Two hours

THE UNIVERSITY OF MANCHESTER

LINEAR ANALYSIS

4 June 2014

09:45

Answer ALL 7 questions in Section A (25 marks in total). Answer 3 of the 4 questions in Section B (75 marks in total). If more than 3 questions from Section B are attempted, then credit will be given for the best 3 answers.

The use of electronic calculators is permitted provided they cannot store text.

SECTION A

Answer **ALL** of the seven questions

A1. What is meant by saying that $\|\cdot\|$ is a *norm* on a vector space V?

[3 marks]

A2. State the definition of the space $\ell^2(\mathbb{C})$ and show that it is infinite dimensional.

[4 marks]

A3. Define the norm $\|\cdot\|_2$ on the space $\ell^2(\mathbb{C})$.

Define $x = (x_i)_{i=1}^{\infty}$ by

$$x = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^i}, \ldots\right).$$

Show that $x \in \ell^2(\mathbb{C})$ and calculate $\|x\|_2$.

[6 marks]

A4. Let H be a Hilbert space over $\mathbb{C}$ and let L be a linear subspace of H. State what is meant by the *orthogonal complement* $L^{\perp}$ of L.

[2 marks]

A5. Let V be a Banach space over $\mathbb{C}$. State what is meant by the *norm* of a bounded linear functional $f : V \to \mathbb{C}$.

[2 marks]

A6. Let V be a Banach space over $\mathbb{C}$. State the definitions of the *dual space* V^* and the *second dual* V^{**}. What is meant by saying that V is *reflexive*?

[4 marks]

A7. Let V be a Banach space and let $T : V \to V$ be a bounded linear operator. State what is meant by

(i) the *spectrum* of T;

(ii) the *spectral radius* of T.

[4 marks]

 P.T.O.

SECTION B

Answer **THREE** of the four questions

B8.

(i) State what is meant when we say that a metric space is *complete*.

(ii) Let $C[0,1]$ denote the vector space of continuous functions $f : [0,1] \to \mathbb{R}$. Show that $C[0,1]$ with the norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$$

is a *Banach space*.

(You may assume that $\mathbb{R}$ is complete and that the uniform limit of a sequence of continuous functions is continuous.)

(iii) State what is meant when we say that two norms on a vector space are *equivalent*.

(iv) Define a norm $\|\cdot\|_2$ on $C[0,1]$ by

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2\right)^{1/2} dx.$$

Show that $\|\cdot\|_2$ is *not* equivalent to $\|\cdot\|_\infty$.

(You do not need to prove that $\|\cdot\|_2$ is a norm.)

[25 marks]

B9.

(i) Let V be a normed vector space over $\mathbb{R}$ and let $f : V \to \mathbb{R}$ be a bounded linear functional. Show that $f : V \to \mathbb{R}$ is *continuous.*

(ii) Let $V = C[0,1]$ with the norm

$$\|\varphi\|_\infty = \sup_{x\in[0,1]} |\varphi(x)|.$$

Define $f : V \to \mathbb{R}$ by

$$f(\varphi) = \int_0^1 \sqrt{x}\varphi(x)\ dx.$$

Show that f is a bounded linear functional and calculate its norm $\|f\|$.

(iii) Let $V = \ell^1(\mathbb{R})$ with the norm

$$\|x\|_1 = \sum_{j=1}^{\infty} |x_j|,$$

where $x = (x_1, x_2, x_3, \ldots)$. Define $g : V \to \mathbb{R}$ by

$$g(x_1, x_2, x_3, \ldots) = \sum_{j=1}^{\infty} \left(3 - \frac{1}{j}\right) x_j.$$

Show, directly from its definition, that g is a bounded linear functional and show that $\|g\| = 3$.

[25 marks]

B10.

(i) Let H be a Hilbert space over $\mathbb{R}$. Prove the *Cauchy-Schwarz inequality*: for all $x, y \in H$,

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}.$$

(ii) State and prove the Parallelogram Law.

(iii) Let H be a Hilbert space over $\mathbb{C}$. For $y \in H$, define

$$f_y(x) = \langle x, y\rangle.$$

Show that $f_y : H \to \mathbb{C}$ is a bounded linear functional and calculate its norm.

(iv) Let $H = \ell^2(\mathbb{C})$ and, for $x = (x_1, x_2, x_3, \ldots) \in \ell^2(\mathbb{C})$, define a linear functional $f : H \to \mathbb{C}$ by

$$f(x) = 2x_1 - 3x_2.$$

Using part (iii), or otherwise, compute $\|f\|$.

(v) Let $H = \ell^2(\mathbb{C})$ and let $L = \ell^1(\mathbb{C})$, regarded as a linear subspace of H. Calculate the orthogonal complement $L^\perp$ in this case.

Deduce that $\ell^1(\mathbb{C})$ is dense in $\ell^2(\mathbb{C})$, with respect to the norm $\|\cdot\|_2$.

[25 marks]

B11.

(i) Let H be a Hilbert space over $\mathbb{C}$. State what is meant by a linear operator $T : H \to H$ being *self-adjoint*.

(ii) Let $H = \ell^2(\mathbb{C})$ and let $a = (a_1, a_2, a_3, \ldots) \in \ell^\infty(\mathbb{C})$. Define

$$T_a(x_1, x_2, x_3, \ldots) = (a_1 x_1, a_2 x_2, a_3 x_3, \ldots).$$

Show that $T_a : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ is a bounded linear operator.

Show that T_a is self-adjoint if and only if a is a real vector (i.e. $a_i \in \mathbb{R}$, for all $i \geq 1$).

(iii) Let $H = \ell^2(\mathbb{C})$ and, for $n \geq 1$, define a sequence of linear operators $S_n : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ by

$$S_n(x_1, x_2, x_3, \ldots) = (x_n, 0, 0, \ldots).$$

Show that $\|S_n\| = 1$, for all $n \geq 1$.

Show that

(a) S_1 has 0 and 1 as eigenvalues and no other eigenvalues;

(b) for $n \geq 2$, S_n has 0 as an eigenvalue and no other eigenvalues.

[25 marks]

END OF EXAMINATION PAPER

Two hours

UNIVERSITY OF MANCHESTER

ANALYTIC NUMBER THEORY

30 May 2014

09.45 – 11.45

Answer **THREE** of the Four questions. (90 marks in total.) If more than THREE questions are answered, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

1.

i. Define Λ, von Mangoldt's function.

ii. Prove that
$$\sum_{d|n} \Lambda(d) = \log n,$$
for all $n \geq 1$.

iii. Prove, by looking at the Euler product for the Riemann zeta function, that
$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$
for $\mathrm{Re}s > 1$.

iv. a. By partial summation show that
$$\sum_{n \leq x} \frac{\Lambda(n)}{n^s} = \frac{\psi(x)}{x^s} + s\int_1^x \frac{\psi(t)}{t^{s+1}} dt$$
for $\mathrm{Re}s > 1$, where $\psi(x) = \sum_{n \leq x} \Lambda(n)$.

b. Deduce that
$$\frac{\zeta'(s)}{\zeta(s)} = -s\int_1^{\infty} \frac{\psi(t)}{t^{s+1}} dt$$
for $\mathrm{Re}s > 1$.

You may assume any appropriate form of Chebyshev's inequality.

[30 marks]

2.

i. a. Prove that

$$\sum_{n\le x}\frac{1}{n}=\log x+O\left(1\right),$$

for real $x>1$.

b. Prove that this can be improved by showing there exists a constant γ such that

$$\sum_{n\le x}\frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right),$$

for real $x>1$.

ii. a. If $d\left(n\right)$ is the number of divisors of n prove that

$$\sum_{n\le x}d\left(n\right)=x\log x+O\left(x\right).$$

b. Deduce that for any $C>1$

$$\left|\left\{\frac{x}{2}<n\le x: d\left(n\right)>\log^{C}n\right\}\right|\ll\frac{x}{\log^{C-1}x},$$

as $x\to\infty$.

[30 marks]

3.

i. a. Let $\pi(x) = \sum_{p \le x} 1$ and $\theta(x) = \sum_{p \le x} \log p$. Prove by partial summation that

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t} dt.$$

b. Assuming $\theta(x) = x + O\left(x^{1/2} \log^2 x\right)$ (true if the Riemann Hypothesis holds) prove that

$$\pi(x) = \mathrm{li} x + O\left(x^{1/2} \log x\right).$$

where

$$\mathrm{li} x = \int_2^x \frac{dt}{\log t}.$$

ii. Let f be the multiplicative arithmetic function defined on powers of primes by

$$f(p^a) = \begin{cases} 1 & \text{if } a \equiv 0 \text{ or } 2 \bmod 3 \\ 0 & \text{if } a \equiv 1 \bmod 3. \end{cases}$$

By using Euler Products find an expression for the Dirichlet Series $D_f(s)$ in terms of products and quotients of $\zeta(ks)$ for various $k \in \mathbb{N}$.

[30 marks]

4.

i. a. For arithmetic functions f and g define the Dirichlet Convolution $f * g$.

b. Prove that if f and g are multiplicative than $f * g$ is also multiplicative.

You may assume that if $\gcd(m, n) = 1$ *then the divisors* $d|mn$ *are in one-to-one correspondence with the pairs of divisors* (d_1, d_2) *where* $d_1|m$ *and* $d_2|n$.

ii. a. Define the arithmetic functions Q_2 and μ_2.

b. Prove that $Q_2 = 1 * \mu_2$.

You may assume that the Möbius function, μ, *is multiplicative.*

c. Prove that

$$\sum_{n \le x} Q_2(n) = \frac{1}{\zeta(2)} x + O\left(x^{1/2}\right).$$

You may assume that $\sum_{m>y} 1/m^\theta \ll 1/y^{\theta-1}$ *for* $\theta > 1$.

[30 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

ALGEBRAIC TOPOLOGY

28 May 2014

14:00 - 16:00

Answer **ALL FOUR** questions in Section A (40 marks in total). Answer **THREE** of the FOUR questions in Section B (45 marks in total). If more than THREE questions from Section B are attempted then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** FOUR questions.

A1. (a) Define what is meant by a *geometric simplicial surface.*

[The notions of *triangle* in $\mathbb{R}^n$, and its *vertices* and *edges* may be used without definition.]

(b) Define what is meant by the statement that a simplicial surface is *orientable.*

(c) Define what is meant by the statement that the orientability of a simplicial surface is a *topological invariant.*

[10 marks]

A2. (a) Define what it meant by a *geometric simplicial complex* K and its underlying space $|K|$.

[The notions of *geometric simplex* and *face* of a simplex may be used without definition.]

(b) An abstract simplicial complex has vertices $\{v_1, v_2, v_3, v_4\}$ and simplices $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$ and $\{v_2, v_3, v_4\}$ and their faces. Draw a realization K of this simplicial complex as a geometric simplicial complex in $\mathbb{R}^2$.

(c) Define the *Euler characteristic* of a simplicial complex and calculate the Euler characteristic of the simplicial complex in part (b).

(d) Draw the *first barycentric subdivision* K' of the geometric simplicial complex K in part (b).

(e) Find the Euler characteristic of K'.

[10 marks]

A3. (a) Define what is meant by the *r-chain group* $C_r(K)$, the *r-cycle group* $Z_r(K)$, and the *r-boundary group* $B_r(K)$ of a simplicial complex K.

(b) Write down without proof generators for the groups $Z_1(K)$ and $B_1(K)$ of the simplicial complex K in Question A2(b). Hence find the first homology group $H_1(K)$.

[10 marks]

A4. (a) Let K be the simplicial complex $\bar{\Delta}^7$ consisting of all of the faces of the standard 7-simplex in $\mathbb{R}^8$ (including the 7-simplex itself). Write down the simplicial homology groups of K, indicating a justification for your statement.

(b) Let L be the 3-skeleton of K. Calculate the Euler characteristic of L and hence, or otherwise, find its simplicial homology groups.

[10 marks]

SECTION B

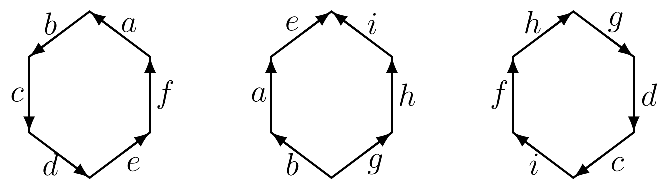
Answer **THREE** of the FOUR questions.

If more than THREE questions are attempted then credit will be given for the best THREE answers.

B5. (a) Explain how a *symbol* may be used to represent a closed surface arising from the identification in pairs of the edges of a polygon.

(b) State the classification theorem for closed surfaces in terms of symbols.

(c) The boundaries of three discs are identified as shown below.



Find a symbol for the resulting closed surface. By reducing the symbol to canonical form, or otherwise, identify the surface up to homeomorphism.

[15 marks]

B6. Let v_i be the ith standard basis vector in $\mathbb{R}^9$, $1 \leqslant i \leqslant 9$. Consider the set K of sixteen triangles with vertices v_i, v_j and v_k where ijk is one of

$$123,\ 124,\ 135,\ 145,\ 236,\ 247,\ 267,\ 347,\ 349,\ 357,\ 368,\ 389,\ 459,\ 579,\ 678,\ 789.$$

Verify that K is a simplicial surface, calculate its Euler characteristic and determine whether it is orientable. Identify the underlying space of K up to homeomorphism.

[15 marks]

B7. Describe a geometrical simplicial complex K whose underlying space is homeomorphic to the *Klein bottle* and indicate why it has this property.

Calculate the simplicial homology groups of K.

[15 marks]

B8. (a) Let K and L be simplicial complexes. Define what is meant by a *simplicial map* $|K| \to |L|$. Define what is meant by a *simplicial approximation* to a continuous map $f\colon |K| \to |L|$. Prove that a simplicial approximation to f is homotopic to f.

(b) Let K be the geometric simplicial complex in $\mathbb{R}$ with vertices 0, 1/2 and 1 and edges $\langle 0, 1/2\rangle$ and $\langle 1/2, 1\rangle$, so that the underlying space $|K|$ is the closed unit interval $[0,1]$. Let $f\colon [0,1] \to [0,1]$ be the map $f(x) = x^2$. Prove that $f\colon |K| \to |K|$ does not have a simplicial approximation but that $f\colon |K'| \to |K|$ does have a simplicial approximation, where K' is the first barycentric subdivision of K.

[15 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY

15-th May 2014
09:45—11:45

ANSWER THREE OF THE FOUR QUESTIONS
If four questions are answered credit will be given for the best three questions
All questions are worth 20 marks

Electronic calculators may not be used

P.T.O.

1.

(a) Explain what is meant by saying that G is a Riemannian metric on a manifold M.

Consider the upper half plane ($y > 0$) in $\mathbf{R}^2$ equipped with the Riemannian metric $G = \sigma(x, y)(dx^2 + dy^2)$.

Explain why $\sigma(x, y) > 0$.

Consider in this Riemannian manifold a curve C such that

$$C\colon \quad \begin{cases} x = 1 \\ y = a + t \end{cases} \quad , \quad 0 \le t \le 1\,,\ (a > 0)\,.$$

Find the length of this curve in the case of $\sigma(x, y) = \frac{1}{y^2}$ (the Lobachevsky metric).

[6 marks]

(b) Consider a surface (the upper sheet of a cone) in $\mathbf{E}^3$

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = 2h \cos \varphi \\ y = 2h \sin \varphi \\ z = h \end{cases} \quad , \quad h > 0, 0 \le \varphi < 2\pi\,.$$

Calculate the Riemannian metric on this surface induced by the canonical metric on Euclidean space $\mathbf{E}^3$.

Show that this surface is locally Euclidean.

[6 marks]

(c) Consider a Riemannian manifold M^n with a metric $G = g_{ik} dx^i dx^k$.
Write down the formula for the volume element on M^n (area element for $n = 2$).

Find the volume of a domain $0 < h < H$ of the cone considered in the part (**b**).

It is well-known that the metric $G = \frac{dx^2 + dy^2}{y^2}$ of the Lobachevsky plane is not locally Euclidean. However show that there exist coordinates $u = u(x, y)$, $v = v(x, y)$ such that in these coordinates the area element is equal to $dudv$.

[8 marks]

P.T.O.

2.

(a) Explain what is meant by an affine connection on a manifold.

Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u, v) all Christoffel symbols vanish except $\Gamma^u_{vv} = u$ and $\Gamma^v_{uu} = v$. Calculate the vector field $\nabla_{\mathbf{X}}\mathbf{X}$, where $\mathbf{X} = \frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$.

[5 marks]

(b) Explain what is meant by the induced connection on a surface in Euclidean space.

Calculate the induced connection on the cylindrical surface in $\mathbf{E}^3$

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} .$$

[6 marks]

(c) Give a detailed formulation of the Levi-Civita Theorem. In particular write down the expression for the Christoffel symbols Γ^i_{km} of the Levi-Civita connection in terms of the Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

Prove that the induced connection on a surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$ is equal to the Levi-Civita connection of the Riemannnian metric induced by the canonical metric on Euclidean space $\mathbf{E}^3$.

A Riemannian metric in local coordinates u, v is $G = e^{-u^2-v^2}(du^2 + dv^2)$.
Calculate the Christoffel symbols of the Levi-Civita connection at the point $u = v = 0$.

[9 marks]

P.T.O.

3.

(a) Define a geodesic on a Riemannian manifold as a parameterised curve.

Write down the differential equations for geodesics in terms of the Christoffel symbols.

Explain why the great circles are the geodesics on the sphere.

[7 marks]

(b) Explain what is meant by the Lagrangian of a "free" particle on a Riemannian manifold.

Explain what is the relation between the Lagrangian of a free particle and the differential equations for geodesics.

Calculate the Christoffel symbols on the Lobachevsky plane.
(You may use the Lagrangian of a "free" particle on this plane $L = \frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2}$.)

Consider an arbitrary geodesic $\mathbf{r} = (x(t), y(t))$ on the Lobachevsky plane.
Show that the magnitude $I(t) = \frac{\dot{x}(t)}{y^2(t)}$ is preserved along the geodesic.

[8 marks]

(c) Consider a plane $\mathbf{R}^2$ equipped with the Riemannian metric $G = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}$.
We know that it is isometric to the sphere of radius R in $\mathbf{E}^3$ (without the North pole) in stereographic coordinates $u = \frac{Rx}{R-z}$, $v = \frac{Ry}{R-z}$, $(x^2 + y^2 + z^2 = R^2)$.

Consider the parallel transport of the vector $\mathbf{A} = \partial_u$ attached at the point $u = R, v = 0$ along the circle $u^2 + v^2 = R^2$ with respect to this Riemannian metric.
Show that during the parallel transport along this circle it will always be orthogonal to this circle.

[5 marks]

P.T.O.

4.

(a) Consider the sphere of radius R in Euclidean space $\mathbf{E}^3$

$$\mathbf{r}(\theta,\varphi)\colon \quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} .$$

Let $\mathbf{e}, \mathbf{f}$ be unit vectors in the directions of the vectors $\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta}$ and $\mathbf{r}_\varphi = \frac{\partial \mathbf{r}}{\partial \varphi}$ and $\mathbf{n}$ be a unit normal vector to the sphere.
Express these vectors explicitly.

For the obtained orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ calculate the 1-forms a, b and c in the derivation formula

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} .$$

Deduce from these calculations the mean curvature and the Gaussian curvature of the sphere.

[7 marks]

(b) Give a definition of the curvature tensor for a manifold equipped with a connection.

On two-dimensional Riemannian manifold with coordinates x^1, x^2 consider the vector fields $\mathbf{A} = \frac{\partial}{\partial x^1}$, $\mathbf{B} = \frac{\partial}{\partial x^2}$, $\mathbf{X} = (1+x^1x^2)\frac{\partial}{\partial x^2}$, and the vector field $\mathbf{Y} = (\nabla_\mathbf{A}\nabla_\mathbf{B} - \nabla_\mathbf{B}\nabla_\mathbf{A})\,\mathbf{X}$, where ∇ is a connection.
Calculate the value of the field $\mathbf{Y}$ at the point $x^1 = x^2 = 0$ if the curvature tensor of the connection ∇ is such that $R^1{}_{212} = 1$ and $R^2{}_{212} = 0$ at this point.

[5 marks]

(c) State the relation between the Riemann curvature tensor of the Levi-Civita connection of a surface in $\mathbf{E}^3$ and its Gaussian curvature K.

Explain why the sphere is not a locally Euclidean Riemannian manifold.

On the sphere of radius R give an example of local coordinates in the vicinity of an arbitrary point $\mathbf{p}$ such that in these coordinates standard Riemannian metric of the sphere is equal to $du^2 + dv^2$ at this point $\mathbf{p}$.

[8 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

COMMUTATIVE ALGEBRA

03 June 2014

09:45 – 11:45

Answer ALL THREE questions in Section A (40 marks in total). Answer TWO of the THREE questions in Section B (40 marks in total). If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

Electronic calculators are permitted, provided they cannot store text.

SECTION A

Answer **ALL** questions in this section (40 marks in total)

A1.

(a) What is meant by saying that a polynomial $f \in \mathbb{Q}[X,Y,Z]$ is reduced modulo a set $F \subset \mathbb{Q}[X,Y,Z] \setminus \{0\}$ with respect to a monomial ordering $\preccurlyeq$?

(b) Let $F = \{f_1 = XY + Y,\ f_2 = YZ + Z,\ f_3 = XZ + Z\}$. Find $q_1, q_2, q_3, r \in \mathbb{Q}[X,Y,Z]$ such that $q_1 f_1 + q_2 f_2 + q_3 f_3 + r = X^2Y + XYZ$ and r is reduced modulo F with respect to the degree lexicographic order with $X \succ Y \succ Z$.

[10 marks]

A2.

(a) Describe, without proof, all irreducible polynomials in $\mathbb{R}[X]$.

(b) Explain why for every positive integer n there is an irreducible polynomial of degree n in $\mathbb{Q}[X]$.

(c) For each of the following polynomials, determine whether it is irreducible in the given ring. Justify your answer.

i. $(2X^2 + 3Y^3)^4 + (5Y^4)^3$ in $\mathbb{Z}[X,Y]$.
ii. $X^{2014} + Y^{2014}$ in $\mathbb{Q}[X,Y]$.
iii. $X^{2014} + Y^{2014} + Z^{2014}$ in $\mathbb{C}[X,Y,Z]$.

