ERRATUM

MATH32062 Introduction to Algebraic Geometry

23 May 2012 9:45 – 11.45

Question 3 (a) should be replaced with the following:

3. In this question K denotes an arbitrary field.

(a) Let $V \subseteq \mathbb{P}^m(K)$, and $W \subseteq \mathbb{P}^n(K)$ be projective algebraic varieties and assume that V is irreducible. Define the notions of a *rational map* $\phi : V \dashrightarrow W$ and a *morphism* $\phi : V \to W$.

Correction to Question 4 (b)

In Question 4 (b), $O = (0 : 0 : 1)$ should be replaced by $O = (0 : 1 : 0)$.

Correction to exam: MATH38152

Social Statistics

A4.
"where m_i is equal to one or zero according to wether $Y_{i,t}$ is observed or not, respectively."
should read:
"where m_i is equal to zero or one according to wether $Y_{i,t}$ is observed or not, respectively."

Two hours

THE UNIVERSITY OF MANCHESTER

LINEAR ANALYSIS

24 May 2013

09:45 - 11:45

Answer **ALL EIGHT** questions in Section A (25 marks in total). Answer **THREE** of the four questions in Section B (75 marks in total). If more than three questions from Section B are attempted, then credit will be given for the best three answers.

Electronic calculators are permitted, provided they cannot store text.

SECTION A

Answer **ALL** eight questions

A1. State what is meant by a *norm* on a vector space over $\mathbb{C}$.

[3 marks]

A2. State what is meant by an *inner product* in a vector space over $\mathbb{C}$.

[3 marks]

A3. Assume $1 \leq p < q < \infty$. Give an example to show that $\ell^p(\mathbb{R}) \neq \ell^q(\mathbb{R})$.

[3 marks]

A4. Put

$$x = \left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, \dots\right).$$

Compute $\|x\|_1$.

[3 marks]

A5. State what is meant by the *norm of a linear functional* on a Banach space.

[3 marks]

A6. State what is meant by the *nth Bernstein polynomial* for a continuous function f on the interval $[0, 1]$.

[3 marks]

A7. Let X be a Banach space, and let $T : X \to X$ be a bounded linear operator. State what is meant by the *spectrum* of T. State what is meant by the *spectral radius* of T.

[4 marks]

A8. Show that $C[0, 1]$ is an infinite dimensional space.

[3 marks]

SECTION B

Answer **THREE** of the four questions

B9.

(a) State what is meant by a *Cauchy sequence* in a metric space. State what is meant by a *Banach space.*

(b) Let $X = \ell^1(\mathbb{R})$ endowed with the $\|\cdot\|_\infty$ norm. Show that this space is *not* a Banach space.

(c) Show that $C[0,1]$ endowed with the $\|\cdot\|_\infty$ norm is a Banach space. *(You can assume without proof that $\mathbb{R}$ is complete, and that the uniform limit of a sequence of continuous functions is continuous.)*

[25 marks]

B10.

(a) Let a map $f : \ell^1(\mathbb{R}) \to \mathbb{R}$ be defined as follows:

$$f(x_0, x_1, \dots) = \sum_{n=0}^{\infty} \frac{x_n}{(n+1)^2}.$$

Show that f is a bounded linear functional on $\ell^1(\mathbb{R})$ and compute its norm.

(b) State what is meant by a *self-adjoint operator* on a Hilbert space. Let $(a_0, a_1, \dots) \in \ell^\infty(\mathbb{R})$ be a fixed sequence and let the operator $T : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ be defined as follows:

$$T(x_0, x_1, \dots) = (a_0 x_0, a_1 x_1, \dots).$$

Show that T is well defined and self-adjoint.

(c) Let $H = \ell^2(\mathbb{R})$ and let $\{e_n\}_{n=0}^{\infty}$ denote the standard basis in H. Define a sequence of linear operators $A_n : H \to H$ as follows:

$$A_n(e_k) = \begin{cases} e_0, & k = n, \\ 0, & k \neq n, \end{cases}$$

and, for $x = (x_0, x_1, \dots) \in H$, put

$$A_n(x) = \sum_{k=0}^{\infty} x_k A_n(e_k).$$

Show that

- $\|A_n\| = 1$ for any $n \geq 1$.
- $A_n(x) \to 0$ for any fixed $x \in H$.

[25 marks]

B11.

(a) State what it means for two norms on a vector space to be *equivalent.*

Prove that $\|\cdot\|_1$ and $\|\cdot\|_2$ on $C[0,1]$ are not equivalent. *(You do not need to prove that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms.)*

(b) Let f denote a linear functional on a normed vector space. State what is meant for f to be *continuous.*

State what is meant for f to be *bounded.*

Prove that f is continuous if and only if it is bounded.

(c) Let $X = C[0,1]$ with the $\|\cdot\|_\infty$ norm and let for any $f \in C[0,1]$,

$$g(f) = \int_0^1 f(x^2)\, dx.$$

Prove that g is a linear bounded functional on X.

[25 marks]

B12.

(a) Let H denote a Hilbert space over $\mathbb{R}$ and, for fixed $y \in H$, put

$$f_y(x) = \langle x, y \rangle.$$

Show that f_y is a bounded linear functional on H and compute its norm. *(You may use the Cauchy-Schwarz inequality.)*

(b) Let $H = \ell^2(\mathbb{R})$ and let for $x = (x_0, x_1, x_2, \dots) \in H$,

$$f(x_0, x_1, x_2, \dots) = 3x_1 + 4x_2.$$

Use the result of part (a) to compute $\|f\|$.

(c) State what is meant by an *isometric isomorphism* between two Banach spaces. State what is meant by the *dual space* for a Banach space. Prove that any Hilbert space over $\mathbb{R}$ is isometrically isomorphic to its dual space. (This is the Riesz Representation Theorem.)

[25 marks]

SECTION C

Answer **ALL** of this section

C13.

(a) State and prove the Cauchy-Schwarz inequality for the inner products over $\mathbb{C}$.

(b) Let H be an infinite dimensional Hilbert space and let $\{f_n\}_{n=1}^{\infty} \subset H$ be a linearly independent subset. Describe the *Gram-Schmidt algorithm* to construct an orthonormal set $\{e_n\}_{n=1}^{\infty} \subset H$ with the same span as $\{f_n\}_{n=1}^{\infty}$. Prove that the resulting set $\{e_n\}_{n=1}^{\infty}$ is indeed orthonormal and that its span is the same as the span of $\{f_n\}_{n=1}^{\infty}$.

(c) Let H be a Hilbert space and let L be a linear subspace. State what is meant by the *orthogonal complement* $L^{\perp}$ of L.

Prove that $L^{\perp}$ is a closed linear subspace of H.

Prove that $H = \overline{L} \oplus L^{\perp}$, where $\overline{L}$ denotes the closure of L.

(d) Let H denote the vector space $C[-1,1]$ endowed with the inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Let L be a subset of $C[-1,1]$ which consists of *even functions*, i.e., $f \in L$ if and only if $f(-x) = f(x)$ for all $x \in [-1,1]$.

Using the definition of $L^{\perp}$, prove that $L^{\perp}$ is the space of *odd functions*, i.e., $g \in L^{\perp}$ if and only if $g(-x) = -g(x)$ for all $x \in [-1,1]$. *(You cannot use the formula $H = \overline{L} \oplus L^{\perp}$ here, since H is not a Hilbert space.)*

[50 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

ANALYTIC NUMBER THEORY

17 May 2013

09:45 - 11:45

Answer **THREE** of the four questions. (90 marks in total.) If more than three questions are answered, then credit will be given for the best three answers.

Electronic calculators may be used, provided that they cannot store text.

1.

i) a) Let f be a function differentiable on $\mathbb{R}$. Prove Euler's summation formula

$$\sum_{1\leq n\leq x} f(n) = \int_1^x f(t)\,dt + f(1) - \{x\} f(x) + \int_1^x f'(t)\{t\}\,dt$$

for real $x > 1$, where $\{x\}$ is the fractional part of x.

b) Deduce that

$$\sum_{1\leq n\leq x} \log n = x\log x - x + O(\log x).$$

c) By applying the formula in Part a with $f(n) = 1/n^s$ deduce that for $\mathrm{Re}s > 1$

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{1+s}}du.$$

Explain how this result can be used to define $\zeta(s)$ for $\mathrm{Re}s > 0$, justifying your claims.

ii) Prove that

$$F(s) = \frac{\zeta'(s)}{\zeta(s)} + \zeta(s)$$

has no pole at $s = 1$.

[30 marks]

2.

i) a) Define the von Mangoldt function $\Lambda(n)$.

b) Prove that

$$\sum_{d|n} \Lambda(d) = \log n$$

for any integer $n \geq 1$.

ii) a) Prove

$$\sum_{1 \leq m \leq x} \Lambda(m) \left[\frac{x}{m}\right] = x \log x - x + O(\log x),$$

for $x > 1$, where $[y]$ is the integer part of y.

You may assume that $\sum_{1 \leq n \leq x} \log n = x \log x - x + O(\log x)$.

b) Deduce that

$$\sum_{1 \leq m \leq x} \Lambda(m) \left(\left[\frac{x}{m}\right] - 2\left[\frac{x}{2m}\right] \right) = (\log 2)\, x + O(\log x),$$

for $x > 1$.

Further deduce that

$$\psi(x) - \psi(x/2) \leq (\log 2)\, x + O(\log x)$$

where $\psi(x) = \sum_{1 < m \leq x} \Lambda(m)$.

[30 marks]

3.

i) a) Prove that

$$\sum_{n=1}^{N} \frac{1}{n} \geq \log(N+1),$$

for any integer $N \geq 1$.

b) Prove

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \geq \sum_{n=1}^{N} \frac{1}{n}.$$

ii) a) Prove by Partial Summation that

$$\theta(x) = \pi(x) \log x + O\left(\frac{x}{\log x}\right),$$

where $\theta(x) = \sum_{p \leq x} \log p$.

You may assume Chebyshev's Theorem in the form $\pi(x) = O(x/\log x)$ *and* $\int_2^x (\log t)^{-\ell}\, dt = O\left(x/\log^{\ell} x\right)$ *for all integers* $\ell \geq 1$.

b) Deduce that

$$\theta(x) \sim x \quad \text{if, and only if,} \quad \pi(x) \sim \frac{x}{\log x}.$$

[30 marks]

4.

i) a) Write down the Euler Product for $\zeta(s)$.

b) Prove that

$$\frac{\zeta(2)\zeta(3)}{\zeta(6)} = \prod_p \left(1 + \frac{1}{p(p-1)}\right).$$

ii) a) Define the arithmetic functions ω and Q_2

b) Write the Dirichlet series $D_{2^\omega}(s)$ in terms of the Riemann zeta function evaluated at ks for various k.

c) Prove that

$$2^\omega = 1 * Q_2$$

You may assume that the convolution of multiplicative functions is multiplicative.

iii) Prove

$$\sum_{n \le x} 2^{\omega(n)} = \frac{1}{\zeta(2)} x \log x + O(x).$$

You may assume that $\sum_{n\le x} Q_2(n) = x/\zeta(2) + O\left(x^{1/2}\right)$, $\sum_{n\le x} 1/n = \log x + O(1)$ *and* $\sum_{n\le x} 1/n^{1/2} = 2x^{1/2} + O(1)$.

[30 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

DIFFERENTIABLE MANIFOLDS

4 June 2012

09:45 - 11:45

Answer any **THREE** of Questions 1–4

You do not need calculators.

Throughout the paper, where the index notation is used, the Einstein summation convention over repeated indices is applied if it is not explicitly stated otherwise.

1.

