_{ij}, for each vehicle type within each mechanical condition was added to the model, the deviance decreased by 9.6. Use the change in deviance to decide whether age should be included in the model.

[4 marks]

[20 marks in total for this question.]

END OF EXAMINATION PAPER

TWO hours

Electronic calculators may be used, provided that they cannot store text.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL PROGRAMMING

22 May 2015
9.45 – 11.45

Answer **THREE** of the **FOUR** questions. If more than THREE questions are attempted, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

1.

A farmer with 500 acres of land wishes to determine the acreage devoted to three crops: wheat (W), corn (C) and soybean (S). The number of man-days, preparation cost and profit per acre of the three crops are summarized below:

Crop	Man-days	Preparation Cost ($)	Profit ($)
Wheat	6	100	60
Corn	8	150	100
Soybean	10	120	80

Suppose there are a maximum of 5000 man-days available and that the farmer has $60,000 for preparation.

(a) Formulate the farmer's decision problem as a linear program (LP) and find the optimal solution by simplex iterations.

[10 marks]

(b) Find the optimal dual solution. Assuming an 8-hour working day, discuss whether it would be profitable to the farmer to hire additional labour at $3 per hour.

[5 marks]

(c) Suppose that the farmer is contracted to grow at least 100 acres of wheat. Use sensitivity analysis to find the new optimal solution.

[5 marks]

[20 marks in total for this question]

2.

(a) The following are a primal/dual pair of linear programming problems in the variables $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^m$

$$P: \quad \begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad\qquad D: \quad \begin{array}{ll} \text{maximize} & \boldsymbol{y}^T\boldsymbol{b} \\ \text{subject to} & \boldsymbol{y}^T\boldsymbol{A} \leq \boldsymbol{c}^T \\ & \boldsymbol{y} \text{ unrestricted} \end{array}$$

State and prove the modified dual to problem P if the primal constraints are replaced by $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$ (keeping $\boldsymbol{x} \geq \boldsymbol{0}$).

[6 marks]

(b) Define the Phase I problem in the solution of

$$LP: \quad \begin{array}{ll} \text{minimize} & 2x + 15y + 5z \\ \text{subject to} & x + 6y + 3z \geq 2 \\ & 2x - 5y + z \geq 3 \\ & x, y, z \geq 0 \end{array}$$

by the two-phase method, and hence find the optimal solution.

[8 marks]

(c) Obtain the dual problem and solve it by applying complementary slackness conditions. Hence verify the optimal solution to LP by the duality theorem.

[6 marks]

[20 marks in total for this question]

3.

(a) Explain briefly what is meant by the *incumbent*, *fathoming* and *pseudocosts* in the context of a branch-and-bound algorithm for solving integer linear programs where the objective is to be maximized.

[5 marks]

(b) Using the branch and bound method

$$
\begin{array}{ll}
\text{Maximize} & 5x_1 + x_2 \\
\text{subject to} & -x_1 + 2x_2 \leq 4 \\
& x_1 - x_2 \leq 1 \\
& 4x_1 + x_2 \leq 12 \\
& x_1, x_2 \geq 0, \text{ integer.}
\end{array}
$$

Show a table of pseudocosts and illustrate your solution by a tree.

[15 marks]

[20 marks in total for this question]

4.

Let $\boldsymbol{r} = (r_1, ..., r_m)^T$ and $\boldsymbol{c} = (c_1, ..., c_n)^T$ represent row and column strategies for a two player zero-sum game $\boldsymbol{\Gamma}_A$ with an $m \times n$ payoff matrix $\boldsymbol{A}$.

(a) Define the expected payoff $E(\boldsymbol{r}, \boldsymbol{c})$ and state the fundamental theorem of matrix games.

[4 marks]

(b) A game is played between Rose (row-player) and Colin (column-player) with the following payoff matrix (paid to Rose).

	c_1	c_2	c_3
r_1	4	3	1
r_2	0	1	2

(i) Formulate Colin's optimal policy as a minimax LP problem.