[15 marks]

A3.

(a) Let K be a field. What is meant by the variety $\mathcal{V}(I)$ of an ideal I of $K[X_1, \ldots, X_n]$? What is meant by the ideal $\mathcal{I}(S)$ of a set $S \subseteq K^n$?

(b) State the weak Nullstellensatz without proof.

(c) Let $K = \mathbb{C}$ and $n = 2$. Is $\mathcal{V}(\mathcal{I}(S)) = S$ for all sets $S \subseteq \mathbb{C}^2$? Give reasons for your answer.

(d) Let the set $S_m \subset \mathbb{C}^2$ be given by $S_m = \{(1,1), (2,2), \ldots, (m,m)\}$, so that $S_1 \subseteq S_2 \subseteq \ldots$ Let I_m be the ideal $\mathcal{I}(S_m)$ of $\mathbb{C}[X,Y]$. Is there an m such that $I_m = I_{m+1}$? Justify your answer.

(e) Let $J = \mathcal{I}(\bigcup_{m=1}^{\infty} S_m)$ where the sets $S_1, S_2, \ldots$ are as in part (d). Write down a finite set which generates the ideal J of $\mathbb{C}[X,Y]$. Justify your answer.

[15 marks]

SECTION B

Answer **TWO** of the three questions in this section (40 marks in total).

If more than TWO questions from this section are attempted, then credit will be given for the best TWO answers.

B4. Denote by $\mathbb{Z}_3$ the ring of integers modulo 3 (which is a field). Let $f_1 = 1 + Y^3$, $f_2 = Y^4 + X^2Y^3$ in $\mathbb{Z}_3[X, Y]$, and let I be the ideal of $\mathbb{Z}_3[X, Y]$ generated by f_1 and f_2.

(a) Find a Gröbner basis of I with respect to the lexicographic order Lex with $X \succ Y$.

(b) List all the monomials in $M(X, Y)$ that do not belong to the ideal $\mathrm{LT}_{\mathrm{Lex}}(I)$.

(c) Explain why the factor ring $\mathbb{Z}_3[X, Y]/I$ is a finite ring.

(d) Give a definition of the radical of an ideal in a commutative ring. What is a radical ideal?

(e) Is I a radical ideal of $\mathbb{Z}_3[X, Y]$? Justify your answer.

[20 marks]

B5.

(a) Define what is meant by a euclidean norm on a commutative domain R.

(b) Prove that a domain with a euclidean norm is a principal ideal domain.

You are given that the set $\mathbb{Z}[\omega] = \{a + b\omega : a, b \in \mathbb{Z}\}$, where $\omega = \frac{-1+\sqrt{-3}}{2}$, is a subring of $\mathbb{C}$. You do not have to prove this.

(c) Let $|z|$ denote the modulus of a complex number z. Explain why $N(z) = |z|^2$ is a euclidean norm on $\mathbb{Z}[\omega]$.

(d) List all the primes in $\mathbb{Z}[\omega]$ that divide the element 7 of $\mathbb{Z}[\omega]$. You do not have to justify your answer.

[20 marks]

B6.

(a) Define what is meant by an irreducible element of a commutative domain.

(b) Prove that in a noetherian domain, every non-zero non-unit is a product of finitely many irreducible elements. You may quote any other results from the course without proof.

(c) Define what is meant by a maximal ideal of a commutative domain.

(d) How many distinct maximal ideals of $\mathbb{C}[X]$ contain the ideal $<X^5 + 20X^2 - 48>$? Justify your answer briefly. You may quote any results from the course without proof.

(e) Can an irreducible element of some noetherian domain R lie in infinitely many maximal ideals of R? Justify your answer briefly. You may quote any results from the course without proof.

[20 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO ALGEBRAIC GEOMETRY

Date: 21 May, 2014

Time: 9:45–11:45

Answer THREE of the FOUR questions.
If more than THREE questions are attempted, then credit will be given for the best THREE answers.

Electronic calculators are permitted, provided they cannot store text.

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1. In this question K denotes an arbitrary field and unless specified otherwise, varieties are defined over K.

(a) (i) Define the *affine algebraic variety* $\mathcal{V}(J) \subseteq \mathbb{A}^n(K)$ determined by an ideal $J \lhd K[x_1, x_2, \ldots, x_n]$.

(ii) Define the *ideal* $\mathcal{I}(X)$ of a set $X \subseteq \mathbb{A}^n(K)$.

(iii) Show that if $X \subseteq \mathbb{A}^n(K)$ is an affine algebraic variety then $\mathcal{V}(\mathcal{I}(X)) = X$.

(b) Let $J = \langle (y-z)^3, x^2 + y^2 - x^2 z \rangle \lhd \mathbb{C}[x, y, z]$. Show that $y - z \in \mathcal{I}(\mathcal{V}(J))$, but $y - z \notin J$.

(c) (i) Define what it means for an affine algebraic variety W to be *irreducible*.

(ii) Prove that an affine algebraic variety W is irreducible if and only if $\mathcal{I}(W)$ is a prime ideal.

(d) Find the irreducible components of the variety

$$W = \mathcal{V}(\langle yz^2 - y, x^2 + y^2 z - z \rangle) \subset \mathbb{A}^3(\mathbb{C}).$$

(You should either correctly identify what kind of variety each component is or prove directly that it is irreducible.)

[20 marks]

2. (a) Let K be a field and let $V \subseteq \mathbb{A}^n(K)$ be an irreducible affine algebraic variety.

(i) Let $P \in V$ be a point and let $\mathbf{v} \in K^n$ be a non-zero vector. Let $\ell = \{P + t\mathbf{v} \mid t \in K\}$ be the line through P with direction vector $\mathbf{v}$. Explain what is meant by saying that ℓ is *tangent* to V at P.

(ii) Define the *tangent space* $T_P V$ to V at P.

(iii) Let $f_1, f_2, \ldots, f_r$ be a generating set for $\mathcal{I}(V)$ and let

$$J_P = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}|_P & \frac{\partial f_1}{\partial x_2}|_P & \cdots & \frac{\partial f_1}{\partial x_n}|_P \\ \frac{\partial f_2}{\partial x_1}|_P & \frac{\partial f_2}{\partial x_2}|_P & \cdots & \frac{\partial f_2}{\partial x_n}|_P \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_r}{\partial x_1}|_P & \frac{\partial f_r}{\partial x_2}|_P & \cdots & \frac{\partial f_r}{\partial x_n}|_P \end{pmatrix}$$

be the Jacobian matrix at $P \in V$. Show that the line $\ell = \{P + t\mathbf{v} \mid t \in K\}$ ($\mathbf{v} \in K^n \setminus \{\mathbf{0}\}$) is tangent to V at P if and only if $J_P\mathbf{v} = 0$.

(b) Let x, y be co-ordinates on $\mathbb{A}^2(\mathbb{C})$ and let t be the co-ordinate on $\mathbb{A}^1(\mathbb{C})$. Let $C \subset \mathbb{A}^2(\mathbb{C})$ be the variety defined by the equation

$$y^2 - (x-2)^2(x+2)^3 = 0.$$

(You may assume without proof that $y^2 - (x-2)^2(x+2)^3$ is an irreducible polynomial, so C is irreducible and $\mathcal{I}(C) = \langle y^2 - (x-2)^2(x+2)^3 \rangle$.)

(i) Find the singular points of C or show that it has none.

(ii) Show that $\phi(t) = (t^2 - 2, t^3(t^2 - 4))$ is a morphism $\mathbb{A}^1(\mathbb{C}) \to C$.

(iii) Let $\psi : C \dashrightarrow \mathbb{A}^1(\mathbb{C})$ be the rational map given by $\psi(x, y) = \dfrac{y}{(x-2)(x+2)}$.
Show that ψ is the inverse of ϕ considered as a rational map. (*Hint:* Show that $\left(\dfrac{y}{(x+2)(x-2)}\right)^2 = x + 2$ in the function field of C.)
Deduce that C and $\mathbb{A}^1(\mathbb{C})$ are birationally equivalent but not isomorphic.

[20 marks]

3. In this question K denotes an arbitrary field.

(a) Let n be a positive integer. Define $\mathbb{P}^n(K)$, the *n-dimensional projective space* over K. Explain what the point with homogeneous co-ordinates $(X_0 : X_1 : \ldots : X_n) \in \mathbb{P}^n(K)$ means.

(b) Give the definition of a *projective transformation* $\mathbb{P}^n(K) \to \mathbb{P}^n(K)$. Explain what it means to say that two sets $V, W \subseteq \mathbb{P}^n(K)$ are *projectively equivalent.*

(c) In this part of the question we identify $\mathbb{P}^1(K)$ with $K \cup \{\infty\}$ by identifying $(X_0 : X_1)$ with $X_0/X_1 \in K$ if $X_1 \neq 0$ and $(X_0 : 0)$ with ∞ as usual.
Recall that if z_1, z_2, z_3, $z_4 \in K \cup \{\infty\}$ with at least 3 of them pairwise distinct, their cross ratio is defined to be

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \in K \cup \{\infty\}$$

with appropriate rules for arithmetic involving ∞ and for division by 0.

(i) Show that any two sets of 3 distinct points in $\mathbb{P}^1(K)$ are projectively equivalent.

(ii) Find $a, b, c, d \in \mathbb{C}$ such that $\phi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, $\phi(z) = \dfrac{az + b}{cz + d}$ satisfies $\phi(-1) = 4$, $\phi(2) = 1$ and $\phi(7) = 6$. (You may use standard properties of the cross ratio if necessary.)

(d) Prove that every isomorphism $\Phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a projective transformation.

[20 marks]

4. (a) Let K be a field. Let $E \subset \mathbb{P}^2(K)$ be a non-singular cubic curve and let O be a point of E. (You should not assume that the equation of E is in a special form.)

(i) Describe without proof how one can define the addition operation on the points of E to obtain a commutative group in which O is the zero element.

(ii) Describe how the $-P$ can be calculated for $P \in E$.

(b) Let F be the elliptic curve given by the affine equation

$$y^2 = x^3 - 7x + 10$$

over $\mathbb{C}$ together with the point $O = (0 : 1 : 0)$ in homogeneous co-ordinates, which is taken to be the zero element of the group law as usual.
Let $P = (-1, -4)$, $Q = (2, 2)$. Calculate $P + Q$ and $2P$.

(c) The curve defined by the equation

$$y^2 = x^3 + 9x + 2$$

in $\mathbb{A}^2(\mathbb{C})$ has two real inflection points, which have an integer x-coordinate. Find these inflection points.

[20 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

LIE ALGEBRAS

16 May 2014

0945 - 1145

Answer the **ONE** question in **Section A** (26 marks in all) and **TWO** of the three questions in **Section B** (27 marks each).

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer the **ONE** question

A1. Let $\Bbbk$ be a field.

(i) Let L be a vector space over $\Bbbk$ with a bilinear operation $[\cdot\,,\cdot]\colon L \times L \to L$. For $x, y, z \in L$ define $j(x, y, z) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]]$ and $j'(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]$.

When is L called *anticommutative*? When is L called a *Lie algebra*? Explain why the identity $[x, x] = 0$ in L implies the identities $[x, y] = -[y, x]$ and $j'(x, y, z) = -j(x, y, z)$.

[4 marks]

From now on assume that L is a Lie algebra over $\Bbbk$.

(ii) Define the *lower central series* $L = L^1 \supseteq L^2 \supseteq \cdots \supseteq L^n \supseteq \cdots$ of L. When is L called *nilpotent*? Prove by induction on $n \in \mathbb{N}$ that $[L^m, L^n] \subseteq L^{m+n}$ and $(L^m)^n \subseteq L^{mn}$ for all $m, n \in \mathbb{N}$.

[6 marks]

(iii) Define the *derived series* of L. State the *lemma on two ideals* for L and use it to show that all members of the derived series are ideals of L. When is L called *solvable*? Use part (ii) to prove that any nilpotent Lie algebra is solvable.

[6 marks]

(iv) When is an endomorphism $D\colon L \to L$ called a *derivation* of L? For any $x \in L$ define the *adjoint endomorphism* $\operatorname{ad} x$ of L and show that it is a derivation of L.

[5 marks]

(v) Suppose L has a basis $\{h, u, v\}$ such that $[h, u] = u$, $[h, v] = -v$ and $[u, v] = 0$. Show that L is solvable, but not nilpotent, and compute the trace of the adjoint endomorphism $\operatorname{ad} h$.

[5 marks]

SECTION B

Answer **TWO** of the three questions

B2. Let $\Bbbk$ be an algebraically closed field and let L be a finite dimensional Lie algebra over $\Bbbk$.

(i) Let V be a vector space over $\Bbbk$. What is the *general linear Lie algebra* $\mathfrak{gl}(V)$? When is a linear map $\rho\colon L \to \mathfrak{gl}(V)$ called a *representation* of L? Show that the map ad: $L \to \mathfrak{gl}(L)$ is a representation of L.

[5 marks]

(ii) When is a linear operator $A \in \mathfrak{gl}(V)$ called *nilpotent*? Prove that if $A \in \mathfrak{gl}(V)$ is nilpotent then so is $\operatorname{ad} A \in \mathfrak{gl}(\mathfrak{gl}(V))$.

[5 marks]

(iii) Explain what is meant by a *flag of subspaces* in V and state *Engel's theorem* for Lie subalgebras of $\mathfrak{gl}(V)$ in the form involving flags of subspaces in V. Use this theorem to prove that if all adjoint endomorphisms $\operatorname{ad} x$ with $x \in L$ are nilpotent then L is a nilpotent Lie algebra.

[7 marks]

(iv) Define the *radical*, $\operatorname{rad} L$, of the Lie algebra L and explain what is meant by "L is semisimple". Prove that L is semisimple if and only if it does not contain nonzero abelian ideals.

[4 marks]

(v) Let H_1 be the first *Heisenberg Lie algebra* over $\Bbbk$ which has basis $\{u, v, z\}$ such that $[u, v] = z$ and $[z, x] = 0$ for all $x \in L$, and let $R = \Bbbk[X]$ be the polynomial ring in one variable X with coefficients in $\Bbbk$. Write I for the identity operator on R and define $\mu_X, d_X \in \mathfrak{gl}(R)$ by setting $\mu_X(f(X)) = Xf(X)$ and $d_X(f(X)) = \frac{d}{dX}(f(X))$ for all $f(X) \in R$.

(1) Prove that the linear map $\rho\colon L \to \mathfrak{gl}(R)$ such that $\rho(u) = d_X$, $\rho(v) = \mu_X$ and $\rho(z) = I$ is a representation of L.

(2) By (1), we may regard R as an L-module. Prove that if $\operatorname{char}(\Bbbk) = p > 0$ then the ideal (X^p) of R (which is spanned by all X^i with $i \geq p$) is an L-submodule of R. Is the factor module R/I irreducible? Justify your answer.

[6 marks]

B3. Let L be a finite dimensional Lie algebra over an algebraically closed field $\mathbb{k}$.

(i) Define the *Killing form* κ of L and show that κ is L-invariant. What is $\operatorname{Rad}\kappa$, the *radical of* κ? Show that $\operatorname{Rad}\kappa$ is an ideal of L.

[6 marks]

(ii) Suppose in this part that $L = H_1$, the first Heisenberg algebra defined in the previous question. What is the radical of the Killing form of L? Justify your answer.

[3 marks]

(iii) Let I be an ideal of L. Show that the Killing form of the Lie algebra I coincides with the restriction of κ to $I \times I$.

[6 marks]

(iv) Suppose in this part that $\operatorname{char}(\mathbb{k}) = 0$. Let R be the radical of the Killing form κ of L. State Cartan's criterion for L in the form involving the Killing form of L and use it to prove that R is a solvable ideal of L.

[6 marks]

(v) Let $L = \mathfrak{sl}(2, \mathbb{k})$ and let κ be the Killing form of L.

(1) Compute the matrix of κ with respect to the standard basis $S = \{e, h, f\}$ of L and show that κ is non-degenerate if and only if $\operatorname{char}(\mathbb{k}) \neq 2$.

(2) Prove that if $\operatorname{char}(\mathbb{k}) = 2$ then $\mathfrak{sl}(2, \mathbb{k}) \cong H_1$ as Lie algebras.

[6 marks]

B4. Let $\Bbbk$ be an a algebraically closed field and let V be a finite dimensional vector space over $\Bbbk$.

(i) Let $x \in \mathfrak{gl}(V)$. When is x called *semisimple*? When is x called *nilpotent*? Define the *Jordan decomposition* of x and explain why such a decomposition is unique.

[6 marks]

(ii) Suppose $\dim V = 2$ and let $B = \{u, v\}$ be a basis of V. Let $x \in \mathfrak{gl}(V)$ be such that $x(u) = u$ and $x(v) = u + v$. Find the matrix of x with respect to B and determine the semisimple part x_s and the nilpotent part x_n of x.

[4 marks]

(iii) When is V called an L-*module*? When is an L-module called *completely reducible*? State *Weyl's theorem* on representations of semisimple Lie algebras.

[6 marks]

(iv) Let A be a finite dimensional algebra over $\Bbbk$ (not necessarily associative or Lie). Let D be a derivation of A and let $\Lambda \subset \Bbbk$ be the set of all eigenvalues of the linear operator $D \in \mathfrak{gl}(A)$. Then $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$ where A_λ is the generalised eigenspace for D corresponding to eigenvalue λ.

(1) Prove that $A_\lambda \cdot A_\mu \subseteq A_{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda$.

(2) Describe the restriction of the semisimple part D_s of D to the generalised eigenspace A_λ and use part (1) to show that D_s is a derivation of A.

[7 marks]

(v) Suppose L is a Lie subalgebra of $\mathfrak{gl}(V)$. When is L called *separating*? Prove that the Lie subalgebra $\mathfrak{sl}(V)$ of $\mathfrak{gl}(V)$ is separating.

[4 marks]

END OF EXAMINATION PAPER

Two hours

UNIVERSITY OF MANCHESTER

GREEN'S FUNCTIONS, INTEGRAL EQUATIONS AND APPLICATIONS

19 May 2014

14:00 - 16:00

Answer ALL five questions (100 marks in total)

Electronic calculators may be used, provided that they cannot store text.

Useful equation:

- For functions f and g defined in either two or three dimensions, and a domain D with smooth boundary in either two or three dimensions

$$\int_D \left(f(\mathbf{x})\nabla^2 g(\mathbf{x}) - g(\mathbf{x})\nabla^2 f(\mathbf{x})\right)\, d\mathbf{x} = \int_{\partial D} \left(f(\mathbf{x})\nabla g(\mathbf{x}) - g(\mathbf{x})\nabla f(\mathbf{x})\right)\cdot \mathbf{n}(\mathbf{x})\, ds$$

where the boundary ∂D is oriented by the outward pointing unit normal $\mathbf{n}(\mathbf{x})$ at each point $\mathbf{x}$ in ∂D.

1. This question concerns the boundary value problem

$$\begin{cases} u''(x) - u'(x) = f(x), \quad x \in (0, L), \\ u(0) + u'(0) = 0, u(L) = 0. \end{cases} \tag{1}$$

(a) Calculate the adjoint operator and adjoint boundary conditions for this boundary value problem.

[7 marks]

(b) For $L \neq \ln(2)$, find the Green's function for (1), and write an expression for the solution u of (1) using that Green's function.

[15 marks]

2. This question concerns the Fredholm integral equation

$$u(x) = \lambda \int_0^1 xy^2 u(y) \mathrm{d}y + f(x). \tag{2}$$

(a) For what values of $\lambda \in \mathbb{C}$ is there a unique solution to (2) for any continuous function f?

[7 marks]

(b) For the values of λ found in part (a), find a formula for the solution u of (2).

[7 marks]

(c) When λ is not one of the values found in part (a) find a condition on f so that there is a solution for (2).

[4 marks]

(d) Write down the first three terms in the Neumann series solution of (2) simplified as far as possible for general functions f.

[7 marks]

3. Let p be a function which is real-valued, positive, continuous, and has continuous first derivative on the interval $[0,1]$. This question concerns the boundary value problem

$$\begin{cases} \mathcal{L}u(x) := \dfrac{\mathrm{d}}{\mathrm{d}x}\left(p\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)(x) = f(x), \quad x \in (0,1) \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{3}$$

(a) For any u not identically equal to zero satisfying the boundary conditions $u(0) = 0$, $u(1) = 0$ show that

$$\langle u, \mathcal{L}u \rangle < 0.$$

[7 marks]

(b) Explain how part (a) can be used to apply the Fredholm alternative to the boundary value problem (3) in order to conclude that there exists a unique solution for any continuous function f.