(a) For a set X, explain what is meant by a *chart* (or local *coordinate system*) on X and what is meant by an *atlas* on X.
Define the notion of a *change of coordinates* between two charts and explain what is meant by a *smooth* atlas.
Consider the real projective line $\mathbb{R}P^1$ as the set of equivalence classes $(x^1 : x^2)$ of non-zero vectors $(x^1, x^2) \in \mathbb{R}^2 \setminus \{0\}$ with respect to the equivalence relation $(x^1, x^2) \sim (\lambda x^1, \lambda x^2)$ for arbitrary $\lambda \neq 0$. Let $U_i = \{(x^1 : x^2) \mid x^i \neq 0\} \subset \mathbb{R}P^1$, $i = 1, 2$. Define maps $\varphi_i \colon \mathbb{R} \to U_i$ by $\varphi_1(u) = (1, u)$ and $\varphi_2(v) = (v, 1)$.
Show that each φ_i is a chart and that the collection $(\varphi_i)_{i=1,2}$ is an atlas.
Calculate the change of coordinate $\varphi_{12} = \varphi_1^{-1} \circ \varphi_2$ and deduce that the atlas is smooth.

[7 marks]

(b) Explain what is meant by a *closed submanifold* of dimension k of a smooth manifold M^n.
Consider a subset $\Gamma \subset \mathbb{R}^n$ specified by equations $f^i(x) = 0$, where $i = 1, \ldots, p$.
Explain what is meant by saying that these equations are *independent*.
Show that if the equations $f^i(x) = 0$ are independent, then the subset Γ is a closed submanifold of $\mathbb{R}^n$ of dimension $k = n - p$.

[7 marks]

(c) Consider the set S consisting of all pairs $(\boldsymbol{a}, \boldsymbol{b})$ of vectors in $\mathbb{R}^n$ such that $\boldsymbol{a} \cdot \boldsymbol{a} = \boldsymbol{b} \cdot \boldsymbol{b} = 1$ and $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ (for the standard scalar product on $\mathbb{R}^n$).
Check that the system of equations specifying S is of constant rank and therefore the set S has a natural structure of a smooth manifold. Find its dimension.
(Hint: to analyze the auxiliary system, consider the expansion of the vectors $\dot{\boldsymbol{a}}$ and $\dot{\boldsymbol{b}}$ over an orthonormal basis $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n}_1, \ldots, \boldsymbol{n}_{n-2}$ in $\mathbb{R}^n$, where $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_{n-2}$ are arbitrary unit vectors pairwise orthogonal and orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$.)

[6 marks]

2.

(a) Give the definition of a *tangent vector* $\boldsymbol{u}$ at a point $\boldsymbol{x} \in M$ in terms of the transformation law for its components u^i.
Define the tangent space $T_{\boldsymbol{x}}M$.
Show that the addition of vectors in $T_{\boldsymbol{x}}M$ is well-defined.

[7 marks]

(b) Explain what is meant by a *partition of unity* on a manifold M and when a partition of unity (f_μ) is said to be *subordinate* to an open cover (U_α) .
For a compact manifold M, show the existence of a finite partition of unity subordinate to an arbitrary open cover.

[7 marks]

(c) Consider the unitary group $U(n) = \{g \in \mathrm{Mat}(n, \mathbb{C}) \mid gg^* = E\}$ as a smooth manifold. Here E denotes the identity and $g^* = \bar{g}^T$ (transpose and complex conjugate).
Describe the tangent space $T_gU(n)$ at an arbitrary point $g \in U(n)$.
Show that the tangent bundle $TU(n)$ is diffeomorphic to the product $U(n) \times T_EU(n)$.

[6 marks]

3.

(a) Explain what is meant by a *derivation* over an algebra homomorphism $\alpha\colon A_1 \to A_2$, where A_1 and A_2 are commutative associative algebras.
State Hadamard's lemma in $\mathbb{R}^n$ (you do not have to give a proof) and deduce from it that, for arbitrary $\boldsymbol{x}_0 \in \mathbb{R}^n$, all derivations $D\colon\ C^\infty(\mathbb{R}^n) \to \mathbb{R}$ over the evaluation homomorphism $\mathrm{ev}_{\boldsymbol{x}_0}\colon\ C^\infty(\mathbb{R}^n) \to \mathbb{R}$ have the form $D = \partial_{\boldsymbol{v}}$ for some $\boldsymbol{v} \in \mathbb{R}^n$.

[7 marks]

(b) Explain what is meant by the *commutator of vector fields* on a manifold M and give a coordinate expression for the commutator of vector fields $\boldsymbol{u} = u^i \boldsymbol{e}_i$ and $\boldsymbol{w} = w^j \boldsymbol{e}_j$. (Here $(\boldsymbol{e}_i)$ is the basis of tangent vectors associated with a local coordinate system, so that $\boldsymbol{e}_i = \frac{\partial \boldsymbol{x}}{\partial x^i}$.)
Show that the commutator of vector fields satisfies the identity

$$[\boldsymbol{u}, f\boldsymbol{w}] = \partial_{\boldsymbol{u}} f\, \boldsymbol{w} + f[\boldsymbol{u}, \boldsymbol{w}]\,,$$

where $f \in C^\infty(M)$.

[7 marks]

(c) Consider on $\mathbb{R}^3$ with standard Cartesian coordinates the vector fields

$$L_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\,, \quad L_2 = -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\,, \quad L_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\,.$$

Find the pairwise commutators $[L_i, L_j]$ expressing them in terms of original vector fields L_1, L_2 and L_3.

[6 marks]

4.

(a) Give the axioms defining the *exterior differential* $d\colon\ \Omega^p(M) \to \Omega^{p+1}(M)$, for all $p = 0, 1, 2, \dots n$, on a manifold M^n.
Deduce from the axioms the expression for the differential of a 1-form $A = A_i dx^i$ on an n-dimensional manifold.
Write out this expression in full for the case $n = 3$.

[7 marks]

(b) Give the definition of the *integral* of an n-form ω over a compact oriented manifold M^n.
Prove that the integral $\int_M \omega$ does not depend on choices of an atlas (with a given orientation) and a partition of unity.

[7 marks]

(c) Consider on $\mathbb{R}^2 \setminus \{0\}$ the 1-form

$$\omega = r^\alpha (xdy - ydx)\,.$$

depending on a parameter α. Here x, y are standard Cartesian coordinates and $r = \sqrt{x^2 + y^2}$ is the polar radius.
Find the value α_0 of the parameter α such that the form ω is closed.
For $\alpha = \alpha_0$, consider the closed form ω and show that its de Rham cohomology class is non-zero. (Hint: you may use the Stokes theorem.)

[6 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY

16 May 2013

09:45 - 11:45

Answer **THREE** of the FOUR questions.

If four questions are answered, credit will be given for the best three questions.

All questions are worth 20 marks.

Electronic calculators are permitted, provided they cannot store text.

1.

(a) Explain what is meant by saying that G is a Riemannian metric on a manifold M.

Consider the plane $\mathbf{R}^2$ with standard coordinates x, y equipped with the Riemannian metric $G = \frac{4(dx^2+dy^2)}{(1+x^2+y^2)^2}$.

Find the cosine of the angle between the vectors $\mathbf{A} = 2\partial_x + \partial_y$ and $\mathbf{B} = \partial_x + 3\partial_y$ attached at the point $(1, 1)$.

Let $C_1\colon \begin{cases} x = f(t) \\ y = h(t) \end{cases}$, $0 \le t \le 1$, be a curve such that its length with respect to the metric G is equal to $a > 0$. Show that the length of the curve $C_2\colon \begin{cases} x = h(t) \\ y = f(t) \end{cases}$ also equals a.

[8 marks]

(b) Explain what is meant by saying that a Riemannian manifold is locally Euclidean.

Show that the surface of the cylinder $x^2 + y^2 = a^2$ in $\mathbf{E}^3$ with the induced Riemannian metric is locally Euclidean.

[6 marks]

(c) Consider a Riemannian manifold M^n with a metric $G = g_{ik}dx^i dx^k$.

Write down the formula for the volume element on M^n (area element for $n = 2$).

Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with the Riemannian metric

$$G = e^{-x^2-y^2}(dx^2 + dy^2)\,.$$

Calculate the total area of the plane with respect to this metric.

(You may use the formula $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.)

Show that this Riemannian manifold is not isometric to the plane $\mathbf{R}^2$ equipped with the standard Euclidean metric $G_{\text{Euclid.}} = du^2 + dv^2$, i.e. there is no diffeomorphism $\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$ of $\mathbf{R}^2$ to itself, such that under this diffeomorphism the Euclidean metric $G_{\text{Euclid.}} = du^2 + dv^2$ transforms into the metric $G = e^{-x^2-y^2}(dx^2 + dy^2)$. [6 marks]

P.T.O.

2.

(a) Explain what is meant by an affine connection on a manifold.

Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u,v) we have that $\nabla_{\frac{\partial}{\partial u}}\left(\frac{\partial}{\partial u}\right) = \frac{\partial}{\partial v}$.

Calculate the Christoffel symbols Γ^u_{uu} and Γ^v_{uu}.

Let ∇ be an arbitrary affine connection on a manifold M. Show that for arbitrary vector fields $\mathbf{X}$, $\mathbf{Y}$ and an arbitrary function F,

$$e^{-F}\nabla_{\mathbf{X}}\left(e^F\mathbf{Y}\right) = \left(\partial_{\mathbf{X}}F\right)\mathbf{Y} + \nabla_{\mathbf{X}}\mathbf{Y}\,.$$

[8 marks]

(b) Explain what is meant by the induced connection on a surface in Euclidean space.

Consider a sphere in $\mathbf{E}^3$:

$$\mathbf{r}(\theta,\varphi):\quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}\,.$$

Let ∇ be the induced connection on the sphere.

Calculate $\nabla_{\partial_\theta}\partial_\theta$. [6 marks]

(c) Give a detailed formulation of the Levi-Civita Theorem. In particular write down the expression for the Christoffel symbols Γ^i_{km} in terms of a Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

Let $\mathbf{E}^2$ be the Euclidean plane with the standard Euclidean metric $G_{\text{Eucl.}} = dx^2 + dy^2$.

Show that for the Levi-Civita connection of this metric the Christoffel symbols vanish in the Cartesian coordinates x, y.

Let ∇ be a symmetric connection on the Euclidean plane $\mathbf{E}^2$ such that its Cristoffel symbols satisfy the condition $\Gamma^y_{xy} = \Gamma^y_{yx} \neq 0$.

Show that for vector fields $\mathbf{A} = \partial_x$ and $\mathbf{B} = \partial_y$, $\partial_{\mathbf{A}}\langle\mathbf{B},\mathbf{B}\rangle \neq 2\langle\nabla_{\mathbf{A}}\mathbf{B},\mathbf{B}\rangle$, i.e. the Euclidean scalar product $\langle\ ,\ \rangle$ is not preserved with respect to this connection.

[6 marks]

P.T.O.

3.

(a) Define a geodesic on a Riemannian manifold as a parameterised curve.

Write down the differential equations for geodesics in terms of Christoffel symbols.

What are the geodesics of the surface of a cylinder $x^2 + y^2 = a^2$?

[7 marks]

(b) Explain what is meant by the Lagrangian of a "free" particle on a Riemannian manifold.

Explain what is the relation between the Lagrangian of a free particle and the differential equations for geodesics.

Calculate the Christoffel symbols for the Euclidean plane $\mathbf{E}^2$ in polar coordinates r, φ (where $x = r\cos\varphi, y = r\sin\varphi$).

(You may use the Lagrangian of a free particle in $\mathbf{E}^2$ in polar coordinates.)

[6 marks]

(c) Consider the Lobachevsky plane as an upper half plane ($y > 0$) in $\mathbf{R}^2$ equipped with the metric $G = \frac{dx^2+dy^2}{y^2}$.