[2 marks]

(ii) Solve Colin's equivalent LP problem by simplex iterations and deduce Colin's optimal strategy.

[6 marks]

(iii) Formulate Rose's equivalent LP problem and solve it using duality theory to determine Rose's optimal strategy.

[6 marks]

(iv) To make it a fair game, what entry fee should be paid and by whom?

[2 marks]

[20 marks in total for this question]

END OF EXAMINATION PAPER

<u>2 hours</u>

Tables of the cumulative distribution function are provided.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL MODELLING IN FINANCE

1 June, 2015

2:00pm – 4:00pm

Answer **all** 4 questions in **Section A** (60 marks in all) and
2 of the 3 questions in **Section B** (20 marks each). If all three questions from Section B are attempted then credit will be given for the two best answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** 4 questions

A1.

Consider a contract $V(S,t)$ which satisfies the Black Scholes equation, namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

(in the usual notation), with a payoff at expiry $t = T$

$$V(S,T) = S^2 - K^2.$$

By seeking a solution of the form $V(S,t) = S^2 f_1(t) + f_2(t)$, determine $V(S,t)$. Explain how to hedge the contract.

[15 marks]

A2.

(i) A European binary put option $B_p(S,t)$ on an underlying of value S has the payoff

$$B_p(S,T) = \begin{cases} 0 & \text{if} \quad S > X \\ K & \text{if} \quad S < X \end{cases}.$$

State the payoff for the corresponding European binary call $B_c(S,T)$.

(ii) By considering $B_c(S,T) + B_p(S,T)$, determine the put-call parity relationship between $B_c(S,t)$ and $B_p(S,t)$, where the (constant) risk-free interest rate is r.

(iii) Suppose that it is only possible to buy and sell European vanilla calls and European binary calls (with $K = 1$) on the same underlying S with exercise prices £70, £80, £90 and £100 all with exercise time T. Construct a portfolio of options such that

$$\Pi(S,T) = \begin{cases} 0, & S < 70 \\ 80 - S, & 70 < S < 80 \\ 0, & 80 < S < 90 \\ S - 90, & 90 < S < 100 \\ 0, & S > 100 \end{cases}.$$

[15 marks]

A3.

Consider an asset whose value 6 months before the expiration of an option is £22. The exercise price of the option is £21, the risk-free interest rate (constant) is 4% per annum, and the volatility (constant) is 15% per $(\text{annum})^{\frac{1}{2}}$.

By what amount does the asset price have to rise for the purchaser of a European call option to break even, and by what amount must the asset price have to fall for the purchaser of a European put option to break even?

You may assume (in the usual notation) that the solution of the Black-Scholes equation for a European put option paying no dividends is

$$P(S,t) = Xe^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where

$$d_1 = \frac{\log(S/X) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\log(S/X) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and $N(x)$ should be determined by interpolation from the tables of the cumulative distribution function, provided herewith. You may assume put-call parity for vanilla options, i.e.

$$S + P - C = Xe^{-r(T-t)},$$

without proof.

[15 marks]

A4.

Draw the expiry payoff diagram for each of the following portfolios (all options are vanilla and have the same expiry date):

(i) Long two assets, short three puts with exercise X.

(ii) Long two calls, short three puts, all with exercise X.

(iii) Long two puts both with exercise price X_1 and long two calls both with exercise price X_2. Compare the cases $X_1 > X_2$, $X_1 = X_2$, $X_1 < X_2$.

[15 marks]

SECTION B

Answer **2** of the 3 questions

B5.

Consider the Black-Scholes equation for an option $V(S,t)$ (in the usual notation), namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where the volatility σ and interest rate r are both constants.

(i) By using the substitution $V(S,t) = S^\alpha V_1(S,t)$, where $\alpha = 1 - 2r/\sigma^2$, find the equation satisfied by $V_1(S,t)$.