[7 marks]

(c) Recall that for the case $p = 1$, the Green's function for the boundary value problem (3) is

$$G(x, x_0) = (x_0 - 1)xH(x_0 - x) + (x - 1)x_0 H(x - x_0)$$

where $H(x)$ is the Heaviside function. Use this to show that for general functions p satisfying the hypotheses, the solution of boundary value problem (3) also satisfies the integral equation

$$u(x) - \int_0^1 \mathcal{K}(x, x_0)u(x_0)\mathrm{d}x_0 = \int_0^1 \frac{G(x, x_0)}{p(x_0)} f(x_0)\mathrm{d}x_0$$

where

$$\mathcal{K}(x, x_0) = \left(\big(x - H(x - x_0)\big)\frac{\mathrm{d}}{\mathrm{d}x_0}\big(\ln(p)\big)(x_0) + G(x, x_0)\frac{\mathrm{d}^2}{\mathrm{d}x_0^2}\big(\ln(p)\big)(x_0)\right).$$

Hint: Note that

$$\frac{d}{dx_0}\big(\ln p\big)(x_0) = \frac{p'(x_0)}{p(x_0)}.$$

[6 marks]

4. The *free-space Green's function* for the Laplacian operator $\mathcal{L} = \nabla^2$ in three dimensions is

$$G_{3\infty}(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} = -\frac{1}{4\pi} \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}.$$

(a) Confirm that this function satisfies Laplace's equation when $\mathbf{x} \neq \mathbf{x}_0$. **Hint:** You can use that, for $\mathbf{x} \neq \mathbf{x}_0$,

$$\nabla_{\mathbf{x}} |\mathbf{x} - \mathbf{x}_0| = \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}.$$

[6 marks]

(b) Let D be a bounded domain with smooth boundary in $\mathbb{R}^3$ and suppose that for all $\mathbf{x}$ in the interior of D the function $V(\mathbf{x}, \mathbf{x}_0)$ satisfies the boundary value problem

$$\begin{cases} \nabla^2_{\mathbf{x}_0} V(\mathbf{x}, \mathbf{x}_0) = 0, \quad \mathbf{x}_0 \in D, \\ V(\mathbf{x}, \mathbf{x}_0)|_{\mathbf{x}_0 \in \partial D} = -G_{3\infty}(\mathbf{x}, \mathbf{x}_0)|_{\mathbf{x}_0 \in \partial D}. \end{cases}$$

We call $G(\mathbf{x}, \mathbf{x}_0) = G_{3\infty}(\mathbf{x}, \mathbf{x}_0) + V(\mathbf{x}, \mathbf{x}_0)$ the *Dirichlet Green's function for D*. Show that for any function $u(\mathbf{x})$ twice continuously differentiable on D

$$u(\mathbf{x}) = \int_D G(\mathbf{x}, \mathbf{x}_0) \nabla^2 u(\mathbf{x}_0) \mathrm{d}\mathbf{x}_0 + \int_{\mathbf{x}_0 \in \partial D} u(\mathbf{x}_0) \nabla_{\mathbf{x_0}} G(\mathbf{x}, \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}_0)\, ds$$

for all $\mathbf{x}$ in the interior of D where $\mathbf{n}(\mathbf{x}_0)$ is the outward point unit normal at each point $\mathbf{x}_0 \in \partial D$.

[7 marks]

5. The *free-space Green's function* for the Laplacian operator $\mathcal{L} = \nabla^2$ in two dimensions is

$$G_{2\infty}(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \ln \left(|\mathbf{x} - \mathbf{x}_0| \right) = \frac{1}{2\pi} \ln \left(\sqrt{(x - x_0)^2 + (y - y_0)^2} \right).$$

(a) Use the method of images to find a Green's function $G(\mathbf{x}, \mathbf{x}_0)$ for the two dimensional Laplacian $\mathcal{L} = \nabla^2$ on the domain $D = \{(x, y) \in \mathbb{R}^2 \ : \ x \geq 0\}$ such that $G(\mathbf{x}, \mathbf{x_0}) = 0$ for $\mathbf{x}_0$ on the boundary of this domain.

[7 marks]

(b) Ignoring the contribution from the integral at infinity, use the answer to part (a) to show that a solution of the boundary value problem

$$\begin{cases} \nabla^2 u(x, y) = 0, & x > 0, \\ u(0, y) = h(y), & y \in \mathbb{R} \end{cases} \tag{4}$$

is given by

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} h(s) \frac{x}{x^2 + (y - s)^2} \, ds. \tag{5}$$

[8 marks]

(c) Use equation (5) to show that when $h(y) = H(y) - H(y - 1)$, where $H(y)$ is the Heaviside function, a solution of (4) is

$$u(x, y) = \frac{1}{\pi} \left(\operatorname{atan} \left(\frac{y}{x} \right) - \operatorname{atan} \left(\frac{y - 1}{x} \right) \right)$$

[5 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

WAVE MOTION

04 June 2014

09:45 – 11:45

Answer **ALL** six questions.

Electronic calculators are permitted, provided they cannot store text.

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1. You are given that a harmonic signal (in the usual notation)

$$y(x,t) = A\exp\Big\{i(kx - \omega t)\Big\}\,,$$

satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} + aky - bk\frac{\partial^2 y}{\partial x^2} = 0\,.$$

Here a and b are positive constants, x is a spatial coordinate and t is time.

Obtain the dispersion relation and sketch the graph of c^2 as a function of k, where c is the wave speed. Obtain the group velocity c_g and show that there is a critical wavenumber $k = k_c$ at which $(c_g/c - 1)$ changes sign. Find the value k_c in terms of a and b.

[8 marks]

2. A plane wave signal, f, takes the form:

$$f(x,t) = A\exp\Big\{i\Big(Kx - \omega(K)t\Big)\Big\}\,,$$

where A is a constant amplitude, $K > 0$ is the wavenumber, $\omega(K)$ is the frequency, x is a spatial coordinate and t is time.

Consider two such plane waves of equal amplitude, but with slightly different wavenumbers $K = k + \Delta$ and $K = k - \Delta$, where Δ is small.

(i) Show that the sum of these two waves can be viewed as a slow amplitude modulation of a plane wave of wavenumber k. Define the 'envelope' and 'carrier' waves in the resulting sum.

(ii) Give the speed of propagation of the envelope of this amplitude modulated wave.

(iii) Sketch the (real part of the) signal at a fixed instant in time, indicating the direction of propagation for $\omega > 0$.

[8 marks]

3. Two-dimensional surface-wave motions are taking place on a semi-infinite layer ($z > 0$) of incompressible fluid. When surface tension T is introduced, the appropriate linearised conditions at the free surface ($z = 0$) are

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{and} \quad -\rho\frac{\partial \phi}{\partial t} = -\rho g \eta + T\frac{\partial^2 \eta}{\partial x^2}\,.$$

Here ρ is the constant fluid density, g is the local gravitational acceleration, $\phi(x, z, t)$ is the velocity potential and $z = \eta$ describes the free surface displacement during the motion.

(i) Show that the free surface condition, including the effects of surface tension, is

$$\frac{\partial^2 \phi}{\partial t^2} = g\frac{\partial \phi}{\partial z} - \frac{T}{\rho}\frac{\partial^3 \phi}{\partial^2 x \partial z} \quad \text{on } z = 0\,.$$

(ii) State the equation satisfied by $\phi(x, z, t)$ in the fluid region $0 < z < \infty$.

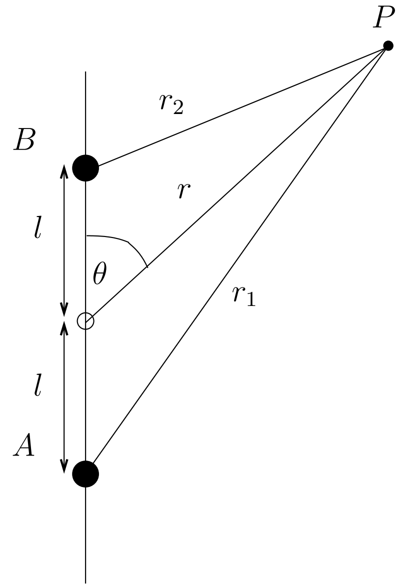
(iii) A travelling surface wave exists on this fluid layer, and has a velocity potential of the form

$$\phi(x, z, t) = \Phi(z)\cos(Kx - \omega t)\,,$$

where K and ω are positive constants. Determine $\Phi(z)$ such that Φ decays as $z \to \infty$. Obtain the dispersion relation for such a wave when surface tension effects are included.

(iv) If such a wave has a phase speed of U, obtain the possible wavelength(s) of the wave as a function of g, T, ρ and U.

[12 marks]



A schematic diagram for question 4.

4. Two monopoles (A, B), of equal and opposite sign, are separated by a distance $2l$ and placed in a compressible fluid of sound speed c, as shown in figure 1. A solution of the three-dimensional wave equation exists in the form of a velocity potential:

$$\phi(r, \theta, t) = \frac{1}{4\pi r_1} m(t - r_1/c) - \frac{1}{4\pi r_2} m(t - r_2/c)\,,$$

for a given function m, where θ, r, r_1 and r_2 are as shown in the figure.

By considering the limit of $l/r \to 0$, where $2l\, m(t) \to d(t)$, give a simplified expression for ϕ in terms of $d(t)$.

Hint: The cosine rule for a triangle of side lengths a, b, c, gives

$$c^2 = a^2 + b^2 - 2ab\cos\alpha\,,$$

where α is the angle between the sides of length a and b.

[12 marks]

5. A compressible fluid of sound speed c fills a pipe with rigid impermeable walls.

(i) State the equation to be satisfied in the fluid and the boundary condition to be imposed at the walls for a velocity potential ϕ. Hence verify that, for a pipe with a constant cross section, a velocity potential solution exists in the form

$$\phi(x,t) = B\exp\Big\{i\omega(t - x/c)\Big\}\,,$$

for any constant B, where ω is a frequency, t is time and x is measured along the axis of the pipe.

(ii) The pressure perturbation associated with this velocity potential is

$$p = -\rho_0\frac{\partial\phi}{\partial t}\,,$$

for a gas of mean density ρ_0. Give a corresponding expression (in terms of the velocity potential) for the downstream volume flux $Q = Au$, where u is the velocity component along the pipe and A the cross sectional area of the pipe.

(iii) Now consider a pipe that has a piecewise constant cross sectional area, being A_1 for $x \le 0$, A_2 for $x > 0$, with a velocity potential of the form

$$\phi(x,t) = \phi_I(x,t) + \phi_R(x,t)\,,$$

for $x \le 0$ and

$$\phi(x,t) = \phi_T(x,t)\,,$$

for $x > 0$. For an incoming wave $\phi_I(x,t) = B_I\exp\{i\omega(t - x/c)\}$, give similar forms for ϕ_T and ϕ_R. Hence determine the amplitudes of ϕ_T and ϕ_R relative to ϕ_I, given that p and Q must be continuous at $x = 0$.

(iv) Interpret your results in the special cases $A_2 \ll A_1$ and $A_2 \approx A_1$.

[20 marks]

6. An incompressible, inviscid fluid layer of depth h is in (linear) irrotational motion. The fluid occupies the space $x \ge 0$, and $0 < z < h$, with z measured downwards into the fluid. The free surface has a mean position $z = 0$ and the linearised condition to be imposed there is

$$\frac{\partial^2\phi}{\partial t^2} = g\frac{\partial\phi}{\partial z}\,,$$

where ϕ is the velocity potential, g is the local gravitational acceleration and t denotes time.

(i) If $z = h$ is a rigid, impermeable base, state the appropriate boundary condition to be applied there. State the governing equation to be satisfied in the fluid.

(ii) By looking for a solution in the form $\phi(x,z,t) = Z(z)\cos(Kx - \omega t)$ where $K, \omega > 0$, obtain the appropriate solution for $Z(z)$. Give the corresponding dispersion relation. How many possible wavelengths can satisfy this dispersion relation for a fixed wave frequency ω?

(iii) By seeking a more general separable form of $\phi(x,z,t) = X(x)Z(z)\cos(\omega t)$, find alternative solutions to $X(x)$ and $Z(z)$ such that ϕ remains bounded in the fluid. Give the corresponding dispersion relation. How many possible wavelengths can satisfy this dispersion relation for a fixed wave frequency ω?

(iv) Hence give the most general form of solution as a linear combination.

[20 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL BIOLOGY

30 May 2014

9:45–11:45

Answer **ALL THREE** questions in Section A (30 marks in total).
Answer **TWO** of the THREE questions in Section B (50 marks in total).
If all THREE questions from Section B are attempted,
credit will be given for the TWO best answers.

Electronic calculators are permitted, providing they cannot store text.

SECTION A

Answer **ALL** 3 questions

A1. One way to control an insect pest is to release sterile insects. If, through steady release, we maintain a constant population n of sterile insects within the wild population, then a model for the population $N(t)$ of fertile insects is

$$\frac{dN}{dt} = N\left(\frac{aN}{N+n} - b\right) - cN(N+n) \tag{1}$$

where $a > b > 0$, $c > 0$ and $n > 0$ are parameters.

(a) Consider the case where there are no sterile insects and show that (1) reduces to the logistic growth law

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$

Your answer should include formulae for the growth rate r and the carrying capacity K in terms of the parameters a, b and c.

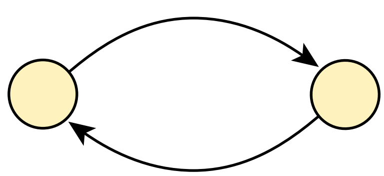
(b) Determine the critical number n_c of sterile insects whose addition would guarantee the pest's extinction.

[10 marks]

A2. Explain what is meant by the following terms

- a *motif* in a genetic regulatory network
- a *coherent 3-node feed-forward loop*: your answer should include a sketch of such a loop.

In a randomly-assembled regulatory network with $N = 5$ nodes and $E = 3$ interactions, what is the probability of finding an example of the subgraph illustrated below?



[10 marks]

A3. The following differential delay equation (DDE) models the growth of a population whose birth rate at time t depends on both the current population $N(t)$ and on the population at an earlier time, $N(t-T)$

$$\frac{dN}{dt} = rN(t-T)\left(1 - \frac{N(t)}{K}\right). \tag{2}$$

Here the parameters r, K and T are all positive real numbers.

(a) Explain briefly what the parameters r and K mean.

(b) Show that by a suitable choice of dimensionless variables x and τ one can reduce the DDE (2) to the form

$$\frac{dx}{d\tau} = x(\tau - \widehat{T})\,(1 - x(\tau)) \tag{3}$$

and give expressions for x, τ and $\widehat{T}$ in terms of N, t, r, K and T.

(c) Write (3) in the form

$$\frac{dx(\tau)}{d\tau} = f(x(\tau), x(\tau - \widehat{T}))$$

and then, by linearizing the function f, show that small perturbations $x(\tau) = 1 + \eta(\tau)$ evolve according to the linearized dynamics

$$\frac{d\eta(\tau)}{d\tau} = -\eta(\tau) \tag{4}$$

(d) Show that (4) implies that all perturbations of the form $\eta(\tau) = \eta_0 e^{\lambda\tau}$ die away exponentially: what does this suggest about the possibility of oscillatory solutions near the equilibrium $x_\star = 1$?

[10 marks]

SECTION B

Answer **2** of the 3 questions

B4. The equation below models a population that is self-limiting, but has spatial structure

$$\partial_t u = D\partial_{xx} u + \mu u(1-u). \tag{5}$$

Consider this PDE with the boundary conditions $u(0,t) = u(L,t) = 0$.

(a) Find the ODE obeyed by a steady state solution to (5).

(b) By introducing $v = du/dx$, reduce the ODE from part (a) to a system of two first order equations.

(c) Sketch the null clines of the system from part (b).

(d) Show that

$$H(u,v) = \frac{v^2}{2} + \frac{\mu}{2D}\left[u^2 - \frac{2u^3}{3}\right]$$

is a constant of the motion for the system of ODEs from part (b) in the sense that if $((u(x), v(x))$ is a solution to the system with initial values $(u(0), v(0)) = (u_0, v_0)$ then

$$H(u(x), v(x)) = H(u_0, v_0).$$

(e) For the case $\mu = D = 1$, find and classify (as local minima, local maxima or saddles) all the critical points of the function $H(u,v)$, then use these results to sketch a contour map of $H(u,v)$.

(f) Relate the contour map from part (e) to solutions of the corresponding ODE from part (a). In particular, if $(u(x), v(x))$ is a solution to the ODE from part (a), how does the curve traced by $(u(x), v(x))$ as x varies $0 \leq x \leq L$ appear on your contour map?

[25 marks]

B5. The following system of equations describes the progress of an epidemic of an infectious disease such as meningitis. Imagine that there is a total population of N_0 people that we can divide among two categories:

Susceptible people who do not currently have the disease

Infectious people who currently have the disease and so can infect susceptibles

If we say that the numbers of persons in these two groups are $S(t)$ and $I(t)$, respectively, then one commonly-used model is defined by the ODEs

$$\frac{dS}{dt} = -\beta IS + \gamma I, \qquad \frac{dI}{dt} = \beta IS - \gamma I \tag{6}$$

where the parameters β and γ are positive real numbers.

(a) Show that if the initial population is $N_0 = S(0) + I(0)$ then $S(t) + I(t) = N_0$ for all times $t \geq 0$.

(b) Explain what the parameters β and γ mean and describe the course of the disease implied by (6): in particular, what fates are possible for those who become infected?

(c) Show that by a suitable choice of dimensionless variables one can reduce the system (6) to the form

$$\frac{du}{d\tau} = -(R_0 u - 1)v \qquad \text{and} \qquad \frac{dv}{d\tau} = (R_0 u - 1)v \tag{7}$$

where $u(\tau)$ and $v(\tau)$ are nonnegative quantities satisfying $u(\tau) + v(\tau) = 1$. Your answer should include expressions for u, v, τ and R_0 in terms of the original variables and parameters.

(d) Using the results from parts (a)–(c), or by any other means, show that (7) is equivalent to

$$\frac{dv}{d\tau} = (R_0 - 1)v - R_0 v^2 \tag{8}$$

(e) A disease is said to be *endemic* if there is an equilibrium $(S_\star, I_\star)$ for (6) in which $I_\star > 0$. Prove the following

Theorem (Threshold Theorem of Epidemiology)**.** *A disease obeying (6) is endemic if and only if $R_0 > 1$, where R_0 is as in (7).*

Your answer should include an analysis of the stability of the equilibria of (8).

[25 marks]

B6. Consider the system of reactions

$$\begin{aligned} X + X &\underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} D \\ X + D &\underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} T \end{aligned} \tag{1}$$

which describes the formation of dimers D and trimers T from monomers X. Here $k_{\pm 1}$ and $k_{\pm 2}$ are rate coefficients for the Law of Mass Action.

(a) Construct a stoichiometric matrix N for the system of reactions (1), explaining your work thoroughly.

(b) Using the Law of Mass Action, write down a system of ODEs that describes the temporal evolution of the concentrations of the chemical species in (1).

(c) By performing Gaussian elimination on N, or by any other means, find a complete set of conserved quantities of the form

$$a_1[X] + a_2[D] + a_3[T]$$

for (1). Here square brackets indicate concentrations, so, for example, $[X]$ is the concentration of monomers.

(d) Find the rank r of the stoichiometric matrix N and decompose N as a product $N = LN_R$ where N_R, the reduced stoichiometric matrix, has full rank and L is of the form

$$L = \begin{bmatrix} I_r \\ L_0 \end{bmatrix},$$

where I_r is the $r \times r$ identity matrix. Your answer should include expressions for L, L_0 and N_R.

Now imagine that we are interested in a very small, isolated volume μ in which there are so few molecules that the ODEs found in part (b) become unrealistic and we are obliged to model the reactions using the Gillespie algorithm.

(e) Suppose that at $t = 0$ there are 4 molecules of the monomer X and none of species D or T. Using the notation (N_X, N_D, N_T), where N_X, N_D and $N_T \in \mathbb{N}$ are non-negative integers giving, respectively, the number of monomers, dimers and trimers, list the set of states the system can occupy.

(f) Using the notation $p_{i,j,k}(t)$ to indicate the probability that the system is, at time t, in a state with i copies of the monomer, j copies of the dimer and k copies of the trimer, write down a system of differential equations for $p_{i,j,k}(t)$.

[25 marks]

END OF EXAMINATION PAPER

2 hours

UNIVERSITY OF MANCHESTER

MATHEMATICAL MODELLING AND REACTIVE FLOW

28 May 2014

14:00 - 16:00

Answer ALL THREE questions.

Electronic calculators may be used, provided that they cannot store text.

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1. Consider the following (non-dimensional) model for the propagation of a premixed flame in a combustible solid:

$$\begin{aligned} \frac{d\theta}{dx} &= \frac{d^2\theta}{dx^2} + \mathrm{Da}\, y_F \exp[\beta(\theta-1)] \\ \frac{dy_F}{dx} &= -\mathrm{Da}\, y_F \exp[\beta(\theta-1)] \\ & \quad y_F = 1\,, \quad \theta = 0 \quad \text{as} \quad x \to -\infty \\ & \quad y_F = 0\,, \quad \theta = 1 \quad \text{as} \quad x \to +\infty \end{aligned}$$

where y_F is the fuel mass fraction, θ the temperature, Da a positive eigenvalue of the problem (proportional to the inverse of the square of the propagation speed) and β the Zeldovich number. It is assumed that β is large and its inverse is used as an expansion parameter; orders in the expansions will be indicated by superscripts.

1. By neglecting the reaction term outside the thin reaction layer, assumed to be located around the origin, determine the outer profiles for θ and y_F. Choose the origin of x to coincide with the location where the outer profiles of θ intersect.

[6 marks]

2. Derive the leading order equations for the inner problem using the inner coordinate $X = \beta x$ and the inner expansions

$$\theta = \Theta(X) = 1 + \Theta^1(X)/\beta + \cdots\,, \quad y_F = F(X) = F^0(X) + F^1(X)/\beta + \cdots$$

and determine the conditions on Θ^1 and F^0 as $X \to \pm\infty$ required by the matching with the outer solutions.

[10 marks]

3. Show that F^0 can be eliminated and write down the differential equation satisfied by Θ^1 and its boundary conditions.

[5 marks]

4. Find the eigenvalue Da to leading order.

[5 marks]

2. Consider the following (non-dimensional) model for a spherical diffusion flame:

$$
\begin{aligned}
\frac{S}{r^2}\frac{dy_F}{dr} &= \frac{1}{r^2}\frac{d}{dr}(r^2\frac{dy_F}{dr}) - \mathrm{Da}\,\omega \\
\frac{S}{r^2}\frac{dy_O}{dr} &= \frac{1}{r^2}\frac{d}{dr}(r^2\frac{dy_O}{dr}) - s\mathrm{Da}\,\omega \\
\frac{S}{r^2}\frac{d\theta}{dr} &= \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\theta}{dr}) + \mathrm{Da}\,\omega
\end{aligned}
$$

with the boundary conditions

$$
\begin{aligned}
&y_F = 1\,, \quad y_O = 0\,, \quad \theta = 0 \qquad \text{at} \quad r = 1\,, \\
&y_F = 0\,, \quad y_O = 1\,, \quad \theta = 0 \qquad \text{as} \quad r \to \infty\,.
\end{aligned}
$$

Here y_F and y_O are the fuel and oxidiser mass fractions, θ the temperature, $r \geq 1$ the radial coordinate, S the fuel supply rate, Da the Damköhler number, s the stoichiometric coefficient and ω the reaction rate. It is assumed that $\omega = y_F y_O \exp(\beta(\theta - 1))$ where β is the Zeldovich number.