Consider a vertical ray C: $x(t) = 1, y(t) = 1 + t$, $0 \le t < \infty$ on the Lobachevsky plane.

Find the parallel transport $\mathbf{X}(t)$ of the vector $\mathbf{X}_0 = \partial_y$ attached at the initial point $(1, 1)$ along the ray C at an arbitrary point of the ray.

Find the parallel transport $\mathbf{Y}(t)$ of the vector $\mathbf{Y}_0 = \partial_x + \partial_y$ attached at the same initial point $(1, 1)$ along the ray C at an arbitrary point of the ray.
(You may use the fact that the vertical ray C is a geodesic in the Lobachevsky plane.)

[7 marks]

P.T.O.

4.

(a) Let M be a surface embedded in the Euclidean space $\mathbf{E}^3$. Let $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface. Consider the derivation formula

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}, \tag{1}$$

where a, b and c are 1-forms on the surface M.
Express the mean curvature and the Gaussian curvature of M in terms of these one-forms and vector fields.
Show that $da + b \wedge c = 0$.

[6 marks]

(b) Consider the surface of a cylinder in Euclidean space $\mathbf{E}^3$,

$$\mathbf{r}(\varphi, h): \quad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} .$$

Find vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ defined at the points of the cylinder such that they form an orthonormal basis at any point, the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface.
For the obtained orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ calculate the 1-forms a, b and c in the derivation formula (1).
Deduce from these calculations the mean curvature and the Gaussian curvature of the cylinder.

[5 marks]

(c) Give a definition of the curvature tensor for a manifold equipped with a connection.

State the relation between the Riemann curvature tensor of the Levi-Civita connection of a surface in $\mathbf{E}^3$ and its Gaussian curvature K.

Consider a surface M in $\mathbf{E}^3$ with the induced Riemannian metric. Explain why this surface is not locally Euclidean in the case if its Gaussian curvature does not vanish.

[9 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

COMMUTATIVE ALGEBRA

31 May 2013

14:00 - 16:00

Answer **ALL FOUR** questions in Section A (40 marks in total). Answer **TWO** of the THREE questions in Section B (40 marks in total). If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

The use of electronic calculators is permitted provided they cannot store text.

SECTION A

Answer **ALL** questions in this section (40 marks in total)

A1.

(a) What is meant by a remainder of a polynomial $f \in \mathbb{Q}[X,Y,Z]$ modulo a set $F \subset \mathbb{Q}[X,Y,Z]$ with respect to a monomial ordering $\preccurlyeq$?

(b) Give an example of a set $F \subset \mathbb{Q}[X,Y,Z]$ such that the polynomial X has more than one remainder modulo F with respect to Lex with $X \succ Y \succ Z$.

(c) Reduce the polynomial $f = X^4YZ^2 \in \mathbb{Q}[X,Y,Z]$ modulo the set $F = \{f_1 = YZ^2 + Z^3, f_2 = 2XY - X^3\}$ with respect to Lex with $X \succ Y \succ Z$. Hence write $f = q_1f_1 + q_2f_2 + r$ where q_1, q_2 are polynomials and r is the remainder that you found.

[12 marks]

A2. Let R be a commutative ring.

(a) What is meant by saying that R is noetherian?

(b) Explain what is meant by saying that R satisfies the ascending chain condition (acc) on ideals.

(c) Prove that a noetherian ring satisfies the ascending chain condition on ideals.

(d) State, without proof, the general Hilbert Basis Theorem.

[12 marks]

A3.

(a) If D is a commutative domain, define the notion of an irreducible element of D.

(b) Describe, without proof, all the irreducible elements of $\mathbb{C}[X]$.

(c) Write down a factorisation of $X^6 + 64Y^6$ into irreducibles in $\mathbb{C}[X,Y]$. You do not have to justify your answer.

[8 marks]

A4. For each of the following polynomials, determine whether it is irreducible in the given ring, and briefly justify your answer.

(a) $(X+1)^8 + X^2 + 2$ in $\mathbb{R}[X]$.

(b) $50 + 75X + 100X^3 + 120X^5$ in $\mathbb{Q}[X]$.

(c) $1 + 2X + X^2 + X^5$ in $\mathbb{Z}_3[X]$.

[8 marks]

SECTION B

Answer **TWO** of the three questions in this section (40 marks in total).

If more than TWO questions from this section are attempted, then credit will be given for the best TWO answers.

B5. Let $R = K[X_1, X_2, \ldots, X_n]$ where K is a field, and let $\preccurlyeq$ be a monomial ordering for R.

(a) Give the definition of a Gröbner basis with respect to $\preccurlyeq$ of an ideal of R.

(b) Give a simple description of all the possible Gröbner bases of the ideal $R = <1>$ of R with respect to $\preccurlyeq$. You do not have to justify your answer.

(c) i. Let now $f_1 = Y^2 + XY^3$, $f_2 = 2 - X + Y^3$ in $\mathbb{R}[X, Y]$, and let I be the ideal of $\mathbb{R}[X, Y]$ generated by f_1 and f_2. Find a Gröbner basis of I with respect to Deglex with $X \succ Y$.

ii. Prove that the subset $\mathcal{V}(I)$ of the euclidean plane $\mathbb{R}^2$ lies on a circle of radius 1 in $\mathbb{R}^2$. Find the centre of that circle.

[20 marks]

B6. Let I be an ideal of a commutative domain R.

(a) What is meant by saying that I is a prime ideal?

(b) Define the radical $\sqrt{I}$ of I.

(c) Prove that if I is a prime ideal, then I is a radical ideal (meaning $\sqrt{I} = I$).

(d) Describe, without proof, all radical ideals of $\mathbb{Z}$ which are not prime ideals.

(e) Show that $I = <2, 1+\sqrt{-7}>$ is a prime ideal of $\mathbb{Z}[\sqrt{-7}]$. You may use results from the course, and you do not need to reproduce their proofs.

[20 marks]

B7. Let K be a field.

(a) What is a monomial ideal of $K[X_1, \ldots, X_n]$? Explain why the ideal $\{0\}$ is a monomial ideal.

(b) Show that $<X^2Y + XY^2>$ is not a monomial ideal of $K[X, Y]$. You may use any results from the course, and you do not need to reproduce their proofs.

(c) List all the subsets V of $K^2 = \{(a, b) \mid a, b \in K\}$ such that $V = \mathcal{V}(I)$ for some monomial ideal I of $K[X, Y]$. For each V listed, give one example of a monomial ideal I such that $V = \mathcal{V}(I)$.

(d) Let I be an arbitrary ideal of $K[X_1, \ldots, X_n]$.

i. Define $\mathrm{LT}_{\preccurlyeq}(I)$, the ideal of leading terms of I with respect to a monomial ordering $\preccurlyeq$.

ii. Show that if $I \subseteq \mathrm{LT}_{\preccurlyeq}(I)$, then I is a monomial ideal. (*Hint*: consider the reduced Gröbner basis of I.)

[20 marks]

END OF EXAMINATION PAPER

ERRATUM

MATH32062 Introduction to Algebraic Geometry

23 May 2012 9:45 – 11.45

Question 3 (a) should be replaced with the following:

3. In this question K denotes an arbitrary field.

(a) Let $V \subseteq \mathbb{P}^m(K)$, and $W \subseteq \mathbb{P}^n(K)$ be projective algebraic varieties and assume that V is irreducible. Define the notions of a *rational map* $\phi : V \dashrightarrow W$ and a *morphism* $\phi : V \to W$.

Correction to Question 4 (b)

In Question 4 (b), $O = (0 : 0 : 1)$ should be replaced by $O = (0 : 1 : 0)$.

Two hours

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO ALGEBRAIC GEOMETRY

23 May 2013

09:45 - 11:45

Answer **THREE** of the FOUR questions.

If four questions are answered, credit will be given for the best three answers.

Electronic calculators are permitted, provided they cannot store text.

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1. (a) Let K be an arbitrary field.

(i) Define the *affine algebraic variety* $V(J) \subseteq \mathbb{A}^n(K)$ determined by an ideal $J \lhd K[x_1, x_2, \ldots, x_n]$.

(ii) Define the *ideal* $I(X)$ of a set $X \subseteq \mathbb{A}^n(K)$.

(b) Let $J = \langle (z-1)^2, x^2 + xy + y - x^2 z \rangle \lhd \mathbb{C}[x, y, z]$.
Show that $\left.\dfrac{\partial f}{\partial z}\right|_{(0,0,1)} = 0$ for any $f \in J$.
Deduce that $z - 1 \notin J$. Show that $z - 1 \in I(V(J))$.

(c) (i) Define what it means that an affine algebraic variety V is *irreducible.*

(ii) Prove that every affine algebraic variety V can be written as a finite union of irreducible affine algebraic varieties.

(d) Find the irreducible components of the variety

$$W = V(\langle x(x-1), x^2 + yz - 1 \rangle) \subset \mathbb{A}^3(\mathbb{C}).$$

[20 marks]

2. (a) Let K be a field. Let $V \subseteq \mathbb{A}^n(K)$ be an affine algebraic variety and let $P \in V$.

(i) Explain what it means for a line to be *tangent to* V *at* P.

(ii) Define the *tangent space* $T_P V$ to V at P.

(iii) Describe without proof how $T_P V$ and its dimension can be calculated by using the Jacobian matrix.

(b) Let x, y be co-ordinates on $\mathbb{A}^2(\mathbb{C})$ and let t be the co-ordinate on $\mathbb{A}^1(\mathbb{C})$. Let $C \subset \mathbb{A}^2(\mathbb{C})$ be the variety defined by the equation $y^5 - 2y^3 + y - x^2 = 0$. (You may assume without proof that $y^5 - 2y^3 + y - x^2 = 0$ is an irreducible polynomial, so C is irreducible and $I(C) = \langle y^5 - 2y^3 + y - x^2 \rangle$.)

(i) Find the singular points of C. Deduce that C is not isomorphic to $\mathbb{A}^1(\mathbb{C})$.

(ii) Show that $\phi(t) = (t(t^4 - 1), t^2)$ is a morphism $\mathbb{A}^1 \to C$.

(iii) Let $\psi : C \dashrightarrow \mathbb{A}^1$ be the rational map given by $\psi(x, y) = \dfrac{x}{y^2 - 1}$. Show that ψ is the inverse of ϕ considered as a rational map. (*Hint:* Show that $\left(\dfrac{x}{y^2 - 1}\right)^2 = y$ as rational functions on C.)

[20 marks]

3. In this question K denotes an arbitrary field.

(a) Let $V \subseteq \mathbb{P}^m(K)$, and $W \subseteq \mathbb{P}^n(K)$ be projective algebraic varieties and assume that V is irreducible. Define the notions of a *rational map* $\Phi : V \dashrightarrow W$ and a *morphism* $\Phi : V \to W$.

(b) Show that any rational map $\Phi : \mathbb{P}^1(K) \dashrightarrow \mathbb{P}^1(K)$ of the form $\Phi(X_0 : X_1) = (aX_0 + bX_1 : cX_0 + dX_1)$ for some $a, b, c, d \in K$ with $ad - bc \neq 0$ is in fact an isomorphism $\mathbb{P}^1(K) \to \mathbb{P}^1(K)$.

(c) In this part of the question we identify $\mathbb{P}^1(K)$ with $K \cup \{\infty\}$ by identifying $(X_0 : X_1)$ with $X_0/X_1 \in K$ if $X_1 \neq 0$ and $(X_0 : 0)$ with ∞ as usual.
Recall that the cross ratio of four distinct elements of $K \cup \{\infty\}$ is defined to be

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

with appropriate rules for arithmetic involving ∞ and for division by 0.