(ii) By using the further substitution $V_1(S,t) = V_2(\xi, t)$, where $\xi = S_d^2/S$, where S_d is constant, show that $V_2(\xi, t)$ satisfies the Black-Scholes equation.

(iii) Explain why $C_{DO} = AV(S,t) + B(\frac{S}{S_d})^{1-2r/\sigma^2} V(S_d^2/S, t)$, where A and B are constants, must also satisfy the Black-Scholes equation.

(iv) Consider a down-and-out call option C_{DO}, which, in the usual notation, has a payoff $\max(S - X, 0)$, but is worthless if the asset value falls below a prescribed value, S_d, where $S_d < X$ (the strike price). Find appropriate values for the coefficients A and B in (iii) where $V(S,t)$ is a vanilla European call option with strike X, such that C_{DO} values a down-and-out call and confirm that the final condition is satisfied.

[20 marks]

B6.

(i) Consider an asset of value S which pays just one dividend, Sd_y per share, during the lifetime of a put option, at $t = t_d$. In the absence of factors such as taxes and transaction costs, how is the value of the asset after payment $S(t_d^+)$ related to the value before payment $S(t_d^-)$? Justify your answer.

(ii) How does the value of a put option on the asset, $P_d(S,t)$ change across the dividend date? Justify your answer.

(iii) Using the above, compare the value of a European put option $P_d(S,t)$ on a single dividend-paying asset, with that of a non-dividend paying (vanilla) European put option $P(S,t,X)$, both with exercise price X. Consider the regimes $0 < t \leq t_d^-$ and $t_d^+ \leq t < T$ (the expiry date) separately. What effect does the dividend payment have on the value of the put?

(iv) Suppose that the asset pays n dividends $Sd_1, Sd_2, \ldots, Sd_n$ per share during the lifetime of the option, at corresponding times $t_1, t_2, \ldots, , t_n$. What is the value of this dividend-paying option at time t, where t is between two dividend dates ($0 \leq t_{k-1} < t < t_k \leq t_n$), compared with the corresponding vanilla European put option?

[20 marks]

B7.

Consider an option $V(S_1, S_2, t)$ where the two underlyings S_1 and S_2 have volatilities σ_1 and σ_2 respectively, and drifts μ_1, μ_2 respectively; the risk-free interest rate is r.

(i) Justify the use of the following stochastic processes to model underlyings ($i = 1, 2$):

$$dS_i = \mu_i S_i dt + \sigma_i S_i dW_i$$

where the dW_i are Wiener processes, such that

$$E[dW_i^2] = dt, \quad E[dW_1 dW_2] = \rho dt.$$

(ii) By considering a portfolio (using the dS_i in (i) above)

$$\Pi = V(S_1, S_2, t) - \Delta_1 S_1 - \Delta_2 S_2,$$

find the choices of Δ_1 and Δ_2 for which the portfolio is perfectly hedged.

(iii) Equating the hedged portfolio in (ii) above to the risk-free return on the portfolio, show that the option value is determined from

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV = 0.$$

(iv) Now consider a perpetual American put option on a basket of two assets (S_1, S_2). Assuming that there is no correlation between the two assets (i.e. $\rho = 0$), show that a solution exists of the form

$$V = A_1 S_1^{\lambda_1} + A_2 S_2^{\lambda_2}$$

where you are to determine λ_1 and λ_2.

(v) If the payoff is $\max(X - S_1 - S_2, 0)$, show that the critical asset prices S_1^* and S_2^* at which it is optimal to exercise satisfy the condition

$$B_1 S_1^* + B_2 S_2^* = X,$$

where you are to determine B_1 and B_2. Why is this solution not valid for small values of S_1 or S_2?

[20 marks]

END OF EXAMINATION PAPER

Two hours

UNIVERSITY OF MANCHESTER

CONTINGENCIES 2

15 May 2015

09.45 - 11.45

Answer ALL EIGHT questions
The total number of marks in the paper is 100.