1. Determine the mixture fraction Z as a function of the spatial coordinate r.

[6 marks]

2. Express $\theta + y_F$ and $y_F - y_O/s$ in terms of Z and determine the stoichiometric mixture fraction Z_s.

[6 marks]

3. In the frozen limit $\mathrm{Da} = 0$, determine and plot θ, y_F and y_O versus Z and versus r.

[4 marks]

4. In the Burke-Schumann limit $\mathrm{Da} \to \infty$, determine and plot θ, y_F and y_O versus Z. Determine the flame temperature θ_{flame} and its location r_{flame}.

[9 marks]

3. Consider the following one-dimensional model for heat-diffusion in a semi-infinite bar heated from $x = 0$:

$$\begin{aligned}
\frac{\partial T}{\partial t} &= D\frac{\partial^2 T}{\partial x^2} && 0 < x < \infty \\
T &= 0 && \text{at} \quad t = 0 \\
\frac{\partial T}{\partial x} &= -Q && \text{at} \quad x = 0 \\
T &= 0 && \text{as} \quad x \to \infty\,.
\end{aligned}$$

Here T is the temperature, D the heat-diffusivity, and Q a positive constant representing the prescribed heat-flux.

1. Use dimensional analysis to show that $T(x,t)$ must take the form

$$T(x,t) = QxF(\eta) \qquad \text{where} \quad \eta = \frac{x}{2\sqrt{Dt}}\,.$$

[6 marks]

2. Write down the differential equation satisfied by $F(\eta)$ and its boundary conditions.

[6 marks]

3. Find the solution $T(x,t)$ of the problem.

[7 marks]

END OF EXAMINATION PAPER

Two Hours

THE UNIVERSITY OF MANCHESTER

NUMERICAL ANALYSIS 2

3 June 2014

1400 - 1600

Answer **ALL** five questions in **SECTION A** (40 marks in total)
and
Answer **TWO** of the three questions in **SECTION B** (20 marks each)
If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

Electronic calculators are permitted provided that they cannot store text.

SECTION A

Answer **ALL** five questions

A1. Let $T_3(x)$ denote the Chebyshev polynomial of degree 3.

(a) State the leading term (the term with the highest power of x) of $T_3(x)$ and list all the values of x in $[-1, 1]$ such that $T_3(x) = \pm 1$.

(b) State the Chebyshev Equioscillation Theorem concerning the best L_∞ approximation $p_n(x)$ (from the set of polynomials of degree $\leq n$), to a given function $f \in C[-1, 1]$.

(c) Write down the best L_∞ approximation to $f(x) = 1 - 3x^2 + 2x^3$ on $[-1, 1]$ from the set of polynomials of degree less than or equal to two. Use parts (a) and (b) to justify your answer.

[10 marks]

A2. Define

$$\langle f, g \rangle_w := \int_a^b w(x) f(x) g(x)\, dx, \qquad \| f \|_{2,w}^2 := \langle f, f \rangle_w$$

and consider the sequence of polynomials: $\phi_0(x) = 1$, $\phi_1(x) = x - \alpha_0$, and

$$\phi_{n+1}(x) = (x - \alpha_n)\phi_n(x) - \beta_n \phi_{n-1}(x), \qquad n \geq 1.$$

(a) State a choice for α_n and β_n that guarantees that $\{\phi_n(x)\}$ is a system of orthogonal polynomials with respect to $w(x)$.

(b) Derive the polynomials $\phi_0(x), \phi_1(x)$ and $\phi_2(x)$ of degrees $0, 1$ and 2, respectively, that are orthogonal with respect to $w(x) = 1$ on the interval $[-1, 1]$.

(c) Describe, very briefly, why orthogonal polynomials are beneficial in constructing best weighted L_2 approximations to functions $f(x) \in C[a, b]$.

[8 marks]

A3. Divide $[0, 1]$ into n equal intervals $[x_i, x_{i+1}]$ of width h, where $x_i = ih$, $i = 0, 1, \ldots, n$ and consider the repeated trapezium rule

$$I(f) = \int_0^1 f(x)\, dx \approx \sum_{i=0}^{n-1} \frac{h}{2}\big(f(x_i) + f(x_{i+1})\big) = T(h).$$

The Euler-Maclaurin summation formula gives an asymptotic expansion for the error of the form

$$I(f) - T(h) = a_2h^2 + a_4h^4 + a_6h^6 + \dots$$

for some real constants $a_2, a_4, a_6, \dots$.

(a) Use the Euler-Maclaurin formula to determine what happens to the error when we double the number of intervals $[x_i, x_{i+1}]$.

(b) By taking an appropriate linear combination of the rules $T(h)$ and $T(\frac{h}{2})$, derive a new quadrature scheme that gives an error that is $\mathcal{O}(h^4)$.

[7 marks]

A4. Consider the third-order ordinary differential equation

$$y'''(x) - y''(x) + y'(x) - 2xy^2(x) = 0, \qquad x \geq 0,$$

together with the initial conditions $y(0) = 2$, $y'(0) = 1$ and $y''(0) = 3$. By introducing appropriately defined variables z_1, z_2, z_3, rewrite this ODE as a system of three first-order equations. Next, apply one step of the standard Euler method with stepsize $h = 0.1$ to estimate $y(0.1)$, $y'(0.1)$ and $y''(0.1)$.

[7 marks]

A5. Consider the numerical method

$$y_{n+1} = y_n + \frac{h}{2}\big(f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\big),$$

for the initial value problem $y'(x) = f(x, y(x)), x \geq x_0$ with $y(x_0) = y_0$, where h is the stepsize and y_n is an approximation to $y(x_n)$ at the grid point $x_n = nh$ for $n = 0, 1, \dots$. Explain how the method can be interpreted as a 2-stage Runge-Kutta scheme. Derive a quadratic polynomial $p(\lambda h)$ such that the condition $|p(\lambda h)| < 1$ provides the region of absolute stability for the method.

[8 marks]

SECTION B

Answer **TWO** of the three questions

B6.

(a) State two reasons why Padé approximants offer more flexibility than single polynomials for approximating functions. In addition, explain briefly why it is beneficial to consider continued fraction representations of Padé approximants.

[4 marks]

(b) Consider

$$r_n(x) = \cfrac{c_1 x}{1 + \cfrac{c_2 x}{1 + \cfrac{c_3 x}{1 + \cdots + \cfrac{c_{n-2} x}{1 + \cfrac{c_{n-1} x}{1 + c_n x}}}}}$$

with coefficients

$$c_1 = 1, \qquad c_{2j} = \frac{j}{2(2j-1)}, \quad c_{2j+1} = \frac{j}{2(2j+1)}, \qquad j = 1, 2, \ldots$$

Verify that $r_3(x)$ is the [2/1] Padé approximant of $f(x) = \log(1+x)$.

[7 marks]

(c) The Top-Down algorithm for evaluating the continued fraction

$$r_n(x) = b_0 + \cfrac{a_1 x}{b_1 + \cfrac{a_2 x}{b_2 + \cfrac{a_3 x}{b_3 + \cdots + \cfrac{a_{n-1} x}{b_{n-1} + \cfrac{a_n x}{b_n}}}}} \tag{1}$$

is given by

$p_{-1} = 1$, $q_{-1} = 0$, $p_0 = b_0$, $q_0 = 1$
for $j = 1\colon n$
    $p_j = b_j p_{j-1} + a_j x p_{j-2}$
    $q_j = b_j q_{j-1} + a_j x q_{j-2}$
end
$r_n = p_n q_n^{-1}$

Prove (e.g., by induction) that p_n/q_n, where p_n, q_n are computed via the Top-Down algorithm does indeed return $r_n(x)$ defined by equation (1).

[9 marks]

B7. Consider the integral

$$I(f) = \int_a^b w(x)f(x)dx,$$

where $w(x) \geq 0$ on $[a,b]$ is a weight function. An n-point quadrature rule uses n weights w_i and n points x_i and has the form

$$J(f) = \sum_{i=1}^{n} w_i f(x_i).$$

(a) Let $w(x) = 1$. Define, and briefly compare, the n-point closed Newton-Cotes rule and the n-point Gauss rule for approximating $I(f)$. In particular, comment on: the choice of weights w_i, the choice of quadrature points x_i, and the degree of precision.

[6 marks]

(b) Derive the two-point Gauss-Chebyshev rule

$$I(f) = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \approx w_1 f(x_1) + w_2 f(x_2)$$

using the method of undetermined coefficients. That is, determine the choice of w_1, w_2 and x_1, x_2 to maximize the degree of precision.

HINT: You may find it useful to define a quadratic polynomial $\Phi(x) = x^2 + ax + b$ whose roots coincide with x_1 and x_2, and find a and b.

[9 marks]

(c) Newton-Cotes rules can be applied in repeated form by dividing $[a,b]$ into subintervals, and applying the rule on each one. Let $J_1(f)$ and $J_2(f)$ denote Newton-Cotes rules, such that $J_2(f)$ is more accurate than $J_1(f)$. Justify the following error estimate

$$|I(f) - J_2(f)| \approx |J_2(f) - J_1(f)|$$

and briefly explain how to construct an adaptive quadrature scheme.

[5 marks]

B8. An ℓ-step linear multistep method for the initial value problem $y'(x) = f(x, y(x))$, $x \geq x_0$, $y(x_0) = y_0$ has the form

$$y_{n+1} = \sum_{i=1}^{\ell} a_i y_{n+1-i} + h \sum_{i=0}^{\ell} b_i f_{n+1-i},$$

where a_i, b_i are given constants, h is the stepsize, and $f_n = f(x_n, y_n)$.

(a) By replacing the integrand in

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))\, dx,$$

with a suitable interpolating polynomial $p_1(x)$ of degree one, derive the two-step Adams–Bashforth method,

$$y_{n+1} = y_n + \frac{h}{2}\left[3f_n - f_{n-1}\right].$$

[6 marks]

(b) The two-step Adams–Moulton method is

$$y_{n+1} = y_n + \frac{h}{12}\left[5f_{n+1} + 8f_n - f_{n-1}\right].$$

By examining the local truncation error $\tau(h)$, determine the order of accuracy of this method.

[9 marks]

(c) Briefly describe how to develop a predictor-corrector method using the two-step Adams–Bashforth method as predictor, and the two-step Adams–Moulton method as corrector. Explain why, in general, it is beneficial to combine Adams–Bashforth and Adams-Moulton methods as predictor-corrector pairs.

[5 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

MARKOV PROCESSES

27th May 2014

14.00 - 16.00

Answer 3 questions. If more than 3 questions are attempted, credit will be given for the best 3 answers. Each question is worth 20 marks.

Electronic calculators may be used, provided that they cannot store text.

Answer 3 questions

1. a)

(i) Define the terms "recurrent state" and "transient state" in the context of discrete time Markov Chains.

[2 marks]

(ii) Define the stationary distribution of a discrete time Markov Chain.

[2 marks]

(iii) Give an example of a Markov Chain for which a stationary distribution exists yet, for some i, j, $\lim_{n\to\infty} p_{ij}(n)$ does not exist.

[2 marks]

(iv) Give an example of a Markov Chain which has more than one stationary distribution.

[2 marks]

b) Consider the discrete time Markov Chain $\{X_n : n \geq 0\}$ with state space {1,2,3,4,5} and transition probability matrix

$$\begin{bmatrix} 0.4 & 0 & 0 & 0.6 & 0 \\ 0 & 0.3 & 0 & 0 & 0.7 \\ 0.5 & 0.1 & 0.4 & 0 & 0 \\ 0.4 & 0 & 0 & 0.6 & 0 \\ 0 & 0.7 & 0 & 0 & 0.3 \end{bmatrix}$$

(i) Classify each state as either recurrent or transient.

[2 marks]

(ii) Calculate $\lim_{n\to\infty} p_{ij}(n)$ for i and $j \in \{1, 2\}$.

[3 marks]

(iii) For $n \geq 1$ let $f_{ij}(n) = P(X_n = j,\ X_k \neq j \quad for \quad 1 \leq k < n \mid X_0 = i)$. Calculate $f_{31}(n)$. Deduce that $P(X_n = 1\ for\ some\ n \geq 1 \mid X_0 = 3) = \frac{5}{6}$.

[3 marks]

(iv) Prove that $\lim_{n\to\infty} p_{31}(n) = \frac{1}{3}$.

[2 marks]

(v) Calculate the expected time of first visit to state 2 given $X_0 = 2$.

[2 marks]

2. Consider the following discrete time Markov chain. A particle moves among the state space $\{0, 1, ...\}$ according to the following transition probabilities;

$$p_{i,i+1} = p, \quad p_{ii} = 1 - p - q, \quad p_{i,i-1} = q \qquad i \geq 1;$$

$$p_{01} = p, \quad p_{00} = 1 - p.$$

where $0 < p, q < 1$ and $p + q < 1$.

(i) Draw the transition diagram.

[2 marks]

(ii) Write down the transition matrix for this Markov chain.

[3 marks]

(iii) Determine the pairs of values of p and q for which a stationary distribution exists. Calculate the stationary distribution.

[6 marks]

(iv) Stating carefully any result to which you appeal, determine, for each pair (p, q), the expected time of first return to state 0, given $X_n = 0$.

[4 marks]

(v) Calculate the expected time of the first visit to state 2, given initial state 0.

[5 marks]

3. At time 0 a population comprises 1 individual. Lifetimes of individuals are independent and each exponentially distributed with parameter λ. At death each individual splits into 2 new individuals. Let $X(t)$ denote the number of individuals alive in the population at time t.

a) (i) With reference to the distribution of the minimum of a number of independent exponential random variables, derive the generator matrix Q for this process.

[5 marks]

(ii) For $X(t) : t \geq 0$, write down the forward equations in matrix form and show they include

$$p'_{1,k}(t) = (k-1)\lambda p_{1,k-1}(t) - k\lambda p_{1,k}(t) \qquad 1 \leq k.$$

[5 marks]

b) A telephone network connects 4 subscribers. When not engaged in a conversation, each makes calls in a Poisson process with rate λ independently of the others. The recipient of a call is equally likely to be any of the other 3 subscribers. If that recipient is not engaged, a conversation begins; otherwise the call is lost. Call durations are independently exponentially distributed with parameter μ.

Let the Markov chain $X(t) : t \geq 0$, with state space $S = \{0, 1, 2\}$, denote the number of conversations in progress at time t.

(i) Stating any results to which you appeal, derive the generator matrix Q for $X(t)$ and calculate its stationary distribution.

[6 marks]

(ii) Find the expected time of first visit to state 2, starting from state 0.

[4 marks]

4. (a) Consider the following $M/M/1$ queueing model. Customers arrive at the queue in a Poisson process with rate λ. There is 1 server. Service times are independent, independent of the arrival process and exponentially distributed with parameter μi, where i is the number of customers in the system.

(i) Stating carefully any results to which you appeal, calculate the stationary distribution of the process, stating for which values of λ and μ it exists.You may quote without proof the expression for the stationary distribution of a birth-death process.

[7 marks]

(ii) In the queueing model above, with $\lambda = \mu = 1$, consider the scenario where a customer, upon finding 1 or more customers in the system on his arrival, joins the queue with probability 0.5. Determine the long-run proportion of arrivals who do not join the queue.

[7 marks]

(b) Derive an expression for the expected waiting time until start of service of a customer arriving at an $M/M/1$ queue where the arrivals are in a Poisson process with rate λ and the service times are independent, each exponentially distributed with parameter μ, where $\lambda < \mu$. You may assume that an arriving customer finds k customers in the system with stationary probability π_k.

[6 marks]

END OF EXAMINATION PAPER

Statistical Tables to be provided

Two hours

THE UNIVERSITY OF MANCHESTER

Time Series Analysis

04 June 2014
14:00 – 16:00

Electronic calculators are permitted, provided they cannot store text.

Answer **ALL** Four questions in Section A (32 marks in all)

and

Two of the Three questions in Section B (24 marks each).

If more than Two questions from Section B are attempted, then credit will be given for the best Two answers.

SECTION A
Answer ALL four questions

A1. *Akaike's information criterion* is often used to choose between competing models fitted to an observed time series of length n. The AIC statistic can be defined by the formula

$$\text{AIC} = \ln \hat{\sigma}^2 + \frac{2k}{n},$$

where $\hat{\sigma}^2$ is an optimal (maximum likelihood or equivalent) estimate of σ^2 and k is the number of estimated parameters (apart from σ^2).

a) Explain concisely the general behaviour of the two terms of the AIC statistic as the number of parameters increases.

b) The table below gives the value of the AIC statistic for several models fitted to an observed time series of length $n = 10$.

Model	AIC
$X_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$	17
$X_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$	16.5
$X_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3}$	18.5

Use the AIC criterion to select a model for this time series.

c) A modification, AIC corrected (AICC), of the AIC statistic is given by the following formula

$$\text{AICC} = \ln \hat{\sigma}^2 + \frac{2k}{n - k - 1}.$$

Calculate AICC for the models in part (b) above and select a model according to AICC.

[8 marks]

A2. Let x_t, $t = 1, \ldots, n$ be an observed time series. The Box-Ljung test is based on the statistic

$$Q_{LB}(h) = n(n+2) \sum_{k=1}^{h} \frac{\hat{\rho}_k^2}{n-k},$$

where n is the length of the observed trajectory and $\hat{\rho}_k$ are sample autocorrelations of $\{x_t\}$. Under the null hypothesis of interest the distribution of $Q_{LB}(h)$ is approximately χ^2_h.

a) State the null and alternative hypotheses in the Ljung-Box test.

b) What is the distribution of the test statistic, under the same null hypothesis, when x_t, $t = 1, \ldots, n$ represent the residuals from a fitted ARMA(p,q) model with intercept?

[8 marks]

A3. Let $X_t = \varepsilon_t + 2\varepsilon_{t-1}$, where $\{\varepsilon_t\} \sim \mathrm{WN}(0, 1)$.

a) Check if the model is invertible.

b) Find $\mathrm{Pred}(X_t|X_{t-1})$, the best linear predictor of X_t given X_{t-1}.

[8 marks]

A4. Consider the following model for a stationary process:

$$X_t = -\tfrac{3}{10}X_{t-1} + \tfrac{4}{10}X_{t-2} + \varepsilon_t - \tfrac{1}{2}\varepsilon_{t-1},$$

where $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$.

a) Write down the model in operator form.

b) Explain why this model is of higher order than necessary and give its reduced form. [8 marks]

SECTION B
Answer 2 of the 3 questions

B5.

a) A stationary time series $\{X_t\}$ follows the AR(2) model

$$X_t = 0.01 + 0.2X_{t-2} + \varepsilon_t, \tag{1}$$

where $\{\varepsilon_t\} \sim \mathrm{WN}(0, 0.02)$.

i) What property of an AR model is referred to as *causality property*?

ii) Find the mean of $\{X_t\}$.

iii) Compute the first two autocorrelations, ρ_1 and ρ_2, and the variance, γ_0, of the process $\{X_t\}$.

iv) Assume that the values $x_{100} = -0.01$ and $x_{99} = 0.02$ of the series are observed at times $t = 100$ and $t = 99$. Compute $\hat{x}_{101|100,99,\ldots}$ and $\hat{x}_{102|100,99,\ldots}$, the 1- and 2-step ahead forecasts at the forecast origin $t = 100$.

v) Give the standard deviations of the forecast errors associated with the predictors in (iv).

b) Let $\{X_t\}$ be an AR(2) process with representation

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t, \qquad \varepsilon_t \sim \mathrm{WN}(0, \sigma^2). \tag{2}$$

Let $\{Y_t\}$ be the "reversed" process defined by $Y_t = X_{-t}$. Show that $\{Y_t\}$ is also an AR(2) process with the same parameters as that of the model for $\{X_t\}$.

[24 marks]

B6.

a) Let $\{V_t\}$ be an I(1) process and $\{Z_t\}$ be a stationary process.

i) Show that $V_t + Z_t$ is not stationary. (*Hint:* You may use the fact that a linear combination of stationary processes cannot be an I(1) process.)

ii) Assume further that the time series $\{V_t\}$ and $\{Z_t\}$ are independent of each other. Show that $V_t + Z_t$ is an I(1) process.

b) Consider a random walk $\{u_t\}$ and a stationary AR(1) time series $\{e_t\}$, which evolve according to the following equations:

$$\begin{aligned} u_0 &= 0, \\ u_t &= u_{t-1} + \varepsilon_{1,t}, \qquad \text{for } t \geq 1, \\ e_t &= \rho e_{t-1} + \varepsilon_{2,t}, \qquad \text{for } t \geq 1, \end{aligned}$$

where $|\rho| < 1$ and $\{\varepsilon_{1,t}\}$ and $\{\varepsilon_{2,t}\}$ are white noise sequences, independent of each other (so, $\mathrm{E}\,\varepsilon_{1,t}\varepsilon_{2,s} = 0$ for all t, s).

Consider also processes $\{X_t\}$ and $\{Y_t\}$ which obey the following equations:

$$\begin{aligned} X_t + \beta Y_t &= u_t \\ X_t + \alpha Y_t &= e_t, \end{aligned}$$

where α and β are some constants, $\alpha \neq \beta$, and $\{u_t\}$ and $\{e_t\}$ are the processes defined above.

i) Show that

$$X_t = \tfrac{\alpha}{\alpha-\beta}u_t - \tfrac{\beta}{\alpha-\beta}e_t$$
$$Y_t = \tfrac{-1}{\alpha-\beta}u_t + \tfrac{1}{\alpha-\beta}e_t.$$

ii) Identify the processes $\{u_t\}$, $\{e_t\}$, $\{X_t\}$ and $\{Y_t\}$ as I(d) processes (e.g. I(1), I(0)). Give your reasons concisely.

iii) For each of the following pairs of processes state, giving your reasons, if they are cointegrated of order one and, if so, give a cointegrating vector.

- $\{u_t\}$ and $\{e_t\}$,
- $\{X_t\}$ and $\{Y_t\}$.

c) An analysis of two time series, $\{X_t\}$ and $\{Y_t\}$, established that they are well described by ARIMA(0,1,0) models. The Johansen test for cointegration was performed with the following results:

```
######################
# Johansen-Procedure #
######################

Test type: maximal eigenvalue statistic (lambda max), with linear trend

Eigenvalues (lambda):
[1] 0.31366569 0.04020297

Values of test statistic and critical values of test:

          test 10pct  5pct  1pct
r <= 1 |  4.02  6.50  8.18 11.65
r = 0  | 36.89 12.91 14.90 19.19

Eigenvectors, normalised to first column:
(These are the cointegration relations)

           x        y
x    1.0000000 1.000000
y   -0.9989006 1.487965
##################################################################
```

Are the two time series cointegrated? If so, give a cointegrating vector and a cointegrating relation. Explain your decision.