(i) Let z_1, z_2, z_3 and z_4 be distinct elements of K and let $\phi : K \cup \{\infty\} \to K \cup \{\infty\}$ be a function of the form $\phi(z) = \dfrac{az + b}{cz + d}$ for some $a, b, c, d \in K$ with $ad - bc \neq 0$. Show that

$$(z_1, z_2; z_3, z_4) = (\phi(z_1), \phi(z_2); \phi(z_3), \phi(z_4)).$$

(ii) Find $a, b, c, d \in \mathbb{C}$ such that $\phi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, $\phi(z) = \dfrac{az + b}{cz + d}$ satisfies $\phi(-1) = 2$, $\phi(0) = 1$ and $\phi(3) = 6$.

[20 marks]

4. (a) Let K be a field. Let $E \subset \mathbb{P}^2(K)$ be a non-singular cubic curve and let O be a point of E. Describe without proof how one can define the addition operation on the points of E to obtain a commutative group in which O is the zero element. (You should not assume that the equation of E is in a special form.)

(b) Let E be the elliptic curve given by the affine equation

$$y^2 = x^3 + 2x^2 - 8x$$

over $\mathbb{C}$ together with the point $O = (0 : 0 : 1)$ in homogeneous co-ordinates, which is taken to be the zero element of the group law as usual.

(i) Let $P = (-1, 3)$, $Q = (-4, 0)$. Calculate $P + Q$ and $2P$.

(ii) Find the three points $X \in E$, $X \neq O$, such that $2X = O$.

(c) The elliptic curve F over $\mathbb{C}$ is given by the affine equation

$$y^2 + 4xy = x^3 - 7x^2 + 3x - 2.$$

Find p, $q \in \mathbb{C}$ such that the curve $y^2 = x^3 + px + q$ is isomorphic to F.

[20 marks]

END OF EXAMINATION PAPER

ERRATUM
MATH32062 Introduction to Algebraic Geometry
23 May 2013, 9:45–11:45

In Question 4 (b), $O = (0 : 0 : 1)$ should be replaced by $O = (0 : 1 : 0)$.

Two hours

UNIVERSITY OF MANCHESTER

GREEN'S FUNCTIONS, INTEGRAL EQUATIONS AND APPLICATIONS

14:00–16:00

Answer **FOUR** of the five questions (80 marks in total)
If more than four questions are attempted, then credit will be given for the best four answers.

Electronic calculators may be used, provided that they cannot store text.

1.
(a) Given the linear differential operator

$$\mathcal{L} = \frac{d^2}{dx^2} + \alpha\frac{d}{dx} + 1$$

with $\alpha \in \mathbb{C}$ and homogeneous boundary conditions

$$\mathcal{B} = \{u(0) = 0, u'(\pi/2) = 0\},$$

use integration by parts and inner product notation to find the adjoint operator $\mathcal{L}^*$ and adjoint boundary conditions $\mathcal{B}^*$.

Define what it means for a linear operator and associated boundary conditions to be fully self-adjoint.

State if $\mathcal{L}$ and $\mathcal{B}$ as given above are self-adjoint when

(i) $\alpha = 0$, (ii) $\alpha = -2$.

[7 marks]

(b) Consider now the boundary value problem (BVP)

$$\mathcal{L}u = f(x), \qquad \mathcal{B} = \{u(0) = 0, u'(\pi/2) = 0\}$$

where $f(x)$ is a given function.

Apply the Fredholm alternative to determine the existence and uniqueness properties of this BVP in *both* cases (a)(i) and (a)(ii) above.

[7 marks]

(c) Using the definition of the Green's function and its adjoint, together with $\langle v, \mathcal{L}w\rangle = \langle \mathcal{L}^*v, w\rangle$, choose v and w appropriately to prove that

$$G(x_1, x_2) = \overline{G^*(x_2, x_1)}.$$

Then make an alternative choice of v and w in order to show that when the Green's function exists, the solution to the BVP considered in (b) can be written in the form

$$u(x) = \int_0^{\pi/2} G(x, x_0) f(x_0)\, dx_0.$$

[6 marks]

2.

This question refers to the *regular Sturm-Liouville boundary value problem*

$$\mathcal{L}u = f(x)$$

for $x \in [0, 1]$ subject to some homogeneous boundary conditions $\mathcal{B}$.

Note that eigenvalues λ_n and associated eigenfunctions $\phi_n(x)$ are defined via

$$\mathcal{L}\phi_n + \lambda_n \phi_n = 0.$$

(a)
(i) Pose an eigenfunction expansion for the solution of the form

$$u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \tag{1}$$

Apply the linear operator to this and use orthogonality, to show that

$$a_n = \frac{-\int_0^1 f(x_0)\phi_n(x_0)\, dx_0}{\lambda_n \int_0^1 \phi_n^2(x_1)\, dx_1}.$$

Substitute this into (1) in order to obtain the solution in the form

$$u(x) = \int_0^1 G(x, x_0) f(x_0)\, dx_0$$

and therefore determine an expression for the eigenfunction expansion of $G(x, x_0)$.

(ii) Find the eigenfunctions and eigenvalues for the specific case

$$\mathcal{L} = \frac{d^2}{dx^2}, \qquad \mathcal{B} = \{u(0) = 0, u(1) = 0\}.$$

[**HINT**: You can assume eigenvalues λ are real. Then consider $\lambda < 0, \lambda = 0$ and $\lambda > 0$ in turn.]

Thus derive the eigenfunction expansion for the associated Green's function using the expression you determined in (i).

[14 marks]

(b) Use the direct (explicit) method to determine the Green's function associated with the operator and boundary conditions in (a)(ii) above.

[6 marks]

3.

(a) The *free-space Green's function* for the Laplacian operator $\mathcal{L} = \nabla^2$ in two dimensions is

$$G_{2\infty}(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \ln(|\mathbf{x} - \mathbf{x}_0|) = \frac{1}{4\pi} \ln((x - x_0)^2 + (y - y_0)^2).$$

Confirm that this function satisfies Laplace's equation when $\mathbf{x} \neq \mathbf{x}_0$.

[4 marks]

(b)

In the following *you may use without proof* the result that for any two functions $f(\mathbf{x})$ and $g(\mathbf{x})$

$$\int_D (f\nabla^2 g - g\nabla^2 f)\ d\mathbf{x} = \int_{\partial D} (f\nabla g - g\nabla f) \cdot \mathbf{n}\ ds \tag{2}$$

where ∂D is the boundary of a domain D with outward pointing normal $\mathbf{n}$.

Consider the following boundary value problem. For $\mathbf{x} \in D$,

$$\nabla^2 u = Q(\mathbf{x})$$

for some given function $Q(\mathbf{x})$, with the associated boundary condition $u(\mathbf{x}) = h(\mathbf{x})$ for $\mathbf{x} \in \partial D$.

Choose f and g appropriately in (2) in order to derive the integral form of solution to this boundary value problem, i.e.

$$u(\mathbf{x}) = \int_D G(\mathbf{x}, \mathbf{x}_0) Q(\mathbf{x}_0)\ d\mathbf{x}_0 + \int_{\partial D} h(\mathbf{x}_0) \nabla_{\mathbf{x}_0} G(\mathbf{x}_0, \mathbf{x}) \cdot \mathbf{n}\ ds. \tag{3}$$

[6 marks]

(c) Consider the boundary value problem in (b) where the domain D is the lower half-space $D = \{-\infty < x < \infty, y \leq 0\}$. Use the method of images to determine the associated Green's function.

[**HINT**: The Green's function should satisfy $G = 0$ on $y = 0$.]

Set $Q(\mathbf{x}) = 0$. Substitute the Green's function you have just determined into the integral solution form (3) together with

$$h(\mathbf{x}_0) = \begin{cases} 1, & -1 \leq x_0 \leq 1, y_0 = 0 \\ 0, & \text{otherwise} \end{cases}$$

to show that

$$u(\mathbf{x}) = \frac{1}{\pi} \left[\arctan\left(\frac{x+1}{y}\right) - \arctan\left(\frac{x-1}{y}\right) \right].$$

[**HINT**: You can assume that the contribution from the boundary integral "at infinity" is zero.]

[10 marks]

4.

This question concerns Fredholm Integral equations, i.e. those of the form

$$u(x) = \lambda \int_a^b K(x, x_0)u(x_0)\ dx_0 + g(x).$$

(a) Construct the Green's function for the boundary value problem

$$u''(x) = f(x) \qquad \mathcal{B} = \{u(0) = 0, u'(1) = 0\}$$

showing that it takes the form $G(x, x_0) = -xH(x_0 - x) - x_0H(x - x_0)$, where $H(x)$ is the Heaviside function.

Write the eigenvalue problem

$$u''(x) + \lambda u(x) = 2, \qquad \mathcal{B} = \{u(0) = 0, u'(1) = 0\} \tag{4}$$

in the form of a Fredholm Integral equation, identifying

$$K(x, x_0) = -G(x, x_0), \qquad g(x) = x^2 - 2x.$$

[10 marks]

(b)
(i) Given $\lambda = \epsilon^2$, solve the differential equation (4) directly to obtain an explicit solution $u(x)$.

Then, using

$$\sin x \approx x - \frac{x^3}{6} + ..., \qquad \cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24} + ..., \qquad \tan x \approx x + \frac{x^3}{3} + ...,$$

expand your explicit solution for $\epsilon \ll 1$, showing that

$$u(x) \approx x^2 - 2x + \frac{\epsilon^2}{12}(-8x + 4x^3 - x^4) + ... \tag{5}$$

(ii) With $\lambda = \epsilon^2 \ll 1$, find the first *two terms* in a Neumann series solution of the corresponding Fredholm Integral equation identified in (a), showing that we recover (5).

[10 marks]

5.
(a) Consider waves in one dimension with time harmonic behaviour $\exp(-i\omega t)$. For a uniform medium they are governed by

$$u''(x) + k_0^2 u(x) = 0$$

where $k_0 \in \mathbb{R}$.

Show that the associated *free space* Green's function is

$$G(x, x_0) = \frac{1}{2ik_0} \exp(ik_0|x - x_0|).$$

[6 marks]

(b) Suppose now that the medium is *non-uniform* with wavenumber $k(x)$:

$$u''(x) + k^2(x)u(x) = 0$$

and $k(x) \to k_0$ as $|x| \to \infty$.

A wave is incident from the left (i.e. it propagates to the right), taking the form

$$u_i(x) = \exp(ik_0 x).$$

By writing the total wave field in the form $u(x) = u_i(x) + u_s(x)$ where u_s is thus the "scattered wave", show that

$$u_s''(x) + k_0^2 u_s(x) = (k_0^2 - k^2(x))u(x).$$

Therefore, using the Green's function from (a), show that $u(x)$ is governed by the integral equation

$$u(x) = u_i(x) + \int_{-\infty}^{\infty} (k_0^2 - k^2(x_0))u(x_0)G(x, x_0)\ dx_0. \tag{6}$$

[10 marks]

(c) Now take the specific case

$$k(x_0) = \begin{cases} k_1, & 0 \le x \le 1, \\ k_0, & \text{otherwise.} \end{cases}$$

Using (6), determine the first two terms in a *Born approximation* to the solution $u(x)$ for $x \ge 1$.

[4 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

WAVE MOTION

28 May 2013

09:45 – 11:45

Answer **ALL** six questions.

Electronic calculators are permitted, provided they cannot store text.

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1. Consider a physical signal $y(x,t)$ in the form

$$y(x,t) = A\sin(kx - \omega(k)t) + A\sin((k+\delta)x - \omega(k+\delta)t)\,,$$

in the usual notation, where $\delta > 0$ is a perturbation to the wavenumber $k > 0$.