Electronic calculators may be used, provided that they cannot store text.

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1. Calculate $A^{\;1}_{55:50:\overline{10}|}$ where the 55 year old is a male and the 50 year old is a female. Use mortality tables PMA92C20 and PFA92C20 respectively and interest at 4% per annum.

[Total 11 marks]

2. An insurance company provides a reversionary annuity to lives x (the failing life) and y (the annuitant) which only provides payments after 5 years from inception.

a) Write down in integral form an expression for the expected present value of this reversionary annuity; assume payments are made continuously.

[2 marks]

b) Write down the integral expression for the expected present value of the reserve that the insurer should hold t years after inception ($t \geq 5$) if:

- if both x and y are alive
- if x is dead and y is alive
- if x is alive and y is dead

[3 marks]

c) Insurance company A issues only reversionary annuity policies and insurance company B issues both annuities for the primary lives (such as x) and reversionary annuities (such as for x and y). Comment on how the occurrence of death affects the reserves held by insurance company A and insurance company B (consider the death of x and y separately). Then comment on the relative risks that insurance company A and insurance company B face with regard to having sufficient funds to cover their reserves.

[4 marks]

[Total 9 marks]

3. An insurance company sells sickness benefit policies which provide an annual payment of £20,000 (assume paid continuously) so long as the policyholder remains ill.

a) Using the standard 3 state model for sickness benefits and defining the probabilities that you use, write in integral form the expected present value of the policy to a healthy life aged x.

[2 marks]

b) For reserving purposes the insurer needs to calculate the expected present value for a person already in receipt of benefits and currently aged y. Write down the expected present value of the reserve in integral form.

[2 marks]

c) The insurer is planning to introduce a waiting period of 12 months for the receipt of benefits under the policy. During the waiting period no benefits are paid and benefits only commence payment if the policyholder has been continuously ill for 12 months. The waiting period applies for each time the person is ill. Write down an integral expression for the expected present value of this policy for healthy life x.

[4 marks]

d) The insurer also has to review the calculation of the expected present value of the reserve for life y, assuming y has been continuously sick for more than 12 months. Explain why the introduction of a waiting period will affect this calculation.

[3 marks]

[Total 11 marks]

4. In a 3 state multiple decrement model with states A (active) and absorbing states R (retired) and D (dead), calculate the entries into a multiple decrement table based on the following:

- construct the table for ages 60 to 63
- assume a radix, $(al)_{60} = 100{,}000$
- transition intensities (force of decrement) are constant each year
- transition intensity for retirement in years 1, 2 and 3 are 0.05, 0.06 and 0.07 respectively
- transition intensity for death in years 1, 2 and 3 are 0.001, 0.0012 and 0.0014 respectively

Express probabilities to 5 decimal places and the entries to the multiple decrement table in whole numbers.

[Total 12 marks]

5. A pension scheme provides a pension of $\frac{1}{60}$ of final average salary at retirement for each year of service. Final average salary is defined as the annual average of salary in the 3 years before the date it is calculated. Consider a member of the pension fund who is aged 40 exact, who has completed 10 years of service and earned £ 30,000 in the previous year.

a) Write out in summation form the expected present value of the pension earned to date for this person (past service pension). This summation should allow for future salary growth, assume a normal retirement age of 65 and the possibility of retirement in normal health occurring at this age and earlier ages. Ignore the valuation of benefits that may be available through other decrements. Notation should be consistent with that used in the Formulae and Tables for Examinations (Orange Book). The pension should be assumed to be payable continuously.

[9 marks]

b) Express this value using standard commutation functions found in the Orange Book and using these factors where relevant, calculate the expected present value of the past service pension.

[5 marks]

c) The pension scheme is introducing a limit of £ 50,000 to the final average salary used when calculating the pension benefit.