[24 marks]

B7.

The number of overseas visitors to New Zealand is recorded for each month over the period 1977–1995. Let x_t be the number of overseas visitors in time period t (in months) and $z_t = \ln x_t$.

a) Comment on the main features of the plots of $\{x_t\}$ and $\{z_t\}$ (see Figure 5.1). Why one might wish to build a model for $\{z_t\}$ rather than directly for $\{x_t\}$?

b) Comment on the main features of the sample autocorrelation function of $\{z_t\}$ (see Figure 5.1).

c) An ARIMA(1,1,0) model was fitted to $\{z_t\}$. Comment on the quality of the fit, using the numerical and graphical information in Figure 5.2.

d) An ARIMA$(1, 1, 0)(0, 1, 0)_{12}$ model was fitted to $\{z_t\}$, see Figure 5.3. Why was this particular modification of the model in part (c) chosen? Did it achieve the desired effect?

e) Identify the model in Figure 5.6 as a seasonal ARIMA$(p, d, q)(p_s, d_s, q_s)_s$ model. Write it in operator form.

f) Choose the best fitting model among those given in Figures 5.2–5.10. Explain the reasons for your choice.

g) Comment on the quality of the fit of the model selected in part (f) above.

Note: Write verbal answers concisely. You normally need one sentence or phrase for each worthy thought. Credit will not be given for irrelevant material.

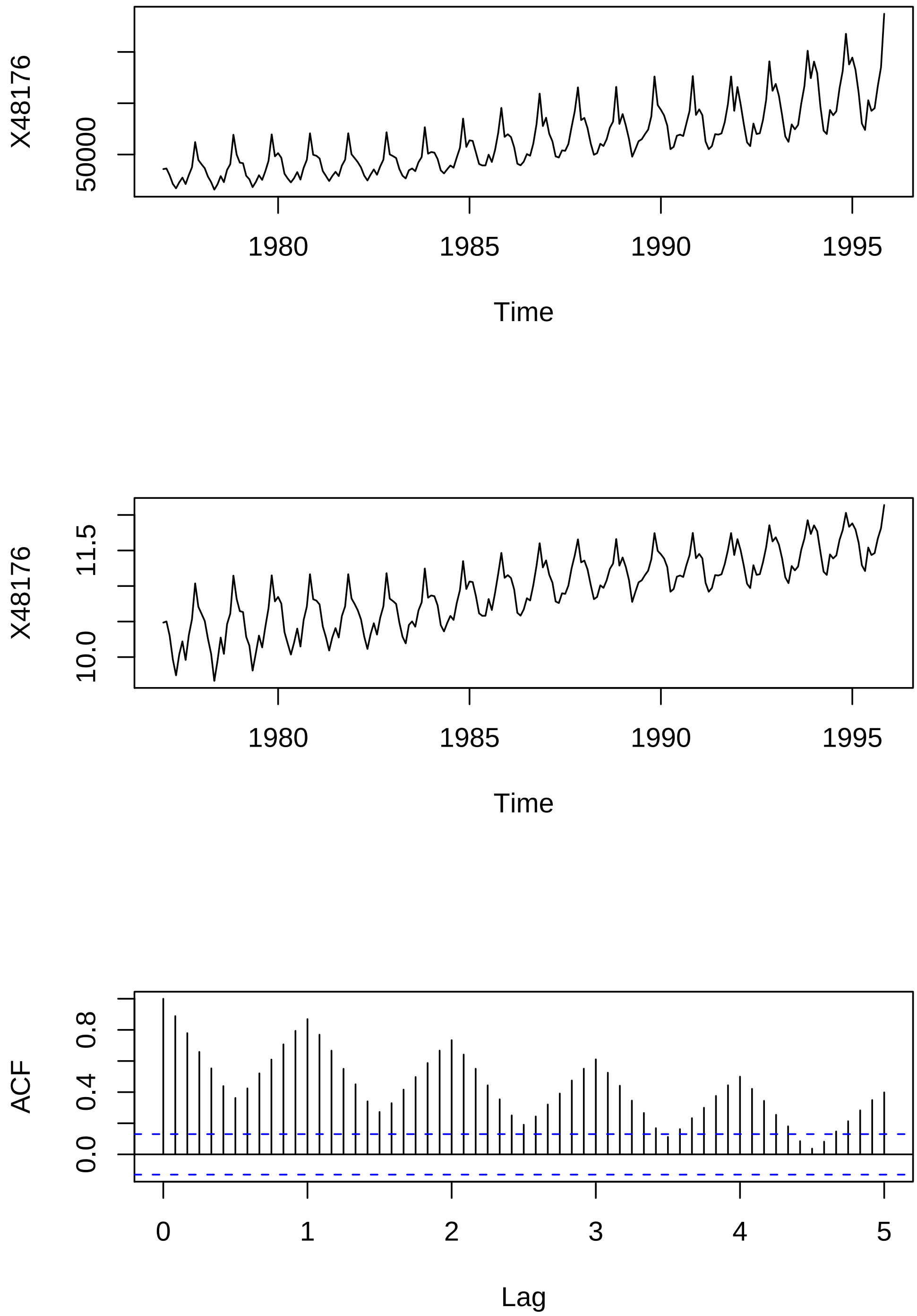

Figure 5.1: Visitors to New Zealand: the time series, $\{x_t\}$ (top), the logged time series, $\{z_t\}$ (middle), and the sample autocorrelation function of $\{z_t\}$ (bottom).

```
Call:
arima(x = visnzln, order = c(1, 1, 0))

Coefficients:
         ar1
      0.0255
s.e.  0.0667

sigma^2 estimated as 0.04985:  log likelihood = 18.17,  aic = -32.34
```

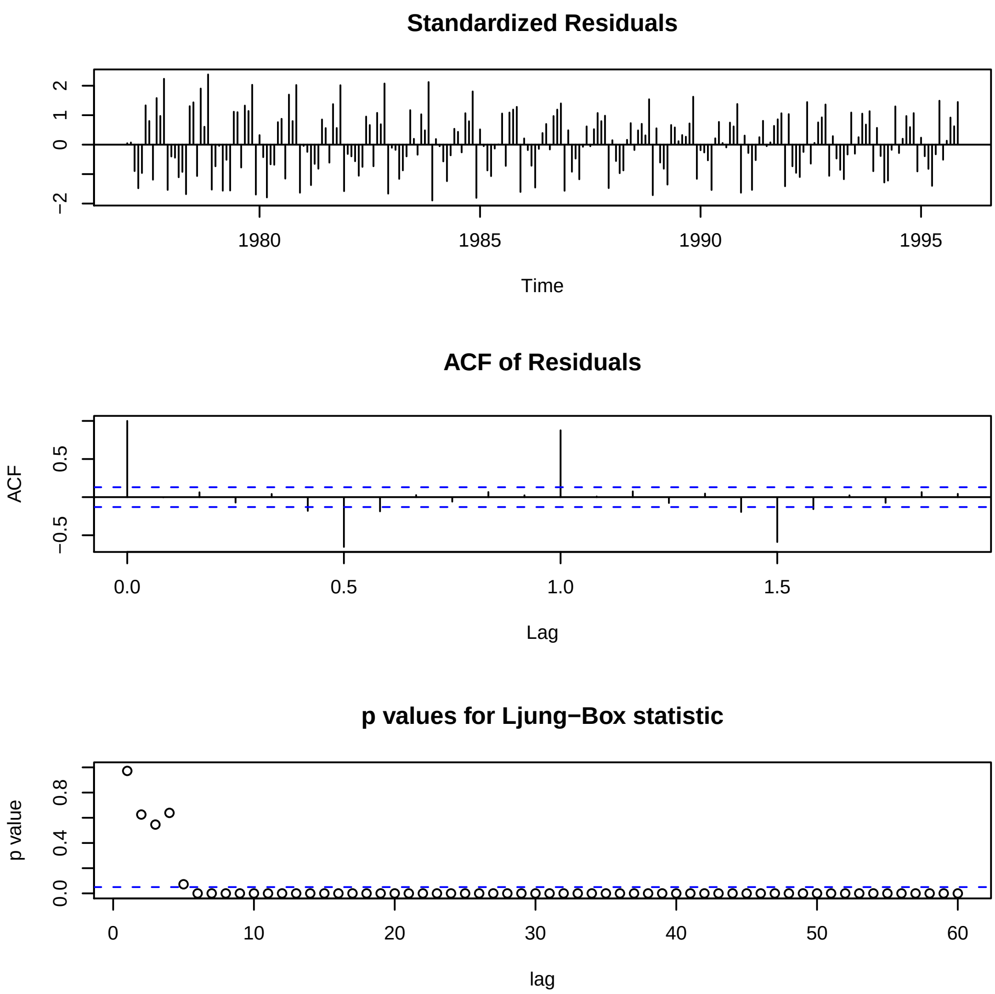

Figure 5.2: Visitors to New Zealand: Model 1

```
Call:
arima(x = visnzln, order = c(1, 1, 0), seasonal = list(order = c(0, 1, 0)))

Coefficients:
          ar1
      -0.4076
s.e.   0.0628

sigma^2 estimated as 0.005184:  log likelihood = 259.3,  aic = -514.6
```

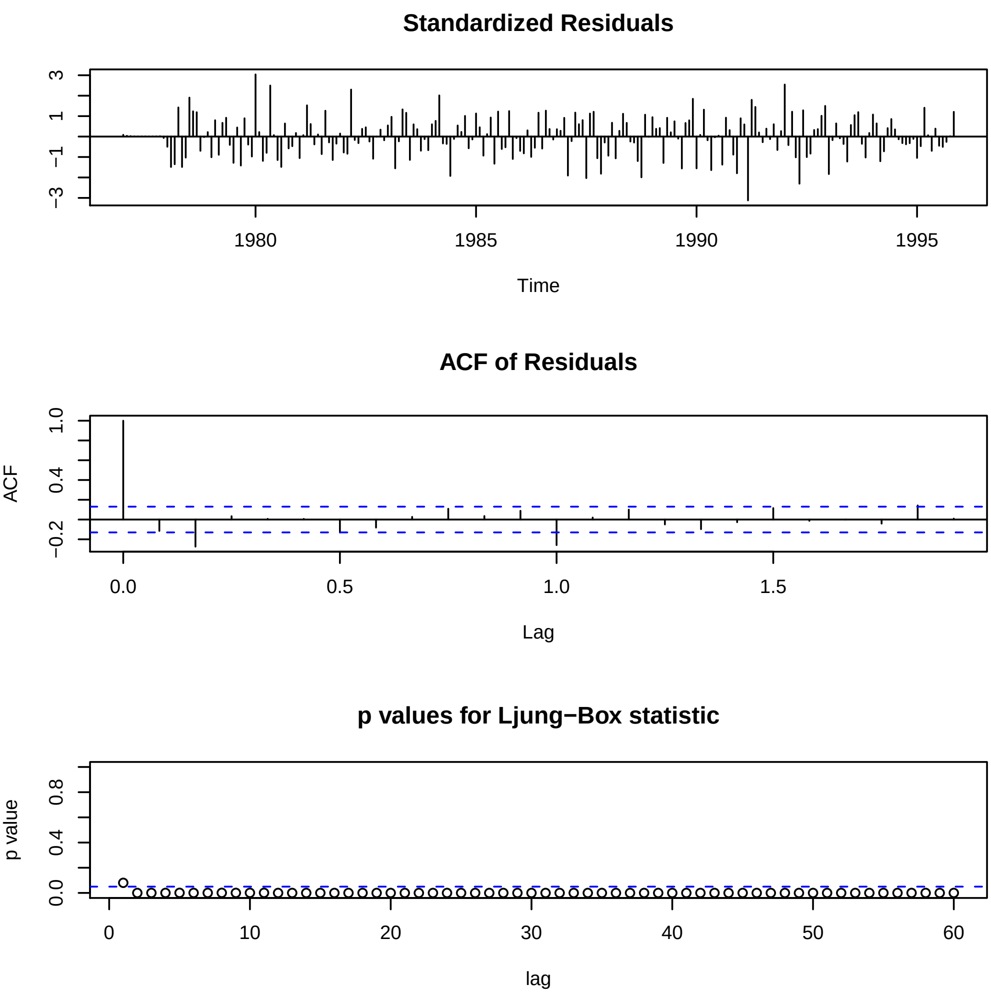

Figure 5.3: Visitors to New Zealand: Model 2

```
Call:
arima(x = visnzln, order = c(1, 1, 0), seasonal = list(order = c(1, 1, 0)))

Coefficients:
          ar1     sar1
      -0.3911  -0.2779
s.e.   0.0633   0.0672

sigma^2 estimated as 0.004784:  log likelihood = 267.44,  aic = -528.87
```

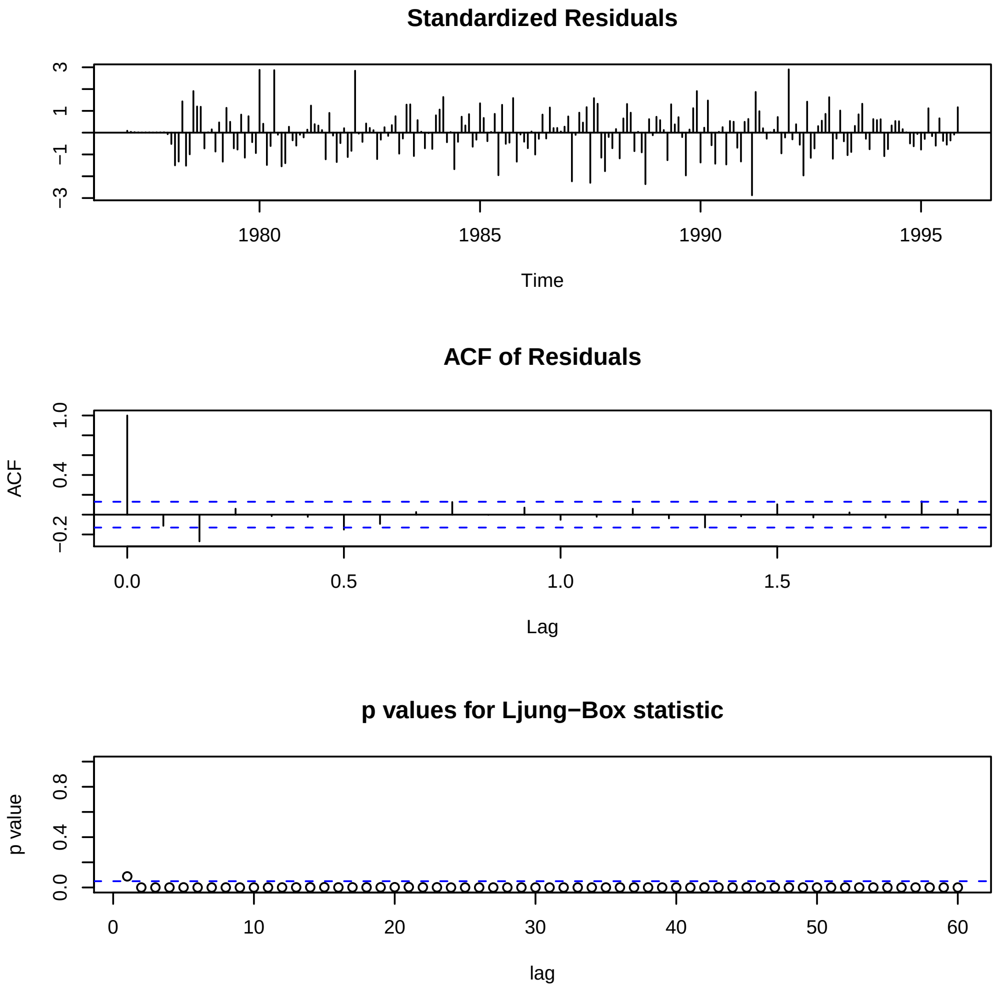

Figure 5.4: Visitors to New Zealand: Model 3

```
Call:
arima(x = visnzln, order = c(0, 1, 1), seasonal = list(order = c(0, 1, 1)))

Coefficients:
          ma1     sma1
      -0.6727  -0.3706
s.e.   0.0726   0.0725

sigma^2 estimated as 0.004050:  log likelihood = 284.64,  aic = -563.27
```

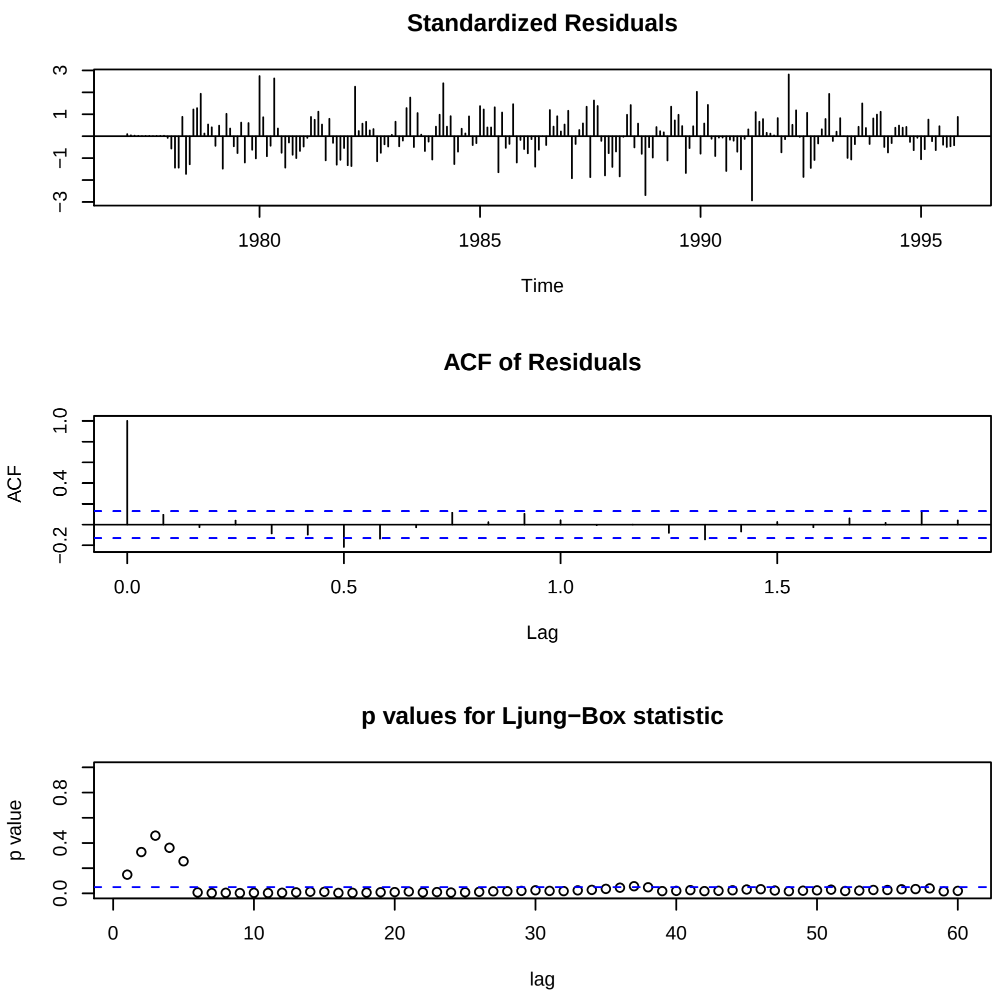

Figure 5.5: Visitors to New Zealand: Model 4


```
Call:
arima(x = visnzln, order = c(1, 1, 0), seasonal = list(order = c(0, 1, 1)))

Coefficients:
          ar1     sma1
      -0.3893  -0.4125
s.e.   0.0632   0.0706

sigma^2 estimated as 0.004556:  log likelihood = 272.01,  aic = -538.02
```

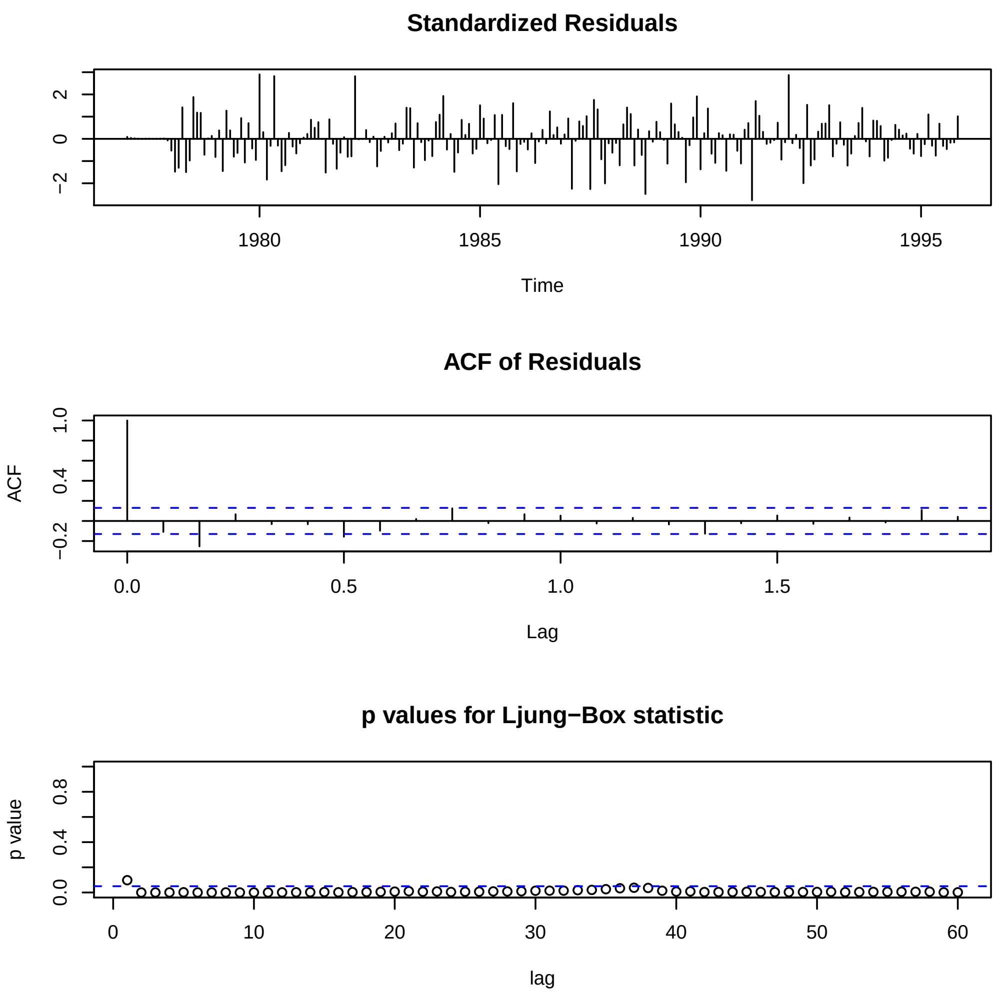

Figure 5.6: Visitors to New Zealand: Model 5

```
Call:
arima(x = visnzln, order = c(0, 1, 1), seasonal = list(order = c(1, 1, 0)))

Coefficients:
          ma1     sar1
      -0.6883  -0.2713
s.e.   0.0685   0.0674

sigma^2 estimated as 0.004182:  log likelihood = 281.6,  aic = -557.19
```