Show that when δ is small, the solution can be written as

$$y(x,t) = 2A\cos\left(\frac{\delta}{2}(x - \omega'(k)t)\right)\sin\left(kx - \omega(k)t\right) + O(\delta)\,.$$

Sketch the signal at a fixed instant in time, indicating the direction of propagation for $\omega > 0$. Indicate the envelope of the wave and the carrier wave in the sketch and give their speed of propagation.

You may use:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)\,.$$

[8 marks]

2. Consider a physical signal $f(x,y,t) = Ae^{i(kx+ly-\omega t)}$, in the usual notation, which satisfies the governing equation

$$\frac{\partial f}{\partial t} - R^2\frac{\partial}{\partial t}\left(\nabla^2 f\right) - \beta R^2\frac{\partial f}{\partial x} = 0\,,$$

where R and β are positive constants and ∇^2 is the two-dimensional Laplacian.

(i) Show that the dispersion relation is

$$\omega = -\beta R^2\frac{k}{1 + R^2(k^2 + l^2)}\,.$$

(ii) Restricting attention to the case of $l = 1$, sketch ω as a function of k for $k > 0$. Determine the group velocity, c_g, and show that $c_g > 0$ for $k > k^*$, where k^* is to be determined.

[10 marks]

3. The dispersion relation for linearised free surface waves under the influence of a surface tension, T, on a deep water layer is

$$\omega^2 = gK + TK^3/\rho\,,$$

in the usual notation.

Show that a wavenumber $K = K_{min}$ exists for which the phase speed is a minimum and determine $c(K_{min})$. Sketch the square of the phase speed as a function of K.

[8 marks]

4. Consider planar sound waves in a slowly varying channel of cross-sectional area $A(x)$. You are given that the sound wave perturbations $\tilde{u}$, $\tilde{p}$, $\tilde{\rho}$ (to the velocity, pressure and density respectively) satisfy:

$$A(x)\frac{\partial \tilde{\rho}}{\partial t} = -\rho_o \frac{\partial}{\partial x}\left(A(x)\tilde{u}\right),$$

and

$$\rho_o \frac{\partial \tilde{u}}{\partial t} = -\frac{\partial \tilde{p}}{\partial x}.$$

Here ρ_o is the undisturbed density of the compressible fluid in which the sound waves are travelling and $\tilde{\rho} = \gamma\tilde{p}$, for some constant γ.

Show that the pressure perturbation $\tilde{p}$ must satisfy the modified wave equation

$$\frac{\partial^2 \tilde{p}}{\partial t^2} = c^2 \frac{1}{A(x)}\frac{\partial}{\partial x}\left(A(x)\frac{\partial \tilde{p}}{\partial x}\right),$$

where c is a constant to be determined.

In the special case of $A(x) = \exp(\alpha x)$, show that for a wave of the form $\tilde{p}(x,t) = a(x)\exp(i\omega t)$ (for some fixed frequency ω), there is a critical value of α above which the waves cease to propagate through the channel.

[12 marks]

5. Consider small disturbances to an inviscid, incompressible fluid of constant density ρ, rotating at constant angular frequency Ω. In a Cartesian coordinate system (x, y, z), a wave motion with velocity field $\underline{u} = (u, v, w)$ and associated pressure p satisfies

$$\frac{\partial \underline{u}}{\partial t} + 2\underline{\Omega} \times \underline{u} = -\frac{1}{\rho}\nabla p,$$

$$\operatorname{div} \underline{u} = 0,$$

where $\underline{\Omega} = (0, 0, \Omega)$.

(i) Using these equations, show that

$$2\Omega\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = -\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right),$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) - 2\Omega\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

(ii) Hence show that the pressure p satisfies

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right) + 4\Omega^2 \frac{\partial^2 p}{\partial z^2} = 0.$$

(iii) Verify that a wave solution exists in the form

$$p = Ae^{i(kx+ly+mz-\omega t)},$$

in the usual notation and give the dispersion relation $\omega = \omega(k, l, m)$. Show that time-harmonic perturbations with a period shorter than π/Ω cannot exist in this form.

(iv) Determine the group velocity from the definition

$$\underline{c}_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m} \right) ,$$

and verify that direction of the group velocity is perpendicular to the wavenumber vector.

[21 marks]

6. Consider small disturbances in an inviscid, compressible fluid of sound speed c that is otherwise uniform and at rest.

(i) In a Cartesian coordinate system x, y, z, the fluid occupies a pipe of square cross section: $z \in (0, a)$, $y \in (0, b)$ and $x \in (-\infty, \infty)$. State the governing equation for the velocity potential $\phi(x, y, z, t)$ of a linear acoustic wave. Given that the boundaries $z = 0, a$ and $y = 0, b$ are impermeable, state the appropriate boundary conditions for ϕ.

(ii) For a velocity potential in the form

$$\phi = \Re \left\{ \hat{\phi}(y, z) e^{i(kx - \omega t)} \right\} ,$$

determine $\hat{\phi}$ by separation of variables. Hence show that

$$\omega^2 = c^2 (m^2 \pi^2 / a^2 + n^2 \pi^2 / b^2 + k^2) ,$$

where $m, n = 0, 1, 2, \cdots$.

(iii) Consider the case of $n = 0$. A naive interpretation of this solution is as a wave propagating in the x-direction with a phase speed

$$\frac{\omega}{k} = c \left(\frac{m^2 \pi^2}{k^2 a^2} + 1 \right)^{\frac{1}{2}} ,$$

which is *greater* than c (the sound speed) when $m \neq 0$. Give the correct interpretation of this solution as the sum of two plane waves, showing that the speed of propagation is (as you should expect) the speed of sound c.

[21 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL BIOLOGY

23 May 2013

09:45 - 11:45

Answer **ALL THREE** questions in Section A (30 marks in total).

Answer **TWO** of the THREE questions in Section B (50 marks in total).

If more than TWO questions from Section B are attempted, then credit will be given for the best TWO answers.

The use of electronic calculators is permitted provided they cannot store text.

SECTION A

Answer **ALL** 3 questions

A1. The Lotka-Volterra predator-prey system is

$$\frac{dN}{dt} = N(a - bP) \qquad \frac{dP}{dt} = P(cN - d) \tag{A1.1}$$

where $N(t)$ is the population of prey, $P(t)$ is that of the predators and a, b, c and d are positive constants.

(a) Find an equilibrium point $(N_\star, P_\star)$ for which both $N_\star$ are $P_\star$ positive.

(b) Say what it means for a function $H(N, P)$ to be a *constant of the motion* for the system (A1.1) and show that

$$H(N, P) = \frac{e^{cN+bP}}{N^d P^a}$$

is a constant of the motion.

(c) Suppose that $(N(t), P(t))$ is a periodic solution to (A1.1) with period T. Define the time-average of the prey population $\overline{N}$ by

$$\overline{N} = \frac{1}{T}\int_0^T N(t)\, dt$$

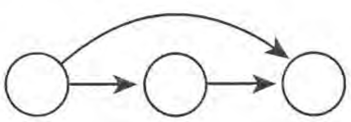
and prove that $\overline{N} = N_\star$, where $N_\star$ is the equilibrium population found in part (a).

[10 marks]

A2. Explain what is meant by the following terms

- a *motif* in a genetic regulatory network
- a *coherent 3-node feed-forward loop*: your answer should include a sketch of such a loop.

In a randomly-assembled regulatory network with $N = 7$ nodes and $E = 10$ interactions, what is the probability of finding an example of the subgraph illustrated below?



You may give either a numerical answer or an expression in terms of binomial coefficients, but you should explain your reasoning thoroughly.

[10 marks]

A3. Consider the following model for the expression of a self-regulating genetic system whose protein product, X is a transcriptional repressor that suppresses production of the mRNA that codes for X.

$$\frac{dX}{dt} = \frac{\beta}{1 + (X/K)} - \alpha X \tag{A3.1}$$

Here α, β and K are positive real numbers.

(a) Show that if $K \ll \beta/\alpha$ the system has an equilibrium $X_\star \approx \sqrt{(\beta/\alpha)K}$.

Under the same conditions—$K \ll \beta/\alpha$—it is reasonable to approximate (A3.1) near equilibrium by

$$\frac{dX}{dt} = \beta\left(\frac{K}{X}\right) - \alpha X. \tag{A3.2}$$

(b) Show that approximate equilibrium $X_\star$ from part (a) is an exact equilibrium for (A3.2).

(c) Linearize (A3.2) near $X_\star$ and solve the resulting ODE for small perturbations of the form $X(t) = X_\star + \delta(t)$, subject to the initial condition that $X(0) = X_\star + \delta_0$ with $\delta_0 \ll X_\star$, then use your result to determine the stability of the equilibrium $X_\star$.

[10 marks]

SECTION B

Answer 2 of the 3 questions

B4. Experiments about the principle of competitive exclusion suggest that qualitative behaviour of an ecosystem can be altered by the regular removal of some animals. The ODEs below model such a system.

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \left(1 - m_1 - \frac{N_1 + a_1 N_2}{K_1}\right)\\ \frac{dN_2}{dt} &= r_2 N_2 \left(1 - m_2 - \frac{N_2 + a_2 N_1}{K_2}\right).\end{aligned} \qquad \text{(B4.1)}$$

Here all the parameters—the r_j, the m_j, the a_j and the K_j—are positive real numbers.

(a) Explain the role of the m_j in (B4.1) and show that if both m_j exceed 1, the model's only biologically meaningful equilibrium has $N_1 = N_2 = 0$.

(b) Show that by suitable choice of dimensionless variables, it is possible to reduce the system (B4.1) above to

$$\frac{du}{d\tau} = f(u,v) = u\,(\gamma_1 - u - \alpha v) \qquad \text{and} \qquad \frac{dv}{d\tau} = g(u,v) = \rho v\,(\gamma_2 - v - \beta u). \qquad \text{(B4.2)}$$

Your answer should include formulae for $u, v, \tau, \alpha, \beta, \rho$ and the γ_j in terms of N_1, N_2, t and the parameters r_j, K_j, m_j and a_j of the original system.

(c) Sketch the null-clines of (B4.2) for the two cases

- $\alpha < 1/\beta < 1$ and $\gamma_1 = \gamma_2 = 1$
- $\alpha < \gamma_1 < 1/\beta < 1$ and $\gamma_2 = 1$

and find all biologically meaningful equilibria.

(d) Linearise the system (B4.2) about an equilibrium point $(u_\star, v_\star)$ and show that if $u_\star, v_\star > 0$, then the community matrix simplifies to

$$A = \begin{bmatrix} \partial_u f & \partial_v f \\ \partial_u g & \partial_v g \end{bmatrix} = \begin{bmatrix} -u_\star & -\alpha u_\star \\ -\rho\beta v_\star & -\rho v_\star \end{bmatrix}$$

(e) Classify the stabilities of the equilibria found in part (c).

(f) Discuss the ecological implications of your results from part (e).

[25 marks]

B5. Consider an enzyme whose active form is a dimer consisting of two monomeric subunits. Imagine that there are two variants of the monomer—called the α and β forms—so that the dimeric enzyme has three possible active forms whose assembly is governed by the following system of reactions

$$\begin{aligned} A + A &\xrightarrow{k_1} D_{\alpha\alpha} \\ B + B &\xrightarrow{k_2} D_{\beta\beta} \\ A + B &\underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} D_{\alpha\beta} \end{aligned} \tag{B5.1}$$

Here the k_j are rate constants and A and B are, respectively, the concentrations of the α- and β-monomers while $D_{\alpha\alpha}$, $D_{\alpha\beta}$ and $D_{\beta\beta}$ are the concentrations of the various dimers.

(a) Construct a stoichiometric matrix N for the system of reactions (B5.1), explaining your work thoroughly.

(b) By performing Gaussian elimination on N, or by any other means, find two conserved quantities for the system (B5.1).