Demonstrate why this limit will not affect the expected pension that the 40 year old with a salary of £ 30,000 will earn.

[2 marks]

d) Another 40 year old who has also completed 10 years service has earned a salary of £ 40,000 in the previous year. What will his past service pension be on retirement for all possible ages of retirement allowed by the pension scheme service table set out in the Orange Book?

[3 marks]

[Total 19 marks]

6. An insurer has issued an insurance policy which has a cash flow vector as follows:

(-200, 50, 150, -100 , -50, 200, 300)

Produce a profit vector on the basis that the company sets up reserves to zeroise future expected negative cash flows. Assume the policyholder is aged 40 exact, mortality is in accordance with AM92 and the interest rate is 5% p.a.

[Total 9 marks]

7. An insurer writes a 2 year unit linked endowment policy for a life aged 60. Level premiums are paid yearly in advance. Benefits are paid at the end of the year.

a) Construct a table setting out the development of the unit fund each year and then a table showing the cash flows each year for the non unit fund. Show all workings so that the derivation of each item in the tables can be seen.

Using this second table show that the profit vector for the policy is (229.78, 11.89)

Details are as follows:

- Premium = £ 1,000 p.a.
- Allocation is 55% in year 1 and 102.5% in year 2
- Bid offer spread = 5%
- Management charges 1% deducted from the unit fund at the end of the year, before any claim payments are made.
- Initial expenses are 10% of first premium and £ 150 on receipt of the first premium.
- Renewal expenses are 1% of the second premium and £ 20 on payment of this premium.
- Probability of death = 0.008 in year 1 and 0.0085 in year 2 .
- A minimum sum assured of £ 1,500 is payable on death or on maturity of the policy.
- Assumed rate of return of 4% p.a. on the unit fund and 2% p.a. on the non-unit fund.

[15 marks]

b) Using the profit vector, calculate the profit signature, net present value and profit margin using a risk discount rate of 6%p.a.

[6 marks]

[Total 21 marks]

8. The person managing the finances of a pension fund is reviewing the mortality assumption used when valuing the pension fund.

Members of the pension fund live in either Manchester or in rural areas of the country. National data (i.e. not data provided solely from the pension fund's membership) supports the assumption that the pension fund uses, which is that mortality rates are higher in Manchester.

a) List 4 of the principal factors which may contribute to the difference in the mortality rates between the two areas.

[4 marks]

b) If a large number of employees were to move from Manchester to a rural area where a new office is opening, comment on whether the mortality experience is likely to change.

[4 marks]

[Total 8 marks]

END OF EXAMINATION PAPER

Two hours

UNIVERSITY OF MANCHESTER

RISK THEORY

14 May 2015

09:45 – 11:45

Answer ALL FOUR questions.
The total number of marks in the paper is 90.

Electronic calculators may be used, provided that they cannot store text.

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1. Consider the Cramér-Lundberg model with initial capital $u \geq 0$, premium rate $c > 0$, claim intensity $\lambda > 0$ and exponentially distributed claim sizes with parameter $\alpha > 0$. Denote by $U = \{U_t : t \geq 0\}$ the corresponding surplus process.

(a) Compute the Laplace exponent ξ of the surplus process U, which is defined as

$$e^{\xi(\theta)} = \mathbb{E}\left[e^{\theta U_1} \middle| U_0 = 0\right], \quad \theta \geq 0.$$

[4 marks]

(b) For this particular surplus process, the ruin probability $\phi(u)$ as a function of the initial capital u, is given by

$$\phi(u) = \frac{\lambda}{\alpha c} e^{-(\alpha - \lambda/c)u}, \quad u \geq 0.$$

Prove this formula. (Hint: use Laplace transforms.)