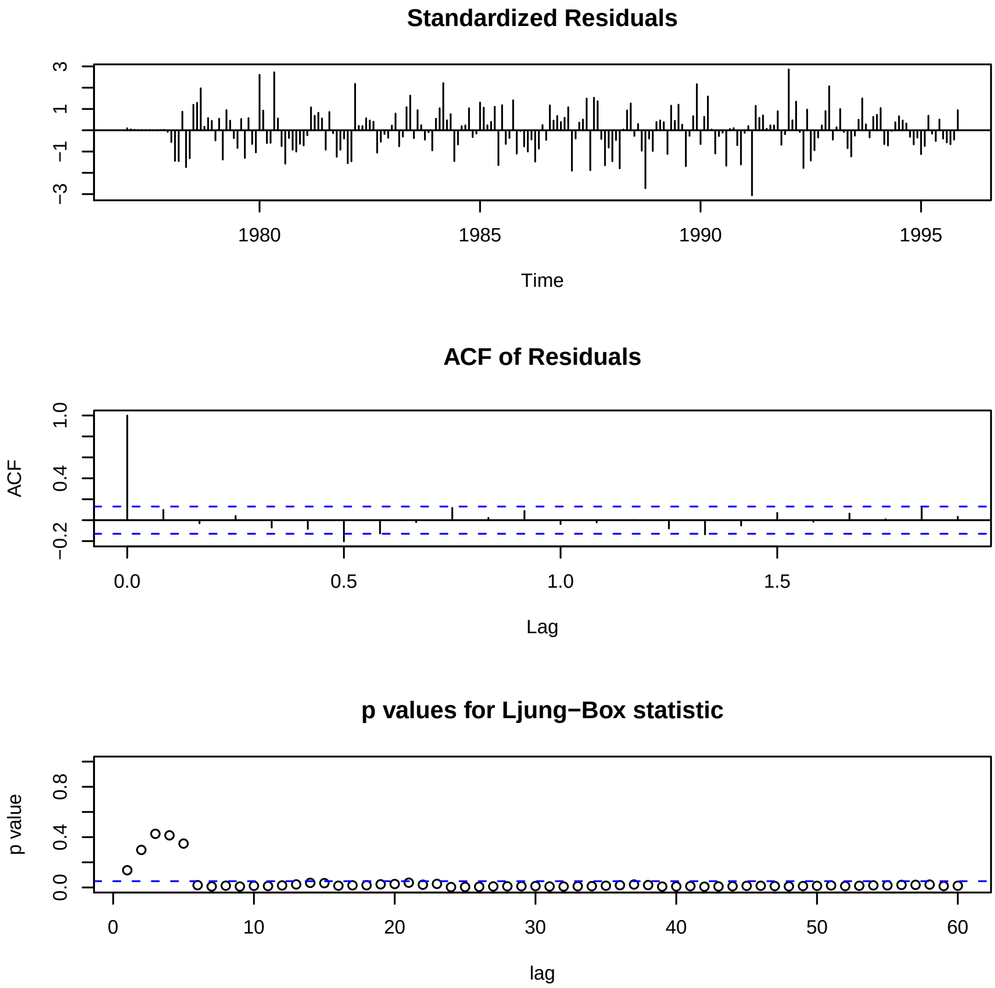

Figure 5.7: Visitors to New Zealand: Model 6

```
Call:
arima(x = visnzln, order = c(1, 1, 1), seasonal = list(order = c(1, 1, 1)))

Coefficients:
         ar1     ma1    sar1     sma1
      0.2546  -0.845  0.2297  -0.5866
s.e.  0.0962   0.059  0.1618   0.1349

sigma^2 estimated as 0.003897:  log likelihood = 288.35,  aic = -566.71
```

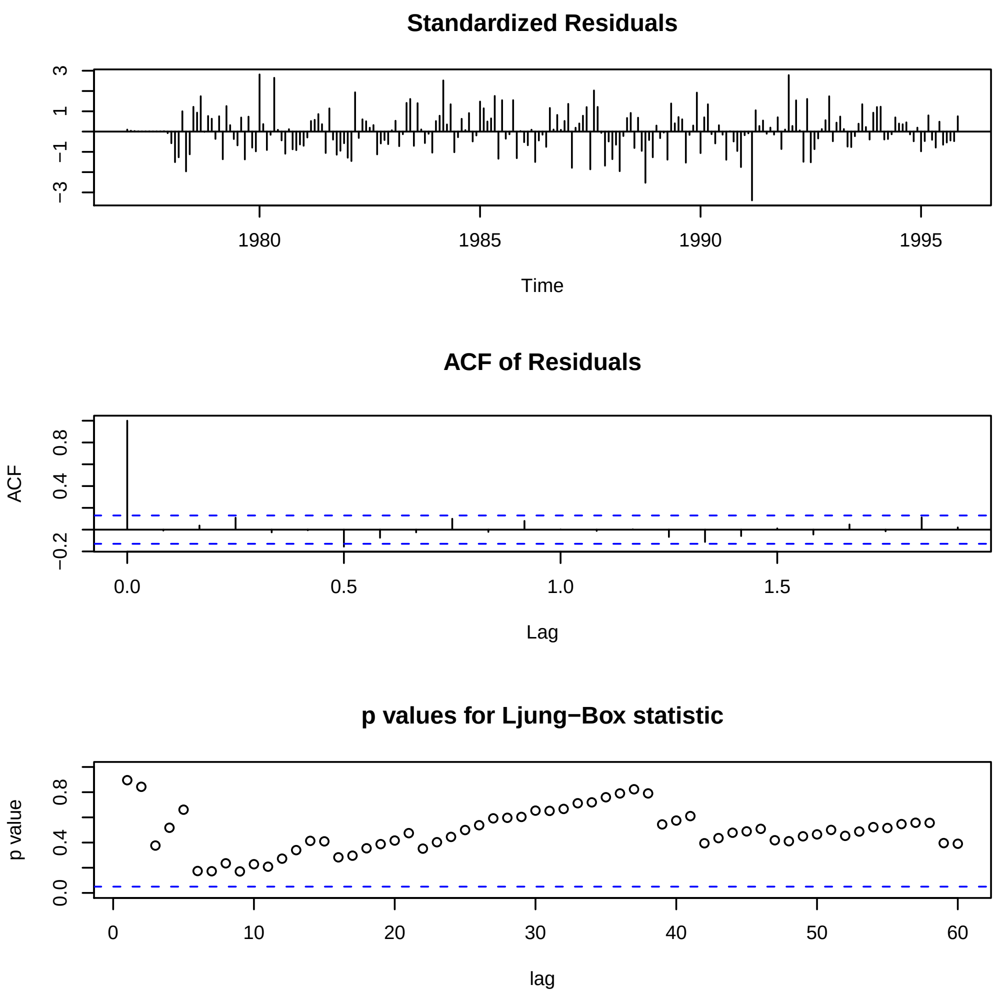

Figure 5.8: Visitors to New Zealand: Model 7

```
Call:
arima(x = visnzln, order = c(1, 1, 1), seasonal = list(order = c(1, 1, 0)))

Coefficients:
         ar1      ma1     sar1
      0.2396  -0.8409  -0.2868
s.e.  0.0972   0.0600   0.0669

sigma^2 estimated as 0.004077:  log likelihood = 284.12,  aic = -560.25
```

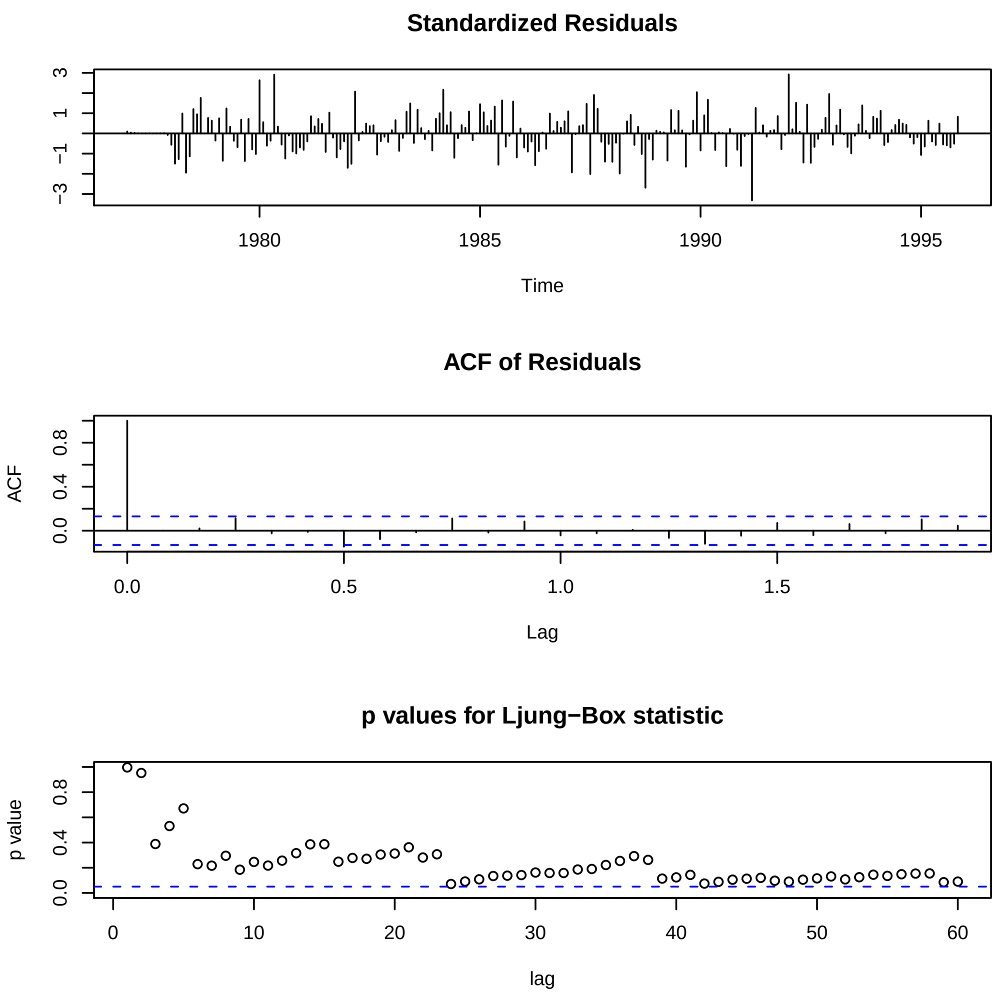

Figure 5.9: Visitors to New Zealand: Model 8


```
Call:
arima(x = visnzln, order = c(1, 1, 1), seasonal = list(order = c(0, 1, 1)))

Coefficients:
         ar1      ma1     sma1
      0.2527  -0.8393  -0.3909
s.e.  0.0948   0.0569   0.0717

sigma^2 estimated as 0.003935:  log likelihood = 287.44,  aic = -566.87
```

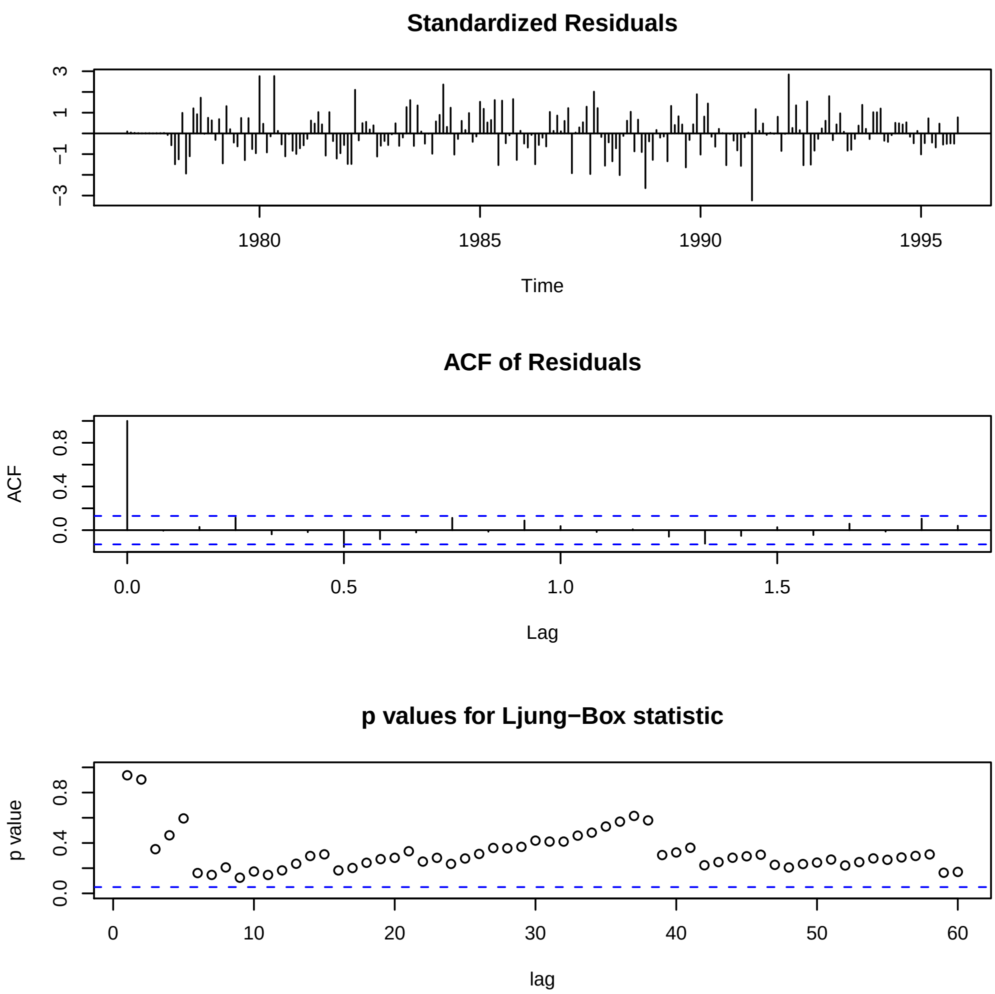

Figure 5.10: Visitors to New Zealand: Model 9

[24 marks]

END OF EXAMINATION PAPER

Two Hours

Mathematical formula books and statistical tables are to be provided

THE UNIVERSITY OF MANCHESTER

GENERALISED LINEAR MODELS

21 May 2014

09:45 – 11:45

Answer ALL TWO questions in Section A.
Answer TWO of the THREE questions in Section B.

If more than TWO questions from section B are attempted
then credit will be given to the best TWO answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** of the two questions

A1.

(a) For a distribution in the exponential family, the probability density/mass function can be written as

$$f(y;\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\},$$

where $a(\cdot)$, $b(\cdot)$ and $c(\cdot,\cdot)$ are given functions, ϕ and θ are parameters. Express the mean and variance of the distribution in terms of θ or ϕ in the case where $b(\theta) = \theta^2/2$ and $a(\phi) = \phi$.

[4 marks]

(b) Show that the geometric distribution with probability mass function

$$P(Y = y) = (1-p)^y p,\ \ y = 0, 1, 2, \ldots$$

belongs to the exponential family, where $p \in [0, 1]$ is a parameter.

[4 marks]

(c) In a generalised linear model, how is the mean of the response variable related to the linear predictor? How is the variance related to the linear predictor?

[4 marks]

(d) Explain what the canonical link is. The Newton-Raphson algorithm becomes identical to which algorithm named after a famous statistician when the canonical link is used?

[4 marks]

(e) Name the distribution in (a) and identify the canonical link for a response variable following this distribution.

[4 marks]

[20 marks in total for this question.]

A2.

(a) Write down a generalised linear model for $y_i \sim Pois(\lambda_i)$ on a single explanatory variable x with values x_i, $i = 1, \ldots, n$ and state any further assumption on $y_1, y_2, \ldots, y_n$. Use the log link and include an intercept or constant term in the model.

[3 marks]

(b) Derive the log-likelihood function for the model in (a).

[4 marks]

(c) Use your result in (b) to derive an equation for the maximum likelihood estimates of the parameters in the model described in (a) and explain how to solve it iteratively.

[6 marks]

(d) Find the Fisher information matrix for the parameters in the model described in (a).

[3 marks]

(e) Derive the deviance of the model in (a) when fitted to data and simplify it.

[4 marks]

[20 marks in total for this question.]

SECTION B

Answer **TWO** of the three questions

B1. The Gamma distribution with probability density function (pdf)

$$f(y;\mu,\alpha) = \frac{1}{\Gamma(\alpha)}\left(\frac{\alpha}{\mu}\right)^{\alpha} y^{\alpha-1}\exp\left(-\frac{\alpha}{\mu}y\right),\ y>0$$

belongs to the exponential family, where μ, $\alpha > 0$ are unknown parameters.

(a) Express the pdf in exponential family form, putting $1/\alpha$ on the denominator.

[4 marks]

(b) Show that the mean of the distribution is equal to μ.

[3 marks]

(c) Find the canonical link function.

[3 marks]

(d) A generalised linear model with Gamma response, linear predictor $\eta = \beta_0 + \beta_1 x + \beta_2 x^2$ and canonical link was fitted to $n = 35$ observations on (x, y). The deviance was 44.28 for the fitted model.

(i) The parameter α is assumed to be unknown but identical for all the observations 1 to 35. Estimate α using the deviance.

[4 marks]

(ii) Can the deviance above be used to check the adequacy of the fitted model? Give reasons for your answer.

[2 marks]

(iii) The deviance was 47.34 for the reduced model with $\eta = \beta_0 + \beta_1 x$. Use an F-test to decide if the change in deviance is significant at the 5% level.

[4 marks]

[20 marks in total for this question.]

B2. In an experiment tobacco budworm moths are exposed to 6 different doses (mg) of the insecticide trans-cypermethrin. For each sex the number of moths killed and not killed at each dose is recorded in y. Logistic models are fitted to the data, where the base-2 logarithm of dose, Ldose=$\log_2$(dose), is taken as a covariate and the variable Sex as a factor. The models (see below) are referred to as fit1 and fit2.

```
> fit1<-glm(y ~ Sex * Ldose, family=binomial(link=logit))

> anova(fit1, test="Chisq")

  Analysis of Deviance Table
  Terms added sequentially (first to last)
               Df    Deviance Resid.   Df   Resid. Dev   P(>|Chi|)
  NULL                                 11    124.876
  Sex          1          6.077        10    118.799      0.014
  Ldose        1        112.042         9      6.757      3.499e-26
  Sex:Ldose    1          1.763         8      4.994      0.184

> fit2 <- glm(y ~ Sex + Ldose, family=binomial(link=logit))
> summary(fit2)
  Coefficients:
                Estimate  Std. Error  z value  Pr(>|z|)
  (Intercept)   -3.4732     0.4685    -7.413   1.23e-13
   SexMale       1.1007     0.3558     3.093   0.00198
   Ldose         1.0642     0.1311     8.119   4.70e-16

   Null deviance:       124.876 on 11 degrees of freedom
   Residual deviance:     6.757 on  9 degrees of freedom
```

(a) Write down the model fit1 as a generalised linear model and explain what the parameters represent.

[4 marks]

(b) What is the deviance of the model fit1? Use it to assess the goodness of fit.

[4 marks]

(c) What is the deviance of the model fit2? Is there a significant difference in deviance at 5% between the two models?

[4 marks]

(d) Is there a significant difference between males and females? Test appropriate hypotheses at the 5% level.

[4 marks]

(e) Explain what is meant by the median lethal dose (LD50). Based on the model fit2, for each sex estimated the LD50 value.

[4 marks]

[20 marks in total for this question.] [20 marks]

B3. A random sample of 227 males were classified according to hair colour (black, brown or fair) and eye colour (brown, grey or blue). A log-linear Poisson model was fitted in R with the following output, where Hair and Eye are factors.

```
> fit<-glm(y ~ Hair * Eye, family=poisson)

> summary(fit)
       Coefficients:
                           Estimate   Std. Error  z value   Pr(>|z|)
      (Intercept)           2.30259     0.31623    7.281    3.30e-13
       HairBrown            0.47000     0.40311    1.166    0.244
       HairFair            -0.69315     0.54772   -1.266    0.206
       EyeGrey              0.87547     0.37639    2.326    0.020
       EyeBlue             -0.22314     0.47434   -0.470    0.638
       HairBrown:EyeGrey    0.06551     0.47808    0.137    0.891
       HairBrown:EyeBlue    0.70865     0.57093    1.241    0.215
       HairFair:EyeGrey     0.98083     0.61067    1.606    0.108
       HairFair:EyeBlue     2.78809     0.66361    4.201    2.65e-05

       Null deviance:      1.0971e+02 on 8 degrees of freedom
       Residual deviance: -3.5527e-15 on 0 degrees of freedom

> anova(fit,test="Chisq")

  Analysis of Deviance Table
  Terms added sequentially (first to last)

                Df  Deviance Resid.  Df   Resid. Dev   P(>|Chi|)
       NULL                          8     109.711
       Hair      2      26.829       6      82.882     1.493e-06
       Eye       2      46.078       4      36.804     9.872e-11
       Hair:Eye  4      36.804       0   -3.553e-15    1.977e-07
```

a) Let y_{ij} be the number of people in cell (i, j), i=1,2,3 (hair), j=1,2,3 (eye). State the joint distribution of $\{y_{ij}\}$.