(c) Find the rank r of the stoichiometric matrix N and decompose N as a product $N = LN_R$ where N_R, the reduced stoichiometry matrix, has full rank and L is of the form

$$L = \begin{bmatrix} I_r \\ L_0 \end{bmatrix},$$

where I_r is the $r \times r$ identity matrix. Your answer should include expressions for L, L_0 and N_R.

(d) Using the Law of Mass Action, write down a system of ODEs that describes the temporal evolution of the concentrations of the chemical species in (B5.1).

Now imagine that we are interested in a very small, isolated volume in which there are so few molecules that the ODEs found in part (d) become unrealistic and we are obliged to model the reactions using the Gillespie algorithm.

(e) Suppose that at $t = 0$ there are 2 molecules of the α monomer, 2 molecules of the β monomer and no dimers. List the set of states the system can occupy.

(f) Draw a directed graph whose nodes are the states listed in part (e) and whose edges represent transitions allowed by the reactions (B5.1). Label the nodes of the graph using the notation j, k, l to indicate that the corresponding state has j molecules of the α monomer, k molecules of the β monomer and l molecules of the dimer $D_{\alpha\beta}$.

(g) Using the same labelling convention as in part (e), let $p_{jkl}(t)$ indicate the probability that the system is, at time t, in a state with j molecules of the α monomer, k molecules of the β monomer and l molecules of the dimer $D_{\alpha\beta}$. Find $\lim_{t\to\infty} p_{jkl}(t)$ for all the $p_{jkl}(t)$ associated with the states found in part (e).

[25 marks]

B6. Consider the following differential delay equation for a population whose birth rate at time t is controlled by the availability of food a certain time T in the past. Here $N(t)$ is the population at time t and it evolves according to the following differential delay equation

$$\frac{dx(t)}{dt} = b\,\Theta\left(1 - \frac{N(t-T)}{K}\right) - cN(t). \tag{B6.1}$$

where the delay T and the parameters b, K and c are positive real numbers, while the function $\Theta\left(1 - N(t-T)/K\right)$ satisfies

$$\Theta\left(1 - \frac{N(t-T)}{K}\right) = \begin{cases} 1 - N(t-T)/K & \text{if } 1 - N(t-T)/K > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Explain the role of the function Θ in the model (B6.1).

(b) Show that a suitable choice of dimensionless variables allows one to reduce the system (B6.1) to the form

$$\frac{du(\tau)}{d\tau} = \alpha\,\Theta(1 - u(\tau - \hat{T})) - u(\tau) \tag{B6.2}$$

Your answer should include formulae for u, $\tau, \hat{T}$ and α in terms of x, t, T and the parameters of the original problem.

(c) Show that the system (B6.2) has a constant solution $u(\tau) = u_\star$ with $0 < u_\star < 1$.

(d) Linearise the system (B6.2) near $u(\tau) = u_\star$ and show that small perturbations of the form $u(\tau) = u_\star + \eta(\tau)$ evolve according to the linearized dynamics

$$\frac{d\eta(\tau)}{d\tau} = -\alpha\eta(\tau - \hat{T}) - \eta(\tau) \tag{B6.3}$$

(e) Consider perturbations of the form $\eta(t) = \eta_0 e^{(\mu + i\omega)t}$ evolving according to (B6.3) and prove

- if $\alpha < 1$, then $\mu < 0$ for all $\hat{T} > 0$ and so all such perturbations die away exponentially;
- if $\alpha > 1$, then there is a critical delay T_0 such that for $\hat{T} > T_0$, $\mu > 0$, implying that the constant solution $u(\tau) = u_\star$ is unstable.

[25 marks]

END OF EXAMINATION PAPER

2 hours

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL MODELLING AND REACTIVE FLOW

04 June 2013

09:45am - 11:45am

Answer ALL THREE questions.

Electronic calculators may be used, provided that they cannot store text.

1.

Consider the non-dimensional model

$$\frac{d\theta}{dt} = -\theta + \mathrm{Da}\,(1-\theta)\exp[\beta(\theta-1)]$$
$$\theta = 0 \quad \text{at} \quad t = 0\,,$$

for a well stirred reactor, where θ is the temperature, Da the Damköhler number, and β the Zeldovich number.

(a) Determine the relationship between the stationary solutions $\theta = \theta_s$ and Da. Show that θ_s is a single-valued function of Da for sufficiently small values of β.

[3 marks]

(b) Determine the critical value β_c of β above which θ_s becomes a multi-valued function of Da, along with the ignition and extinction points (provide a rough sketch of the curve θ_s versus Da).

[6 marks]

(c) Carry out a linear stability analysis of the different branches of the curve θ_s versus Da.

[7 marks]

(d) Carry out an asymptotic analysis of the lower branch of θ_s versus Da in the limit $\beta \to \infty$; in particular determine the scaling of Da (or the distinguished limit) needed to capture the ignition point, which is also to be determined.

[7 marks]

2.

The propagation of a one-dimensional premixed flame in the negative x-direction (in a frame of reference attached to the flame) is described by the non-dimensional model

$$\begin{aligned} \frac{d\theta}{dx} &= \frac{d^2\theta}{dx^2} + \lambda(1-\theta)^n \exp[\beta(\theta-1)] \\ & \quad x \to -\infty \quad \theta = 0 \\ & \quad x \to +\infty \quad \theta = 1\,. \end{aligned}$$

Here θ is the temperature, n a positive integer representing the order of the reaction, λ a positive eigenvalue of the problem (proportional to the inverse of the square of the propagation speed) and β the Zeldovich number assumed to be large.

(a) Assuming that the chemical reaction takes place in an infinitely thin layer around $x = 0$ in the limit $\beta \to \infty$, determine the outer profiles at both sides of the reaction layer choosing the origin at the location where the outer profiles intersect.

[6 marks]

(b) Determine the inner problem to leading order (in an expansion in β^{-1}) using the inner variable $X \equiv \beta x$, and the inner expansion $\theta = 1 + \beta^{-1}\Theta^1(X) + \cdots$. In particular determine the appropriate scaling for λ, and the matching conditions with the outer solution.

[9 marks]

(c) Find λ to leading order by solving the inner problem. [You may find it useful to use the following result involving the Gamma function: $\Gamma(n) \equiv \int_0^\infty x^{n-1}e^{-x}dx = (n-1)!$].

[8 marks]

3.

A non-reactive solvent is released around the origin at $t = 0$ into a parallel flow $u(y)$ in a two-dimensional channel of half-width h. The solvent's concentration $c(x, y, t)$ is governed by the convection-diffusion equation

$$c_t + u(y)c_x = D(c_{xx} + c_{yy}) \quad \text{with} \quad c_y = 0 \ \text{at} \ y = 0 \ \text{and} \ y = h.$$

(a) Rewrite the problem in terms of the non-dimensional variables $\tilde{x}, \tilde{y}, \tilde{t}$ and $\tilde{u}$ given by $x = L\tilde{x}$, $y = h\tilde{y}$, $t = \frac{L}{u_0}\tilde{t}$, and $u = u_0\tilde{u}$. Here u_0 represents the maximum of the flow velocity and L is a longitudinal length scale chosen so that the reference (observation) time L/u_0 is large compared with the transverse diffusion time h^2/D. Hence show that the problem can be written, *after dropping tildes*, as

$$\epsilon \mathrm{Pe}\,(c_t + u(y)c_x) = \epsilon^2 c_{xx} + c_{yy} \quad \text{with} \quad c_y = 0 \ \text{at} \ y = 0 \ \text{and} \ y = 1,$$

and determine the non-dimensional parameters ϵ and Pe in terms of L, h, u_0 and D.

[5 marks]

(b) Denoting transverse averages by bars, and writing

$$c(x, y, t) = \bar{c}(x, t) + c'(x, y, t) \quad \text{where} \quad \bar{c} = \int_0^1 c\, dy \quad \text{and} \quad \overline{c'} = 0,$$

show that $\bar{c}$ is governed by

$$\epsilon \mathrm{Pe}\left(\bar{c}_t + \bar{u}\,\bar{c}_x + \overline{uc'_x}\right) = \epsilon^2 \bar{c}_{xx}\,.$$

[5 marks]

(c) Show that c' is governed by

$$\epsilon \mathrm{Pe}\left(c'_t + u'\bar{c}_x + uc'_x - \overline{uc'_x}\right) = \epsilon^2 c'_{xx} + c'_{yy},$$

where $u' \equiv u(y) - \bar{u}$ is the deviation of the velocity from its mean value.

[3 marks]

(d) Show that in the distinguished limit $\epsilon \to 0$ with $\epsilon\mathrm{Pe} \to 0$, we have $c' = \epsilon \mathrm{Pe}\bar{c}_x F(y)$ to leading order and determine the function $F(y)$.

[8 marks]

(e) Using the result of the previous question, determine $\overline{uc'_x}$ to leading order. Hence, show that $\bar{c}$ is governed by

$$\epsilon \mathrm{Pe}\,(\bar{c}_t + \bar{u}\,\bar{c}_x) = \epsilon^2\left(1 + \alpha \mathrm{Pe}^2\right)\bar{c}_{xx},$$

and determine the number α. What is the value of α in the particular case of a Poiseuille flow given by $u = 1 - y^2$.

[8 marks]

(f) Returning to dimensional variables, show that $\bar{c}$ is governed by a convection-diffusion equation with effective diffusion coefficient D_{eff} which is to be determined.

[5 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

NUMERICAL ANALYSIS 2

5 June 2013

09:45 - 12:45

Answer **ALL SIX** questions in Section A (40 marks in all) and **TWO** of the three questions in Section B (20 marks each).

If more than TWO questions from Section B are attempted, then credit will be given for the best two answers.

Electronic calculators are permitted, provided they cannot store text.

SECTION A

Answer **ALL** six questions

A1. You are given that the Chebyshev polynomial of degree n is defined by $T_n(x) = \cos(n\cos^{-1}(x))$ for $x \in [-1, 1]$ and $n = 0, 1, \ldots$.

Define the set of Chebyshev points x_i, $i = 0, 1, \ldots, n$ associated with $T_{n+1}(x)$ and briefly comment on why it is advantageous to use them as interpolation points when constructing polynomial approximations p_n of degree n to $f \in C[-1, 1]$. In addition, prove that

$$T_n(-x) = (-1)^n T_n(x),$$

so that T_n is an odd function when n is odd and an even function when n is even.

[8 marks]

A2. Let

$$r(x) = x - \cfrac{1}{x - 2 + \cfrac{5}{x+3}}.$$

Write $r(x)$ in the form $p(x)/q(x)$, where p and q are polynomials, and express p and q in the nested form used by Horner's method. Work out the cost, in multiplications (M), divisions (D) and additions/subtractions (A) of evaluating $r(x)$ using

(1) bottom-up evaluation of the continued fraction, and

(2) evaluation of $p(x)/q(x)$ using Horner's method on p and q.

Explain why for evaluating rational polynomials, continued fraction representations may be preferable to representations of the form $p(x)/q(x)$, where p and q are polynomials.

[6 marks]

A3. Use the method of undetermined coefficients to derive the 4-point closed Newton–Cotes rule

$$\int_0^1 f(x)\,dx \approx af(0) + bf(1/3) + cf(2/3) + df(1) := J(f).$$

That is, find the coefficients a, b, c and d. Given that the error is of the form $Af^{(4)}(\eta)$, for some $\eta \in [0, 1]$, determine the constant A.

[9 marks]

A4. Consider the integral

$$I(f) := \int_a^b w(x)f(x)\,dx,$$

where $w(x) \geq 0$ on $[a, b]$ is a weight function. Explain how to choose the nodes $\{x_i\}_{i=1}^n$ and weights $\{w_i\}_{i=1}^n$ so that the n-point Gauss rule

$$G_n(f) := \sum_{i=1}^n w_i f(x_i),$$

has degree of precision $2n - 1$. Show that, for this choice,

$$w_i = \int_a^b w(x)l_i(x)\,dx,$$

where l_i is a Lagrange interpolating polynomial.