[7 marks]

(c) An insurance company is deciding how to allocate capital to two lines of business (lobs). The company assumes that the surplus of each lob evolves through time as a surplus process as defined at the start of this question. The surplus processes are assumed to be independent. The first lob has premium rate 4, claim intensity 2 and parameter of the exponential claim distribution equal to 1, whereas the second lob has premium rate 5, claim intensity 1 and parameter of the exponential claim distribution equal to 0.7. Denote by $u_1 \geq 0$, respectively $u_2 \geq 0$, the initial capital of the first, respectively second lob.

(i) Determine (as a function of u_1 and u_2) the probability that at least one lob gets ruined.

(ii) Assume that the company has in total a capital of size 3 to divide over the two lobs as initial capital, i.e. $u_1 + u_2 = 3$. How should the company choose u_1 and u_2 if their goal is to minimise the probability that at least one lob gets ruined?

[8 marks]

(d) Suppose now that the first lob of the insurance company in part (c) buys excess of loss reinsurance with retention level 0.8. Assume that the reinsurer charges reinsurance premium at rate 2.5. Determine the ruin probability probability of the first lob under this reinsurance scheme given that $u_1 = 0$.

[6 marks]

[Total 25 marks]

2.

(a) What is the exponential premium principle?

[3 marks]

(b) When does a premium principle satisfy the no rip-off property?

[3 marks]

(c) Show that the Esscher premium principle is additive.

[5 marks]

(d) Let X be a discrete-valued risk with probability mass function as given in Table 1. Compute the Value-at-Risk and the Tail-Value-at-Risk at confidence level 0.90.

[7 marks]

x	$\mathbb{P}(X = x)$
200	0.45
300	0.35
400	0.12
500	0.08

Table 1: Table corresponding to Question 2(d).

[Total 18 marks]

3. Suppose that the random variable Θ takes the values -1, 0 and 1 with respective probabilities $1/4$, $1/2$ and $1/4$. Suppose furthermore that given $\Theta = \theta$, the random variable X follows a normal (Gaussian) distribution with mean θ and variance 1.

(a) Show that the posterior distribution is given by

$$f_{\Theta|X}(\theta|x) = \begin{cases} \left(1 + 2\mathrm{e}^{x+1/2} + \mathrm{e}^{2x}\right)^{-1} & \text{if } \theta = -1 \\ 2\left(\mathrm{e}^{-x-1/2} + 2 + \mathrm{e}^{x-1/2}\right)^{-1} & \text{if } \theta = 0 \\ \left(\mathrm{e}^{-2x} + 2\mathrm{e}^{-x+1/2} + 1\right)^{-1} & \text{if } \theta = 1. \end{cases}$$

[5 marks]

(b) Suppose that a sample $X = x$ is observed. Find the Bayesian estimate for θ under the squared error loss function.

[5 marks]

(c) Suppose now that the observed sample has the value $x = 0$. Find the Bayesian estimate for θ under the loss function l given by $l(\theta, a) = |\theta - a|$.

[8 marks]

[Total 18 marks]

4. For a portfolio of insurance policies the risk parameter Θ has the following probability density function:

$$f_{\Theta}(\theta) = \begin{cases} 4\theta^3 & \text{if } 0 < \theta < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The annual claim amount of a policy, denoted X, has the following conditional probability density function given $\Theta = \theta$:

$$f_{X|\Theta}(x|\theta) = \begin{cases} 2x/\theta^2 & \text{if } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

A randomly selected policy has claim amount 0.1 in Year 1.

(a) Determine the Bayesian credibility estimate of the expected claim amount of the selected policy in Year 2.

[11 marks]

(b) Determine the Bühlmann credibility estimate of the expected claim amount of the selected policy in Year 2.

[13 marks]

(c) In Year 1 the portfolio contained 250 policies, together they generated a total claim amount of 135 in that year. In Year 2 the number of policies in the portfolio has grown to 300. Assuming that the policies in this portfolio are independent, use the Bühlmann-Straub model to determine the Bühlmann credibility estimate of the expected total claim amount for this portfolio in Year 2.

[5 marks]

[Total 29 marks]

END OF EXAMINATION PAPER