[4 marks]

b) Give three reasons why a Poisson model can be used in this case.

[3 marks]

c) Is there an association between hair colour and eye colour? Use the Hair:Eye line to test appropriate hypotheses at the 1% significance level. Further interpret the HairFair:EyeBlue line.

[5 marks]

d) What does the estimated intercept 2.30259 correspond to? Answer the question and compute the corresponding mean or expected value.

[4 marks]

e) Use the estimates in summary(fit) to compute the probability that a man has

i) brown hair and grey eyes,

ii) brown hair and blue eyes,

iii) fair hair and grey eyes, or

iv) fair hair and blue eyes.

[4 marks]

[20 marks in total for this question.]

END OF EXAMINATION PAPER

Statistical Tables are provided

Two hours

The University of Manchester
School of Mathematics

Design and Analysis of Experiments

Date: 15 May 2014
Time: 14:00 – 16:00

Answer the question in Section A (20 marks) and TWO of the three questions in Section B (20 marks each). If all three questions in Section B are attempted then credit will be given for the two best answers.

Electronic calculators may be used, provided that they cannot store text.

Section A

Question 1.

Four variables have to be examined for their effects on the yield Y of a chemical process:

x_1 - temperature (2 levels: 30^0 and 60^0C),
x_2 - NaOH concentration (2 levels: 25% and 35%),
x_3 - catalyst concentration (2 levels: 1% and 2%),
x_4 - level of agitation (2 levels: low and high).

For each of the situations described below, construct suitable designs, write down the confounding pattern and what model can be estimated.

(a) Four chemical reactors are available for the experiment. Consider the reactor as a blocking factor and construct a two-level complete factorial design in four blocks.

(9 marks)

(b) Two chemical reactors are available for the experiment. Consider the reactor as a blocking factor and construct a two-level complete factorial design in two levels in two blocks.

(6 marks)

(c) Not more than 8 observations can be taken. All of them should be made using one chemical reactor.

(5 marks)

Total: 20 marks

Section B

Question 2. The percentage of vanadium in a sample of petroleum ash was determined using the standard method (A) and two new methods (B and C) of dissolving the ash. The sample was divided into fifteen specimens, and five specimens were analysed by each of the three methods. The observations are:

Method		
A	B	C
8	6	13
22	12	30
8	13	29
4	18	24
4	26	24

(a) Write down an appropriate model for the data. Complete the general form of the one-way ANOVA table given below.

Source	SS	Df	MS	E[MS]	F-ratio	P-value
Treatment	$\sum_{i=1}^{t} r_i \bar{y}_{i\bullet}^2 - n\bar{y}_{\bullet\bullet}^2$	?	?	$\sigma^2 + \frac{1}{t-1}\sum_{i=1}^{t} r_i \alpha_i^2$	?	?
Residual	?	?	?	?	-	-
Total	$\sum_{i=1}^{t}\sum_{j=1}^{r_i} y_{ij}^2 - n\bar{y}_{\bullet\bullet}^2$	?	-	-	-	-

(4 marks)

(b) Calculate the entries of the ANOVA table above. What assumptions should be satisfied?

(6 marks)

(c) Interpret the results obtained in (b). Is there a statistically significant difference between the results obtained using the three methods?

(6 marks)

(d) Explain why the effects associated with the methods are considered fixed. Give an example of a different study where the effect of treatments should be considered random. Write down the appropriate model for your example.

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Total: 20 marks

Question 3. A balanced incomplete block design (BIBD) has been used to assess the lifetime of tyres made of 4 different rubber compounds (A, B, C and D). Composite tyres were used. They consist of three-part treads enabling three compounds to be tested in the same tyre. All tests have been carried out using one car. The compounds tested on the four tyres were as follows:

Tyre 1 A, B, C
Tyre 2 A, B, D
Tyre 3 A, C, D
Tyre 4 B, C, D

The following table lists the lifetimes (in hours) of car tyres that were measured.

		Compound			
		A	B	C	D
Tyre	1	238	238	279	-
	2	196	213	-	308
	3	254	-	334	367
	4	-	312	421	412

(a) Identify the parameters of the experimental design and its main features.

(6 marks)

(b) State an appropriate model for the data. Complete the general form of the ANOVA table given below by adding the missing entries marked with question marks. Calculate the ANOVA table for the data using the formulas provided in the table and at the beginning of the next page.

(10 marks)

Source of Variation	df	SS
Treatments (adjusted for blocks)	?	$SSA = r_E \sum_{i=1}^{t} \hat{\alpha}_i^2$
Blocks (unadjusted)	?	$SSB = k \sum_{j=1}^{b} \bar{y}_{\bullet j}^2 - CF$
Residual	?	?
Total	?	$\sum_{i=1}^{t} \sum_{j=1}^{r_i} y_{ij}^2 - CF$

(The question continues on the next page)

Note that (in standard notation): $CF = n\bar{y}_{\bullet\bullet}^2$, $r_E = \dfrac{\lambda t}{k}$, $\hat{\alpha}_i = \left(ky_{i\bullet} - C_i\right)/\lambda t$, where

$$C_i = \sum_{j=1}^{b} n_{ij} y_{\bullet j} \quad \text{and} \quad n_{ij} = \begin{cases} 1, & i \in I_j \\ 0, & otherwise \end{cases}$$

and I_j is the set of treatments used in the jth block.

(c) The experimenter intends to develop a new composite tyre that would allow four compounds to be tested on one tyre. However, he also wants to compare five compounds and to use five composite tyres. Suggest an appropriate experimental design and state its parameters and three reasons why you have chosen this particular design.

(4 marks)

Total: 20 marks

Question 4. A pharmaceutical company carries out a study in order to compare its new drug (A) with two drugs (B and C) that are already on the market. Each drug is tested in 4 different hospitals, i.e. 12 hospitals are used in this clinical trial. Equal numbers of 50 patients were given the drugs in each hospital (i.e. n = 600 patients) and the response (y) to the treatment is measured. High values of y are desirable. Previous experience suggests that it is possible to assume that the observations are normally distributed with variance σ^2. The data (not provided here) were used to calculate the following results (in standard notation): SST = 4056, SSA = 1312, SSE = 1874.

Also, $\bar{y}_{\bullet\bullet\bullet} = 225.1; \bar{y}_{1\bullet\bullet} = 232.4; \bar{y}_{2\bullet\bullet} = 221.8; \bar{y}_{3\bullet\bullet} = 221.1$.

Note that some formulas that you may need are given at the end of this question.

(a) Explain why the experimental design that has been used is a nested design. Describe the main features of such designs.

(4 marks)

(b) Write down an appropriate model and construct the ANOVA table for the data. What conclusions can be made based on it? State the assumptions that you have made.

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(7 marks)

(d) Estimate the variance of the differences between hospitals and the correlation between observations taken on two patients from the same hospital.

(3 marks)

Formulas:

- $E[MSA] = \frac{rt_B}{t_A - 1}\sum_{i=1}^{t_A}\alpha_i^2 + r\sigma_B^2 + \sigma^2$ if factor A is treated fixed, or $E[MSA] = rt_B\sigma_A^2 + r\sigma_B^2 + \sigma^2$ if factor A is treated random.
- $E[MSB(A)] = r\sigma_B^2 + \sigma^2$ if factor B(A) is treated random.
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Residual	?	?	?	?	-	-
Total	$\sum_{i=1}^{t}\sum_{j=1}^{r_i} y_{ij}^2 - n\bar{y}_{\bullet\bullet}^2$	?	-	-	-	-

(4 marks)

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(c) Interpret the results obtained in (b). Is there a statistically significant difference between the results obtained using the three methods?

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(**The question continues on the next page**)

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Total: 20 marks

END OF EXAMINATION PAPER

Simple random sampling: In without replacement sampling, (SSWOR), the first and second order inclusion probabilities are given by: $\pi_i \stackrel{def}{=} \sum_{s\in\mathcal{S}} p(s)I_i(s) = \frac{n}{N}$, and $\pi_{ij} \stackrel{def}{=} \sum_{s\in\mathcal{S}} p(s)I_i(s)I_j(s) = \frac{n(n-1)}{N(N-1)}$
Proportional Allocation: number allocated to stratum i, $n_{i\ prop} = \frac{N_i}{N} \times n$.
Optimal Allocation: number allocated to stratum i

$$n_{i\ opt} = \frac{N_i \sigma_i n}{\sum_{i=1}^{H} N_i \sigma_i},$$

where H is the total number of strata.
Post stratified sampling *Preliminaries:* $N = N_1 + N_2$, $P_1 = \frac{N_1}{N}, P_2 = \frac{N_2}{N}$. *Post stratified mean:* $\bar{y}_{pst} = P_1 \times \bar{y}_1 + P_2 \times \bar{y}_2$. *Variance of the Post stratified mean:* $\mathrm{Var}(_{pst}) \approx \frac{1}{n}[P_1 \times s_1^2 + P_2 \times s_2^2] + \frac{1}{n^2}[(1-P_1) \times s_1^2 + (1-P_2) \times s_2^2]$
Normal distribution: For a variable X with expected value $E(X) = \mu$ and variance $V(X) = \sigma^2$ that follows a normal distribution we write $X \sim N(\mu, \sigma^2)$. The variable $Z = (X-\mu)/\sigma$ is a *standard normal* variate $Z \sim N(0,1)$. The transformation $aZ + b \sim N(b, a^2)$. If $X_i \sim N(\mu_i, \sigma_i^2)$, independently for $i = 1, \ldots .n$, $\sum_{i=1}^{n} X_i \sim N(\sum \mu_i, \sum \sigma_i^2)$. A critical value c_α on the $(1-\alpha)\times 100\%$-level is such that

$$\Pr(|(X-\mu)/\sigma| < c_{\alpha/2}) = \Pr(|Z| < c_{\alpha/2}) = 1 - \alpha$$

Common *levels of significance* are $\alpha = .05$ and $\alpha = .01$ with critical values $c_{.025} = 1.96$ and $c_{.005} = 2.56$, respectively (Note: these critical values are commonly approximated as 2 and 2.6 respectively and $(1-\alpha)\times 100\%$-level is also referred to as the $\alpha \times 100\%$-level when there is no ambiguity).
Estimator: For a collection of random variates $X_1, \ldots, X_k$ with support $\mathcal{X}$ and a distribution indexed by a parameter θ that takes values in Θ, a point estimator $\hat{\theta}$ is a function $\hat{\theta} : \mathcal{X} \to \Theta$. An estimator is said to be and unbiased estimator of θ if $E(\hat{\theta}(X_1, \ldots, X_k)) = \theta$. The standard deviation $\sqrt{V(\hat{\theta}(X_1, \ldots, X_k))}$ is called the *standard error of the estimator*. Under the assumption that an estimator is approximately $N(\theta, V(\hat{\theta}))$, a test of $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ on the $(1-\alpha)\times 100\%$-level is to reject H_0 if

$$\left| \frac{\hat{\theta}(X_1, \ldots, X_k) - \theta_0}{\sqrt{V(\hat{\theta}(X_1, \ldots, X_k))}} \right| > c_\alpha$$

A $(1-\alpha)\times 100\%$ *confidence interval* is an interval estimator $(\hat{\theta}_L, \hat{\theta}_R)$ such that $\Pr(\hat{\theta}_L \leq \theta, \hat{\theta}_R \geq \theta) = 1 - \alpha$. If the point estimator $\hat{\theta}$ is approximately $N(\theta, V(\hat{\theta}))$, then an approximate $(1-\alpha)\times 100\%$ confidence interval is given by $\hat{\theta}_L = \hat{\theta} - c_\alpha \sqrt{V(\hat{\theta}(X_1, \ldots, X_k))}$ and $\hat{\theta}_R = \hat{\theta} + c_\alpha \sqrt{V(\hat{\theta}(X_1, \ldots, X_k))}$.
Proportion: If $X_1, \ldots, X_n$ are i.i.d. *Bernoulli*(p), the estimator $\hat{p} = \sum_{i=1}^{n} X_i/n$ is (under some conditions) approximately $N(p, p(1-p)/n)$. A $(1-\alpha)\times 100\%$ confidence interval is given by $\hat{p} \pm c_\alpha \sqrt{\hat{p}(1-\hat{p})/n}$
Regression: When regressing variables $Y_1, Y_2, \ldots, Y_n$ for n independent observations on covariates $x_{i1}, \ldots, x_{ip}$, $i = 1, \ldots, n$, it is assumed that the covariates are fixed, $Y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \epsilon_i$, for a vector β of unknown parameters $\beta_1, \ldots, \beta_p$, and random terms ϵ_i are independent identically distributed as $\epsilon \sim N(0, \sigma^2)$. The ordinary least square (OLS) estimate $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_p)^T$ of β minimises $\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$, where $\hat{y}_i = \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \cdots + \hat{\beta}_p x_{ip}$. Denoting the matrix of covariates $X = (x_{1\cdot}^T, \ldots, x_{n\cdot}^T)^T$, where $x_{1\cdot} = (x_{i1}, \ldots, x_{ip})$, the OLS is given by $\hat{\beta} = (X^T X)^{-1} X^T y$. If we for an individual parameter β_k denote the standard error $s.e.(\hat{\beta}_k) = \sqrt{V(\hat{\beta}_k)}$, and given that the assumptions about the model are true, we can test $H_0 : \beta_k = 0$ against $H_1 : \beta_k \neq 0$ on the $(1-\alpha)\times 100\%$-level and reject H_0 if $|\hat{\beta}_k / s.e.(\hat{\beta}_k)| > c_\alpha$

UNIVERSITY OF MANCHESTER: MATH38152

Social Statistics

Tuesday 20th May 14:00 – Two Hours

Electronic calculators may be used provided that they cannot store text

Mathematical formula sheet provided

Answer **ALL** five questions in SECTION A (40 Marks)
Answer **TWO** of the three questions in SECTION B (20 marks each)

The total number of marks on the paper is 80.
A further 20 marks are available from coursework during the semester making a total of 100.

SECTION A

Answer ALL five questions

A1.

(a) Give an example of a way in which a survey and a census differ and what the differences in aims there might be between them. (4 marks)
(b) What is the only source of randomness in a probability sample. (4 marks)
[8 marks total]

A2.
In early 2014 a YouGov poll of 1100 people suggested that 41% of those sampled would vote to stay in the EU in a referendum, rather than to leave it.

(a) Assuming Normality, calculate a 95% confidence interval for the true proportion who would vote to stay in the EU based on this sample estimate. (2 marks)
(b) If a new poll was to be taken today, what sample size would be needed to estimate the true population proportion with a 4% margin of error, based on:

1. An educated guess? (2 marks)
2. A conservative estimate? (2 marks)

Hint: For both the educated guess and the conservative estimate, assume that N is large such that the Finite Population Correction may be ignored. Normality may also be assumed.

(c) Give one reason why it is advantageous to take a smaller sample to obtain the same degree of accuracy of the estimate, if possible. (2 marks)
[8 marks total]

A3.
The University of Manchester plans to interview a sample of third year students face-to-face about their experiences of studying at Manchester. They hope to conduct the survey quickly, and have a workforce of 5 interviewers.

(a) Where could a sampling frame be found in this case? (2 marks)
(b) How might a stratified sample be taken, based on this information? Give an example of a stratification variable. (2 marks)
(c) If the sampling fraction, f, is 0.1, how would proportional allocation be made for a stratified sample? (2 marks)
(d) Briefly explain why cluster (or multi-stage) sampling might also be useful in this case? (2 marks).
[8 marks total]

A4.

For an $n \times K$ data matrix $(Y_{ij} : i = 1, \ldots, n, j = 1, \ldots, K)$ of $n = 200$ individuals and K variables, we have an associated matrix of missing data indicators $(M_{ij} : i = 1, \ldots, n, j = 1, \ldots, K)$, where M_{ij} is equal to 1 or 0 according to whether the observation Y_{ij} is missing or not. For each individual case i we have either $\prod_{j=1}^{K} m_{ij} = 1$ or $\sum_{j=1}^{K} m_{ij} = 0$

(a) Is this item non-response or unit non-response? (2 marks)

(b) What does it mean to perform a 'mean imputation' for all the missing values? Can you perform a 'conditional mean' imputation? (2 marks)

(c) In the sample, Y_{i1} is 1 or 0 according to whether i is female or not. The variable Y_{i2} is 1 or 0 according to whether i thinks Scotland should become independent or not. Among the $n_W = 50$ women who respond 20% report that they think that Scotland should become independent and among the $n_M = 100$ men that respond this figure is 30%. Using sex as an adjustment class, what would the proportion of people that support Scottish independence be if you found out that 20 of the 50 non-respondents were male and the rest female? (4 marks)

[8 marks total]

A5.

In Chile a number of prison inmates that are later released from prison take part in an educational program. Let Y_i be a binary variable that is 1 if individual i $(i = 1, \ldots, n)$ receives a new prison sentence within 5 years of release from prison, and 0 if i does not receive a new prison sentence within 5 years of release from prison. Let T_i be a binary variable that is 1 if individual i $(i = 1, \ldots, n)$ takes part in the programme and 0 if i does not participate in the programme.

(a) Assuming that $n = 100$, $\sum T_i = 50$ and that 25 indivudals that took the program were sentenced within 5 years of release and 20 individuals that did not take the program were sentenced within 5 years of release. What is the average effect of the programme on re-offending (being sentenced within 5 years of release)? (3 marks)

(b) In words, define what the counterfactual outcome is for individuals i for which $T_i = 1$ is. (2 marks)

(c) Assume that this study was conducted in two different prisons: prison A and prison B. In prison A inmates volunteered to participate in the study and in prison B inmates were allocated to participate in the program at random. Discuss the difference in interpretation of the causal effects for the two prisons. (3 marks)

[8 marks total]

SECTION B

Answer TWO of the three questions

B1.

Think of the research question 'are there ethnic biases in party preference in the UK'? and how you would go about investigating this from data collection to analysis and interpretation. For each of **four** (4) out of the following stages of your choice identify and discuss a potential issue:

(i) Identifying target population
(ii) Design: what to measure
(iii) Design: how to measure
(iv) Sampling frame

(v) Sampling design
(vi) Data quality
(vii) Missing data
(viii) Inference and interpretation
Full marks require a satisfactory treatment of all aspects in a coherent and relevant manner.
[20 marks total]

B2.

Four farms were sampled, with replacement, from a certain area, covering $N = 400$ square kilometres in total, to estimate the total number of workers and farms in the area. There are three types of farm in the area: Arable, Cattle, Mixed.
(a) Explain briefly why the Hansen-Hurvitz (H-H) and Horvitz-Thompson (H-T) estimators are useful in

Table 1: Sampled farms

i	# Employees, m_i	Farm Type	Size, n_i	p_i	π_i
1	10	Arable	5	.0125	.0491
2	20	Mixed	10	.0250	.0963
3	3	Arable	2	.0050	.0199
4	8	Cattle	8	.0200	.0776

survey sampling. (3 marks)
(b) Explain how p_1 and π_1 were calculated in Table 1, and explain in words what p_1 and π_1 represent. (3 marks)
(c) Use the H-H estimator to estimate the total number of workers in the area and the total number of farms in the area. (5 marks)
(d) Use the H-T estimator to estimate the total number of workers in the area and the total number of farms in the area. (5 marks)
(e) In this particular area, most farms are either mixed or arable; there are very few cattle farms. Give brief details of a sample design that would ensure that all three farm types are represented in a sample of farms from this area. Also explain briefly how to assess the precision of such a sampling scheme, relative to that obtained from a simple random sample (SRS). (4 marks)

[20 marks total]

B3.
A researcher is interested in the effect of unemployment on mental health. Self-reported mental health Y_i is measured for n individuals $i = 1, \ldots, n$. Furthermore, their employment is measured as X_i, which is equal to 1 if i is in employement and 0 otherwise. An $n \times 2$ matrix of missing data indicators $M = (m_{ij})$ is provided, where $j = 1$ refers to the outcome variable and $j = 2$ the predictor.
(a) The OLS estimate of the regression coefficients of $y_i = \alpha + \beta X_1 + \epsilon_i$ are

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n} \sum_i x_i \sum_i y_i}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} x_i\right)^2}$$

and

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$$

Re-express the *complete case* OLS estimate of α and β in terms of y_i, x_i, and the missing indicators. (5

marks)
(b) Under the assumption that $\alpha = 0$, $\sum_{i=1}^{n} m_{i2} = 0$, and that y_i has been imputed using $\hat{\beta}_{CC} x_i$, for $\hat{\beta}_{CC}$ being the complete case OLS estimate, what is the OLS estimate for β when regressing y on x? Simplify the expression as far as possible. (5 marks)
(c) Assuming that $Var(\epsilon) = \sigma^2$ is known, what is the distribution of $\hat{Y}_i | x_i$ for $m_{i1} = 1$ and $m_{i2} = 0$ if the missing value is imputed according to the model using $\hat{\alpha}_{CC}$ and $\hat{\beta}_{CC}$ (3 marks)
(d) What is the distribution of $\hat{Y}_i | x_i$ for $m_{i1} = m_{i2} = 1$ if y_i is imputed using $\hat{\alpha}_{CC}$ and $\hat{\beta}_{CC}$, and x_i is imputed using $N(\bar{x}_{CC}, \tau^2)$ (3 marks)
(e) Assuming $\Pr(M_{i1} = 1 | y_i, x_i) = p^{I\{y_i \geq x_i\}} q^{I\{y_i < x_i\}}$, where $0 \leq p \leq 1$, $0 \leq q \leq 1$, $I\{A\}$ is an indicator function equal to 1 if A is true, and 0 otherwise; and that $\sum_{i=1}^{n} m_{i2} = 0$, are data missing completely at random (MCAR), missing at random (MAR), or missing not at random (MNAR)? (4 marks)
[20 marks total]

END OF EXAMINATION PAPER

TWO hours

Electronic calculators may be used, provided that they cannot store text.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL PROGRAMMING

16 May 2014

9.45 – 11.45

Answer **THREE** of the **FOUR** questions. If more than THREE questions are attempted, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

1.