[6 marks]

A5. Consider Van der Pol's second-order differential equation

$$y''(x) - 2(1 - y(x)^2)y'(x) + y(x) = 0, \qquad x \geq 0,$$

together with the initial conditions $y(0) = 2$, $y'(0) = 1$. Write this initial value problem as a first-order system and use one step of Euler's method with stepsize $h = 0.1$ to estimate $y(0.1)$ and $y'(0.1)$.

[5 marks]

A6. The Backward Euler method for the initial value problem $y'(x) = f(x, y(x))$, $y(x_0) = y_0$ is written

$$y_{n+1} = y_n + hf_{n+1},$$

where h is the chosen stepsize, $f_{n+1} = f(x_{n+1}, y_{n+1})$ and y_n is the approximation to $y(x_n)$ at the grid point x_n for $n = 0, 1, \ldots$. Find the region of absolute stability and compare it with that of the standard Euler method. Which scheme has more favourable stability properties?

[6 marks]

SECTION B

Answer **TWO** of the three questions

B7.

(a) Let $w(x)$ be a weight function. Define the $L_{2,w}$ norm of $f \in C[a,b]$ (denoted $\| f \|_{2,w}$), taking care to state any properties that $w(x)$ must satisfy. Explain what is meant by the phrase "a best $L_{2,w}$ approximation" g from a subspace $G \subset C[a,b]$.

[3 marks]

(b) Consider the set of polynomials defined by $\phi_0(x) = 1$, $\phi_1(x) = x - \alpha_0$, and

$$\phi_{n+1}(x) = (x - \alpha_n)\phi_n(x) - \beta_n\phi_{n-1}(x), \qquad n \geq 1,$$

where

$$\alpha_n := \frac{\langle x\phi_n, \phi_n\rangle_w}{\| \phi_n \|_{2,w}^2}, \qquad \beta_n := \frac{\langle x\phi_n, \phi_{n-1}\rangle_w}{\| \phi_{n-1} \|_{2,w}^2},$$

and the inner-product $\langle \cdot, \cdot \rangle_w$ is defined by

$$\langle u, v\rangle_w := \int_a^b w(x)u(x)v(x)dx.$$

First, show that $\langle \phi_1, \phi_0 \rangle_w = 0$. Then, using proof by induction, show that for any $n = 0, 1, \ldots$

$$\langle \phi_n, \phi_j \rangle_w = 0 \qquad \text{for } j = 0, 1 \ldots, n-1.$$

[9 marks]

(c) Using the result in part (b), state a convenient choice for the subspace G in part (a). Define the normal equations and explain how to find the best $L_{2,w}$ approximation g to f from the chosen G.

[4 marks]

(d) Consider the weight function $w(x) = 1/2$ and the interval $[-1, 1]$. Find the polynomials ϕ_0, ϕ_1, ϕ_2 of degrees $0, 1, 2$ that are orthogonal with respect to the inner-product $\langle \cdot, \cdot \rangle_w$.

[4 marks]

B8. Divide $[0,1]$ into n equal subintervals $[x_i, x_{i+1}]$ of width h, where $x_i = ih$, $i = 0, 1, \dots, n$. Let $f(x)$ be twice continuously differentiable on $[0,1]$ and consider the repeated trapezium rule

$$I(f) = \int_0^1 f(x)\,dx \approx \sum_{i=0}^{n-1} \frac{h}{2}\big(f(x_i) + f(x_{i+1})\big) = T(h).$$

(a) State (but do not prove) the Mean Value Theorem for sums. Given that the error in applying the trapezium rule on $[x_i, x_{i+1}]$ satisfies

$$\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{h}{2}\big(f(x_i)+f(x_{i+1})\big) = -\frac{h^3}{12} f''(\theta_i), \qquad \text{for some } \theta_i \in [x_i, x_{i+1}],$$

show that the error $E_{RT}(f) = I(f) - T(h)$ satisfies

$$E_{RT} = \frac{-h^2}{12} f''(\theta),$$

for some $\theta \in [0,1]$.

[6 marks]

(b) The Euler-Maclaurin summation formula gives an asymptotic expansion for the error in the repeated trapezium rule of the form:

$$I(f) - T(h) = a_2 h^2 + a_4 h^4 + a_6 h^6 + \dots$$

for some real constants $a_2, a_4, a_6, \dots$. Use this observation to determine what happens to the error when we double the number of intervals $[x_i, x_{i+1}]$. In addition, find a way to combine $T(h)$ and $T(\frac{h}{2})$ to obtain a new approximation scheme that gives an error that is $\mathcal{O}(h^4)$.

[5 marks]

(c) Define $T_{k,1} = T(2^{-(k-1)})$ for $k \geq 1$ and

$$T_{i,j} = \frac{4^{j-1} T_{i,j-1} - T_{i-1,j-1}}{4^{j-1} - 1}, \quad i, j \geq 2.$$

Starting from the approximation $T_{1,1}$, describe the Romberg Scheme for generating a sequence of increasingly accurate approximations $T_{i,j}$ to $I(f)$. Comment on the cost of the scheme and give a criterion for deciding at what point in the sequence we should stop.

[7 marks]

(d) Define an approximation scheme that leads to an error that is $\mathcal{O}(h^6)$.

[2 marks]

B9. Consider the numerical method

$$y_{n+1} = y_n + \frac{h}{2}\big(f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\big), \qquad (\star)$$

for the initial value problem $y'(x) = f(x, y(x)), x \geq x_0$ with $y(x_0) = y_0$, where h is the chosen stepsize and y_n is an approximation to $y(x_n)$ at the grid point x_n for $n = 0, 1, \ldots$.

(a) Explain how the method in $(\star)$ can be interpreted as:

(1) an 'improved' Euler scheme,

(2) a 2-stage Runge-Kutta scheme.

[6 marks]

(b) Define the terms *local truncation error* (denoted $\tau(h)$) and *order* of a numerical method for initial value problems.

[3 marks]

(c) Show that for the numerical method $(\star)$,

$$\tau(h) = y(x_{n+1}) - y(x_n) - \frac{h}{2}y'(x_n) - \frac{h}{2}f(x_n + h, y(x_n) + hy'(x_n)).$$

[3 marks]

(d) Using part (c), show that the numerical method in $(\star)$ has order two.

HINT: you may find the following identity useful

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f.$$

[8 marks]

END OF EXAMINATION PAPER

Two hours

THE UNIVERSITY OF MANCHESTER

MARKOV PROCESSES

23 May 2013

14:00 - 16:00

Answer 3 questions. If more than 3 questions are attempted, credit will be given for the best 3 answers. Each question is worth 20 marks.

Electronic calculators may be used, provided that they cannot store text.

Answer 3 questions

1. a) The transition matrix of a four state Markov chain with state space $S = \{1, 2, 3, 4\}$ is given by

$$\begin{bmatrix} 1-a & a & 0 & 0 \\ 1-b & 0 & b & 0 \\ 1-c & 0 & 0 & c \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where $0 < a, b, c < 1$.

(i) Draw the transition diagram.

[2 marks]

(ii) Explain why the Markov chain is irreducible and aperiodic.

[2 marks]

(iii) Determine the stationary distribution.

[2 marks]

(iv) Write down the mean recurrence time of state 1.

[2 marks]

(v) Calculate the mean time of first visit to state 2 starting from state 1.

[2 marks]

b) Let $\{X_n, n \geq 0\}$ be a Markov chain with state space S. Prove each of the following identities:

$$\text{(i) For } n \geq 1,\ p_{ij}(n) = \sum_{k=1}^{n} f_{ij}(k) p_{jj}(n-k);$$

[5 marks]

$$\text{(ii) For } n, m \geq 0,\ p_{ij}(n+m) = \sum_{k \in S} p_{ik}(m) p_{kj}(n).$$

[5 marks]

where

$p_{ij}(n) = P(X_n = j \mid X_0 = i)$ for $n \geq 1$, $p_{ii}(0) = 1$ and $p_{ij}(0) = 0$ for $i \neq j$,
$f_{ij}(n) = P(X_k \neq j, 1 \leq k < n, X_n = j \mid X_0 = i)$.

2. a) Let $\{X_n, n \geq 0\}$ be a Markov chain with state space the set of integers and transition probabilities given by

$$p_{i,i+1} = p, \quad p_{i,i-1} = 1 - p = q \quad where\ 0 < p < 1.$$

(i) Show that, for n odd,

$$P(X_n = 0 \mid X_0 = 0) = 0$$

and that, for n even,

$$P(X_n = 0 \mid X_0 = 0) = \binom{n}{\frac{n}{2}} (pq)^{\frac{n}{2}}.$$

[5 marks]

(ii) Show that the Markov chain is transient when $p \neq q$ and null recurrent when $p = q$

[You may assume the asymptotic result $n! \sim n^n e^{-n}(2\pi n)^{1/2}$ as $n \longrightarrow \infty$; and use without proof that the sequence $p_{0,0}(n) = \binom{n}{\frac{n}{2}} (pq)^{\frac{n}{2}}$ for n even and $p_{0,0}(n) = 0$ for n odd has generating function $P_0 = (1 - 4pqs^2)^{\frac{-1}{2}}$.]

[5 marks]

b) N black and N white balls are placed in two urns so that each contains N balls. One ball is selected at random from each urn and the two balls are interchanged. This swop is then repeated many times. Let the number of black balls in urn 1 immediately following the n^{th} swop be denoted by X_n where $\{X_n, n \geq 0\}$ is a Markov chain.

(i) Show that this Markov chain possesses a unique stationary distribution $(\pi_i)_{0 \leq i \leq N}$, say, which satisfies

$$\pi_{i-1} p_{i-1,i} + \pi_i p_{ii} + \pi_{i+1} p_{i+1,i} = \pi_i \quad 1 \leq i \leq N - 1,$$

$$\pi_0 = \pi_1 p_{10}, \quad \pi_N = \pi_{N-1} p_{N-1,N},$$

where $P = (p_{ij})$ is the transition matrix of the Markov chain.

[5 marks]

(ii) Determine the transition matrix P.

[3 marks]

(iii) Show that, for $0 \leq i \leq N$,

$$\pi_i = \binom{N}{i}^2 \pi_0.$$

[2 marks]

3. a) Consider the random process $\{X(t), t \geq 0\}$ with state space $\{0, 1\}$ where, the holding times for states 0 and 1 are independent exponential random variables with parameters λ and μ respectively. Upon leaving one state, the process enters the other.

(i) Explain why this process is a continuous time Markov chain.

[3 marks]

(ii) Write down the system of forward equations $P'(t) = P(t)Q$ for this process.

[3 marks]

(iii) With the initial condition $p_{00}(0) = 1$, use this system to show that, for $t \geq 0$,

$$p'_{00}(t) = -(\lambda + \mu)p_{00}(t) + \mu$$

and hence that

$$p_{00}(t) = \frac{1}{\lambda + \mu} \left\{ \mu + \lambda \exp(-[\lambda + \mu]t) \right\}.$$

[4 marks]

b) Consider the continuous time Markov chain with state space $S = \{1, 2, 3\}$, where $q_j = \lambda$ for $j = 1, 2, 3$; i.e. all holding times are exponentially distributed with parameter λ. The jump-chain has transition matrix $R = (r_{ij})$ equal to

$$\begin{bmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

(i) Write down the matrix Q for this continuous time Markov chain.

[3 marks]

(ii) Show that the unique stationary distribution is given by
$(\pi_1, \pi_2, \pi_3) = (\frac{1}{3}, \frac{5}{18}, \frac{7}{18})$.