(a) The standard form linear programming problem is

$$LP : \min\left\{\mathbf{c}^T\mathbf{x} | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\right\}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{A}$ is a real $m \times n$ matrix with $m < n$; $\mathbf{b}$ $(m \times 1)$ and $\mathbf{c}$ $(n \times 1)$ are constants. Assume that LP has a bounded feasible region $\mathcal{S}$ with a finite set of extreme points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$. By adopting a suitable representation of a general point $\mathbf{x} \in \mathcal{S}$ in terms of these extreme points, prove that the minimum value of LP is achieved at an extreme point.

[6 marks]

(b) Using the simplex method

$$\begin{array}{ll} \text{maximize} & -x_1 + 2x_2 + x_3 \\ \text{subject to} & x_1 + x_2 + 2x_3 \leq 6 \\ & -x_1 + 2x_2 \leq 3 \\ & \qquad x_1, x_2, x_3 \geq 0. \end{array}$$

[6 marks]

(i) Find the optimal basis matrix $\mathbf{B}$ and its inverse $\mathbf{B}^{-1}$.

[2 marks]

(ii) Find the range of values for c_3, the coefficient of x_3 in the objective function, which will preserve the optimality of the current basis.

[3 marks]

(iii) Find the range of values for b_1, the right hand side of the first constraint, which will preserve the optimality of the current basis.

[3 marks]

2.

(a) Define the dual problem D, in m dual variables $\mathbf{y}\in\mathbb{R}^m$ for the primal LP in standard form:

$$P : \min\left\{\mathbf{c}^T\mathbf{x}|\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\right\}$$

where $\mathbf{A}\,(m \times n)\,, \mathbf{b}\,(m \times 1)\,, \mathbf{c}\,(n \times 1)$ are constants. State and prove the (weak) duality lemma for this pair of problems. Hence deduce a test for optimality for a given pair of feasible vectors $\mathbf{x}, \mathbf{y}$.

[10 marks]

(b) Using the transformation $x_2 = u_2 - v_2$, where u_2, v_2 are non-negative variables,

$$\begin{array}{ll} \text{maximize} & 2x_1 - x_2 \\ \text{subject to} & x_1 + x_2 \leq 4 \\ & -x_1 - 2x_2 \leq 6 \\ & x_1 \geq 0,\ x_2 \text{ unrestricted in sign} \end{array}$$

(i) Explain why the variables u_2, v_2 cannot be simultaneously basic.

(ii) Obtain the dual problem and, using the duality theorem or otherwise, find an optimal solution to the dual.

[10 marks]

3.

The LP *relaxation* of the following *pure integer* linear programming problem

$$\begin{array}{lll} \textit{ILP:} & \text{minimize} & 3x_1 + 4x_2 = z \\ & \text{subject to} & 3x_1 + x_2 \geq 4 \\ & & x_1 + 2x_2 \geq 4 \\ & & x_1, x_2 \geq 0 \quad \text{and } x_1, x_2 \in \mathbb{Z} \end{array}$$

has an optimal solution at $x_1 = \frac{4}{5}$, $x_2 = 1\frac{3}{5}$ and $z_0 = \frac{44}{5}$.

(a) Construct an optimal simplex tableau from this information, and use your solution to derive a lower bound for the optimal value of *ILP*.

[5 marks]

(b) State the requirements of a cutting plane, and construct a Gomory cutting plane for this problem *from first principles*, using as a source row criterion $\min_i |f_{i0} - \frac{1}{2}|$, (in the usual notation). Use the cutting plane to solve *ILP*.

[10 marks]

(c) Find the equation of the cut in the $x_1 - x_2$ plane and illustrate the solution graphically.

[5 marks]

4.

Let $\mathbf{r} = (r_1, ..., r_m)^T$ and $\mathbf{c} = (c_1, ..., c_n)^T$ represent row and column strategies for a two player zero-sum game Γ_A with an $m \times n$ payoff matrix $\mathbf{A}$.

(a) Define the expected payoff $E(\mathbf{r}, \mathbf{c})$ and state the fundamental theorem of matrix games.

[5 marks]

(b) A game is played between Rose (row-player) and Colin (column-player) with the following payoff matrix (paid to Rose).

	c_1	c_2	c_3	c_4
r_1	0	0	3	1
r_2	2	1	4	2
r_3	1	3	0	0

Colin tells Rose that he will play his equilibrium strategy, which (he claims) is

$$\mathbf{c}^T = (c_1, c_2, c_3, c_4) = \left(0, \tfrac{1}{2}, 0, \tfrac{1}{2}\right)$$

and that the value of the game to her is $3/2$.

(i) State Colin's optimal policy as a minimax problem and formulate this problem in linear programming (LP) form.

[5 marks]

(ii) Verify that Colin's information is correct, or disprove his assertion.

[5 marks]

(iii) Find Rose's optimal policy.

[5 marks]

END OF EXAMINATION PAPER

2 hours

Tables of the cumulative distribution function are provided.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL MODELLING IN FINANCE

29 May 2014

09:45am – 11:45am

Answer **all** 4 questions in **Section A** (60 marks in all) and **2** of the 3 questions in **Section B** (20 marks each). If all three questions from Section B are attempted then credit will be given for the two best answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** 4 questions

A1.

An asset pays a dividend of £2 at the end of each of the next 2 years. The current asset price is £100 and the constant risk-free interest rate is 10% per annum with continuous compounding. Suppose that a zero value forward contract on this asset with delivery date in 3 years and delivery price of £135 is available. Does this situation create any arbitrage opportunities? Justify your answer. If your answer is yes, then describe the arbitrage strategy and compute the risk-free profit.

[15 marks]

A2.

(i) Consider a contract with value function $V(S,t)$ which satisfies the Black-Scholes equation, namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

in the usual notation, with constant r and σ, with a payoff at expiry $t = T$ given by

$$V(S,T) = S^n,$$

where $n > 0$. By seeking a solution of the form $V(S,t) = S^n f(t)$, determine $V(S,t)$.

(ii) By using (i) above, with $n = 2$, find a lower bound for the contract with payoff

$$V(S,T) = \exp(S) - 1 - S.$$

[15 marks]

A3.

(i) Sketch the expiry payoff diagrams for the following portfolio (all options are vanilla and have the same expiry date): short three calls all with exercise price X_1 and long two puts both with exercise price X_2, for the three cases $X_1 > X_2$, $X_1 = X_2$, $X_1 < X_2$.

(ii) A portfolio Π has the following payoff at time T:

$$
\begin{aligned}
\Pi(S,T) &= 0, \quad S < X_1 && (1)\\
&= 2(S - X_1), \quad X_1 < S < X_2 && (2)\\
&= 2(X_2 - X_1), \quad S > X_2. && (3)
\end{aligned}
$$

Replace this portfolio at time t by buying or selling vanilla calls or puts, all with the same expiry date, but with differing exercise prices.

[15 marks]

A4.

Suppose that the interest rate process r is governed by the general stochastic differential equation

$$dr = w(r,t)dX + u(r,t)dt,$$

where X is a Brownian motion and where $w(r,t)$ and $u(r,t)$ are arbitrary functions. Consider two bonds with different maturities T_1 and T_2, namely V_1 and V_2 respectively. Consider a portfolio comprising one V_1 bond and $-\Delta$ of V_2 bonds. Using Ito's Lemma (which may be used without proof), show that the choice

$$\Delta = \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r}$$

eliminates the random component of the portfolio. Hence show that the value of the bond satisfies the equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} - a(r,t)\frac{\partial V}{\partial r} - rV = 0,$$

where $a(r,t)$ is an arbitrary function (the inclusion of which you should justify).

[15 marks]

SECTION B

Answer **2** of the 3 questions

B5.

Consider the Black-Scholes equation for a European option with value V, where the underlying asset pays a continuous constant dividend yield D, namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0,$$

in the usual notation, where r and σ are also both constants.

(i) Show that

$$S\frac{\partial V}{\partial S}$$

satisfies the Black-Scholes equation. (Hint: you will need to differentiate the Black-Scholes equation with respect to S.)

(ii) Let ρ and ν be defined as

$$\rho = \frac{\partial V}{\partial r}, \quad \nu = \frac{\partial V}{\partial D}.$$

What do ρ and ν measure?

(iii) Show that ρ and ν satisfy the equations

$$\frac{\partial \rho}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \rho}{\partial S^2} + (r - D)S\frac{\partial \rho}{\partial S} - r\rho = V - S\frac{\partial V}{\partial S},$$

$$\frac{\partial \nu}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \nu}{\partial S^2} + (r - D)S\frac{\partial \nu}{\partial S} - r\nu = S\frac{\partial V}{\partial S}.$$

(iv) Write down the final conditions for ρ and ν at $t = T$ (the expiry date) for a call and a put.

(v) Using (i), (iii) and (iv) above, find $f(t)$ and $g(t)$ where

$$\rho = f(t)(S\frac{\partial V}{\partial S} - V), \quad \nu = g(t)S\frac{\partial V}{\partial S}.$$

[20 marks]

B6.

You may assume that the Black-Scholes equation (in the usual notation) for a put option on a non-dividend paying asset is

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0,$$

where the interest rate r and the volatility σ are both constant.

(i) A binary put option $P_b(S,t)$ has the payoff

$$P_b(S,T) = \left\{ \begin{array}{lll} 0 & \text{if} & S > X \\ K & \text{if} & \mathrm{S} < \mathrm{X} \end{array} \right. .$$

Show that its value is given by

$$P_b(S,t) = K\frac{\partial P}{\partial X},$$

where $P(S,t;X)$ is the value of a vanilla put option with strike X.

(ii) Given that the value of a European put option is

$$P(S,t;X) = Xe^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where $N(x)$ is the cumulative distribution function of the standard normal distribution, namely

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy,$$

and

$$d_1 = \frac{\log\frac{S}{X} + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

show that

$$P_b(S,t;X) = Ke^{-r(T-t)}N(-d_2).$$

(iii) Consider a stock whose price (6 months from the expiration of a binary put option) today is £10.50. The option pays £0.5 if the stock value at expiry is less than £11, and nothing if the stock value at expiry is greater than £11. The risk-free interest rate is 4% per annum (fixed) and the volatility (constant) is 15% per (annum)$^{\frac{1}{2}}$. Using the formula above for $P_b(S,t;X)$, determine its value today. The value of $N(x)$ may be determined by interpolation using the tables of the cumulative distribution function, as provided.

[20 marks]

B7.

(i) Let S_1 and S_2 be assets which satisfy equations of the form

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dW_1,$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dW_2,$$

where μ_1, μ_2, σ_1 and σ_2 are constants, and W_1 and W_2 are correlated Brownian motions such that

$$E(dW_1, dW_2) = \rho dt.$$

Derive Ito's lemma for a function $f(S_1, S_2, t)$.

(ii) Consider the portfolio

$$\Pi(S_1, S_2, t) = V(S_1, S_2, t) - \Delta_1 S_1 - \Delta_2 S_2,$$

where $V(S_1, S_2, t)$ is an option with S_1 and S_2 as underlyings, and Δ_1 and Δ_2 are held constant in a (small) time period dt. Find the values of Δ_1 and Δ_2 that render the portfolio riskless.

(iii) Show, using (ii) above, that $V(S_1, S_2, t)$ must satisfy the equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \tfrac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 S_2} + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV = 0,$$

where r is a constant risk-free interest rate.

(iv) By seeking a solution of the form

$$V(S_1, S_2, t) = S_2 + A(t) \log S_1 + B(t),$$

find the value of an option for $t < T$ with the payoff

$$V(S_1, S_2, T) = S_2 + \log S_1.$$

[20 marks]

END OF EXAMINATION PAPER

Two hours

A set of Formulae and Tables for the Actuarial Examinations
is required.

THE UNIVERSITY OF MANCHESTER

CONTINGENCIES 2, FINAL EXAMINATION

2 June 2014

09:45 – 11:45

Answer all SEVEN questions.
The total number of marks in the paper is 100.

Electronic calculators may be used, provided that they cannot store text.

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1. Show that the expected present value of an annuity of £1 per annum payable continuously to life (y) on the death of (x) where the payments are guaranteed to continue for at least 5 years once they have started, can be expressed as:

$$\bar{A}^{1}_{xy}\bar{a}_{\overline{5|}} + v^5\,{}_5p_y\bar{a}_{x|y+5}$$

[8 marks]

2.

(a) Give four reasons, with an example to illustrate each reason, why workers in two different industries might experience different mortality rates. [8 marks]

(b) The mortality experience in one year of workers employed on a major infrastructure project is given below.

Age	No. of employees	No. of deaths
25	5,328	16
26	5,189	14
27	5,067	11

Calculate the crude mortality rate and the standardised mortality ratio with respect to ELT15 for the group of workers listed above. [6 marks]

[Total 14 marks]

3. Life (x) is aged 45 exact and life (y) is aged 43 exact. Calculate:

(a) The present value of a pure endowment of £200,000 payable if both lives survive for 20 years. [4 marks]

(b) The present value of an assurance of £100,000 payable immediately on the death of (x) before age 65 provided (y) is alive. [4 marks]

(c) The present value of an annuity to (y) for life on the death of (x) before age 65. The annuity is £10,000 per annum, payable monthly in arrears, and increases by 1.5% at each anniversary of $(x)'s$ death.

Hint: first show that the net present value of a one-year annuity to life (y) equals 0.9866, use this to calculate the value of an annuity for life, and then use Part (b). [9 marks]

Mortality (x): $\mu_x = 0.0055$

Mortality (y): $\mu_y = 0.005$

Force of interest: 0.02

[Total 17 marks]

4.

(a) Describe the retirement benefits provided under each of the following types of pension scheme:

(i) a final salary scheme

(ii) a career average revalued earnings scheme

(iii) a defined contribution scheme [6 marks]

(b) For each of the types of scheme listed above, explain how the investment and salary risk of providing the pension benefit is shared between employer and employee. [6 marks]

[Total 12 marks]

5. A continuous time Markov model with constant transition rates is used to model three states of a pension scheme. The transition rates are:

Active to Retired: 0.03

Active to Dead: 0.02

No other states or transitions are considered.

(a) Calculate the dependent probability of retirement in one year and the independent probability of death from the active state in one year. [6 marks]

(b) The employer wishes to grant a benefit of a whole life assurance payable immediately on death from the active state only. They are prepared to grant a benefit with an expected present value of £20,000 per member. Calculate the sum assured that can be granted using a force of interest of 0.03. [6 marks]

[Total 12 marks]

6. A five year unit-linked policy issued to a woman aged 50 exact has the following profit vector:

$$(-66.6, -57.2, -42.6, 193.1)$$

(a) Define *profit vector* and *profit signature*. [2 marks]

(b) Explain the meaning of *zeroisation* and explain why an insurance company might choose to zeroise the above profit vector. [3 marks]

(c) What is the main disadvantage of zeroisation? [2 marks]

(d) Calculate the reserves that would need to be held at the end of years 1 and 2, and the revised profit vector after zeroisation.

Mortality: PFA92base

Interest on reserves: 2.5% per annum [7 marks]

[Total 14 marks]

7. An endowment assurance with a sum assured of £150,000 and a term of four years is issued to a man aged 61 exact. The sum assured is payable at the end of the year of death or on survival to the end of the term. A level net premium was calculated on the reserving basis as £34,532, payable annually in advance.

The insurance company holds prospective net premium reserves calculated using the premium above and a reserving basis of AM92 ultimate and interest of 4% per annum.

(a) Show that the reserves held at the end of each of the first three years of the policy are:
$_1V =$£34,862
$_2V =$£71,375
$_3V =$£109,699

[5 marks]

(b) Using the basis given below, calculate the profit vector and profit margin for this policy if the gross annual premium charged is £36,000. [18 marks]

Interest rate on cash flows and reserves: 5% per annum

Mortality: AM92 ultimate

Initial expense: 8% of annual premium

Renewal expense: 1% of annual premium plus £100 (paid from the start of the second year)

Risk discount rate: 6% per annum

[Total 23 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

RISK THEORY

27 May 2014

09:45-11:45

Answer ALL FOUR questions.
The total number of marks in the paper is 90.

Electronic calculators may be used, provided that they cannot store text.

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1. Consider the Cramér-Lundberg model where the risk process $U = \{U_t : t \geq 0\}$ of an insurance company is given by

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i, \quad t \geq 0,$$

where t is time in years, $u \geq 0$ is the initial capital, $c > 0$ is the premium rate, $N = \{N_t : t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$ that counts the number of claims and the claim sizes $X_1, X_2, \ldots$ are independent, identically distributed and strictly positive random variables that are independent of the Poisson process N. Further, the probability density function of X_1 is given by

$$f_{X_1}(x) = \frac{\alpha_1 \alpha_2}{\alpha_2 - \alpha_1}\left(\mathrm{e}^{-\alpha_1 x} - \mathrm{e}^{-\alpha_2 x}\right), \quad x > 0,$$

where $\alpha_2 > \alpha_1 > 0$.

(a) Given that exactly one claim occurs in the first year, calculate the probability that $U_1 < 0$.

[4 marks]

(b) Given that exactly one claim occurs in the first year, is the probability that ruin occurs in the first year (i) smaller than, (ii) bigger than or (iii) equal to the answer of part (a)? Justify your answer.

[3 marks]

(c) Recall that the ruin probability is defined as the probability that $U_t < 0$ for some $t > 0$. Does the ruin probability of the insurance company become higher or lower when the parameter λ is increased (assuming that all the other parameters are kept the same)? Justify your answer.

[3 marks]

(d) We define the function ξ by

$$\xi(\theta) = c\theta - \frac{\theta\lambda}{\alpha_2 - \alpha_1}\left(\frac{\alpha_2}{\theta + \alpha_1} - \frac{\alpha_1}{\theta + \alpha_2}\right), \quad \theta \in \mathbb{R}\backslash\{-\alpha_1, -\alpha_2\}.$$

Show that the Laplace exponent of the risk process U is given by the function ξ, i.e. show that

$$\mathrm{e}^{\xi(\theta)} = \mathbb{E}\left[\mathrm{e}^{\theta U_1} \middle| U_0 = 0\right], \quad \theta \geq 0.$$

[3 marks]

(e) If the net profit condition is satisfied, then the ruin probability $\phi(u)$ as a function of the initial capital u, can be expressed as

$$\phi(u) = -\frac{\xi'(0)}{\xi'(\theta_1)}\mathrm{e}^{\theta_1 u} - \frac{\xi'(0)}{\xi'(\theta_2)}\mathrm{e}^{\theta_2 u},$$

where θ_1 and θ_2 are the two strictly negative roots of the function ξ and where ξ' denotes the derivative of ξ.

Assume now that the parameters of the risk process U are given by $c = 4$, $\lambda = 4$, $\alpha_1 = 2$ and $\alpha_2 = 4$.

(i) Show that the net profit condition is satisfied.

(ii) Determine the ruin probability when the initial capital $u = 5$.

[7 marks]

(f) Suppose now, in addition to the assumptions made in part (e), that the insurance company buys proportional reinsurance with retention level 2/3. The premium rate that the reinsurer charges to the insurance company is denoted by $\bar{c} > 0$. Denote by $\widetilde{U}$ the risk process of the insurance company under the proportional reinsurance scheme with retention level 2/3.

(i) For which values of the reinsurance premium rate $\bar{c}$, is the net profit condition satisfied for the risk process $\widetilde{U}$?

(ii) Assume $\bar{c} = 1$. Determine the Lundberg coefficient corresponding to the risk process $\widetilde{U}$.

(iii) Assume $\bar{c} = 1$. By considering the Lundberg bound for the ruin probability, explain whether or not it is sensible for the insurance company to purchase the reinsurance.

[11 marks]

[Total 31 marks]

2.

(a) What is the expected value premium principle?

[3 marks]

(b) Show that the exponential premium principle satisfies the no rip-off property.

[5 marks]

(c) Construct an example which shows that the exponential premium principle with risk aversion parameter equal to 2, is not scale-invariant.

[4 marks]

(d) Recall that the probability density function of a beta distribution with parameters $a > 0$ and $b > 0$ is given by

$$f(x) = \begin{cases} c * x^{a-1}(1-x)^{b-1}, & 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $c > 0$ does not depend on x. Let X be a risk which has a beta distribution with parameters $a = 1$ and $b = 2$. Compute the Value-at-Risk and the Tail-Value-at-Risk for any confidence level $p \in (0, 1)$.

[7 marks]

[Total 19 marks]

3. Suppose that a parameter Θ takes on values $\theta_1 = 2$, $\theta_2 = 6$, $\theta_3 = 15$ and $\theta_4 = 20$. The distribution of X is discrete and depends on Θ as shown in Table 1. Assume that Θ has a prior distribution given

	θ_1	θ_2	θ_3	θ_4
x_1	0.3	0.6	0.2	0.5
x_2	0.5	0.2	0.7	0.3
x_3	0.2	0.2	0.1	0.2

Table 1: Table corresponding to Question 3.

by $f_\Theta(\theta_1) = 0.3$, $f_\Theta(\theta_2) = 0.25$, $f_\Theta(\theta_3) = 0.25$ and $f_\Theta(\theta_4) = 0.2$. Suppose that x_2 is observed.

(a) What is the posterior distribution of Θ?

[4 marks]

(b) What is the Bayesian estimate under the squared error loss function?

[4 marks]

(c) What is the Bayesian estimate under the 0-1 loss function?

[4 marks]

[Total 12 marks]

4. An insurance company is deciding what premium to charge a certain policyholder. The company assumes that independently in each year, the number of claims made by the policyholder is Poisson distributed with parameter θ. Further, the company believes that θ is itself exponentially distributed with parameter 0.25. In the first year, the policyholder has produced 6 claims.

(a) Determine the Bayesian credibility estimate of the expected number of claims of the policyholder in year two.

[10 marks]

(b) Determine the Bühlmann credibility estimate of the expected number of claims of the policyholder in year two.

[10 marks]

(c) One year later the insurance company observes that the policyholder has made 5 claims in year two. Determine both the Bayesian and the Bühlmann credibility estimate of the expected number of claims in year three corresponding to this policyholder.

[8 marks]

[Total 28 marks]

END OF EXAMINATION PAPER