[3 marks]

(iii) Let a_{ij} denote the expected time of first visit to state j starting from state i. By conditioning on the destination of the first transition, show that

$$a_{31} = \frac{1}{\lambda}\frac{1}{2} + (a_{21} + \frac{1}{\lambda})\frac{1}{2}.$$

Hence, or otherwise, show that $a_{31} = \frac{2}{\lambda}$.

[4 marks]

4. Customers arrive at a two-server shop queue in a Poisson process rate λ per hour. If an arriving customer finds a server free, she presents herself for service. If she finds both servers occupied, but no-one else waiting, she either joins the queue with probability $\frac{1}{2}$, or leaves with probability $\frac{1}{2}$. If she finds both servers occupied and at least one other customer waiting, she does not join the queue. The service time distribution is exponential with mean μ^{-1} and the service times are independent and independent of the arrival process. Let $(\pi_0, \pi_1, \pi_2, \pi_3)$ denote the stationary distribution for the number of customers in the system.

(i) Show that, when there are 2 customers in the system, the time to the next customer joining the queue is exponentially distributed with parameter $\frac{\lambda}{2}$.

[4 marks]

(ii) Using the result of part (i) and carefully stating any result to which you appeal, show that

$$\pi_0 = \frac{1}{1+\rho+\rho^2/2+\rho^3/8} \qquad \pi_1 = \frac{\rho}{1+\rho+\rho^2/2+\rho^3/8} \qquad \pi_2 = \frac{\rho^2}{2(1+\rho+\rho^2/2+\rho^3/8)}$$
$$\pi_3 = \frac{\rho^3}{4(1+\rho+\rho^2/2+\rho^3/8)}$$

where $\rho = \frac{\lambda}{\mu}$.

[6 marks]

(iii) Explain why, when the stationary distribution is the initial distribution, the mean number of customers lost per hour is equal to

$$\frac{\lambda(2\rho^2+\rho^3)}{8(1+\rho+\rho^2/2+\rho^3/8)}.$$

[4 marks]

(iv) At present, $\lambda = 60$ and $\mu = 40$, and the average profit per customer served is *£1*. The shop keeper wishes to employ two replacement cashiers who are each, on average, able to serve twice as quickly ($\mu = 80$). To employ these two cashiers would increase wage costs by *£5* per hour. Calculate whether it would be profitable for the shop keeper to do so.

[6 marks]

END OF EXAMINATION PAPER

Statistical Tables to be provided

Two hours

THE UNIVERSITY OF MANCHESTER

Time Series Analysis

20 May 2013
14:00 – 16:00

Electronic calculators are permitted, provided they cannot store text.

Answer **ALL** Four questions in Section A (32 marks in all)

and

Two of the Three questions in Section B (24 marks each).

If more than Two questions from Section B are attempted, then credit will be given for the best Two answers.

SECTION A
Answer ALL four questions

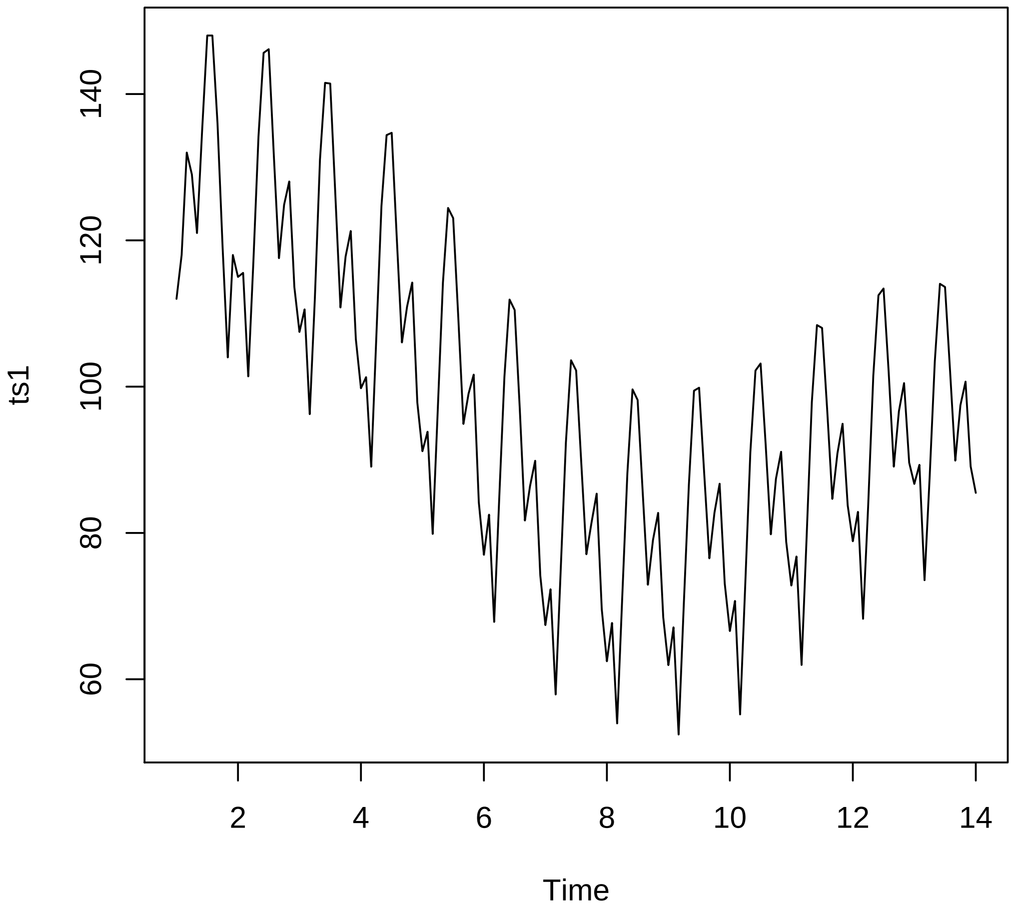
A1.
A graph of a monthly time series, $\{x_t\}$, is shown below. The series spans a period of 14 years. Give concise answers to the following questions.

a) What major features of the data can be inferred from this plot?

b) What might be the effect of differencing $\{x_t\}$?

c) What might be the effect of a seasonal differencing, $(1 - \mathbf{B}^{12})$, of $\{x_t\}$?

d) What feature(s) in a time series plot might prompt you to consider a log-transformation? Is such a transformation appropriate for the time series $\{x_t\}$?



[8 marks]

A2. Let $\{\varepsilon_t\}$ be a white noise process with variance σ^2, i.e. $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$.

a) Write the following model for a process $\{X_t\}$ in operator form.

$$X_t = -0.1X_{t-1} + 0.72X_{t-2} + \varepsilon_t - 0.5\varepsilon_{t-12}.$$

b) Let X_t be a stationary process, such that

$$(1 - 0.3\mathbf{B})X_t = 4.2 + (1 + 0.7\mathbf{B})\varepsilon_t.$$

Find the mean corrected representation of this model. In other words, write this model in terms of the mean corrected process $X_t - \mu$, where $\mu = \mathrm{E}\, X_t$.
Give your answer in: (i) operator form, and (ii) difference equation form.

c) Suppose that the process $\{X_t\}$ is such that

$$(1 - \mathbf{B})(1 - \mathbf{B}^{12})X_t = (1 - 0.5\mathbf{B})(1 + 0.1\mathbf{B}^{12})\varepsilon_t.$$

Let $Y_t = (1 - \mathbf{B})X_t$. Write down the model for $\{Y_t\}$.

d) Give two different multiplicative forms of the filter $1 - \mathbf{B}^4$.

[8 marks]

A3.
A time series is analysed with R and a model is fitted with the `arima` function. Here is a summary of the fitted model.

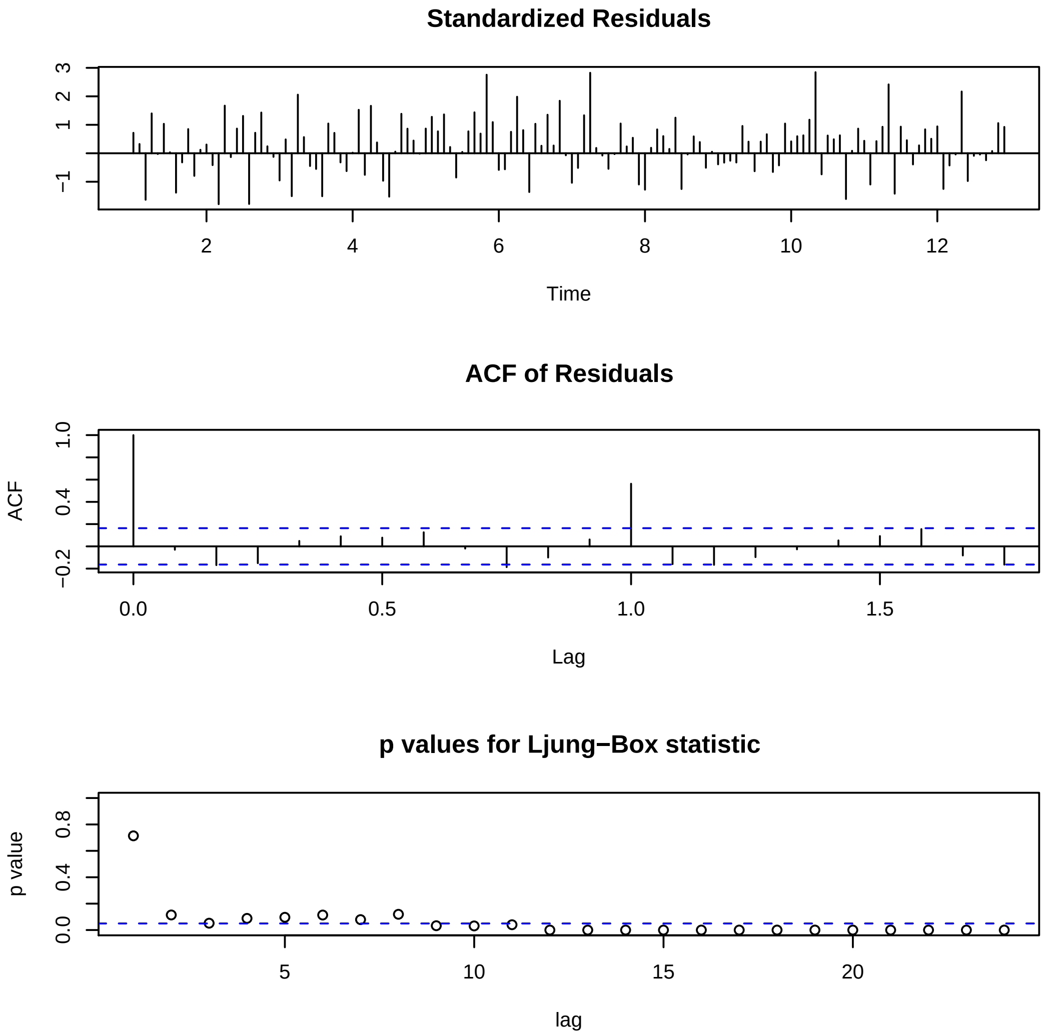
```
Call:
arima(x = tsqu3, order = c(1, 0, 0), seasonal = list(order = c(0, 0, 0), period = 12),
    include.mean = FALSE)

Coefficients:
         ar1
      0.4016
s.e.  0.0765

sigma^2 estimated as 1.798:  log likelihood = -246.67,  aic = 497.34
```

a) Write down the fitted model in operator form.

b) Give the estimated variance of the innovations.

c) What (if anything) can be inferred about the quality of the model's fit from the numerical information above?

d) The following diagnostic plots for the fitted model were obtained.



Explain what each of these plots say about the quality of the model's fit.

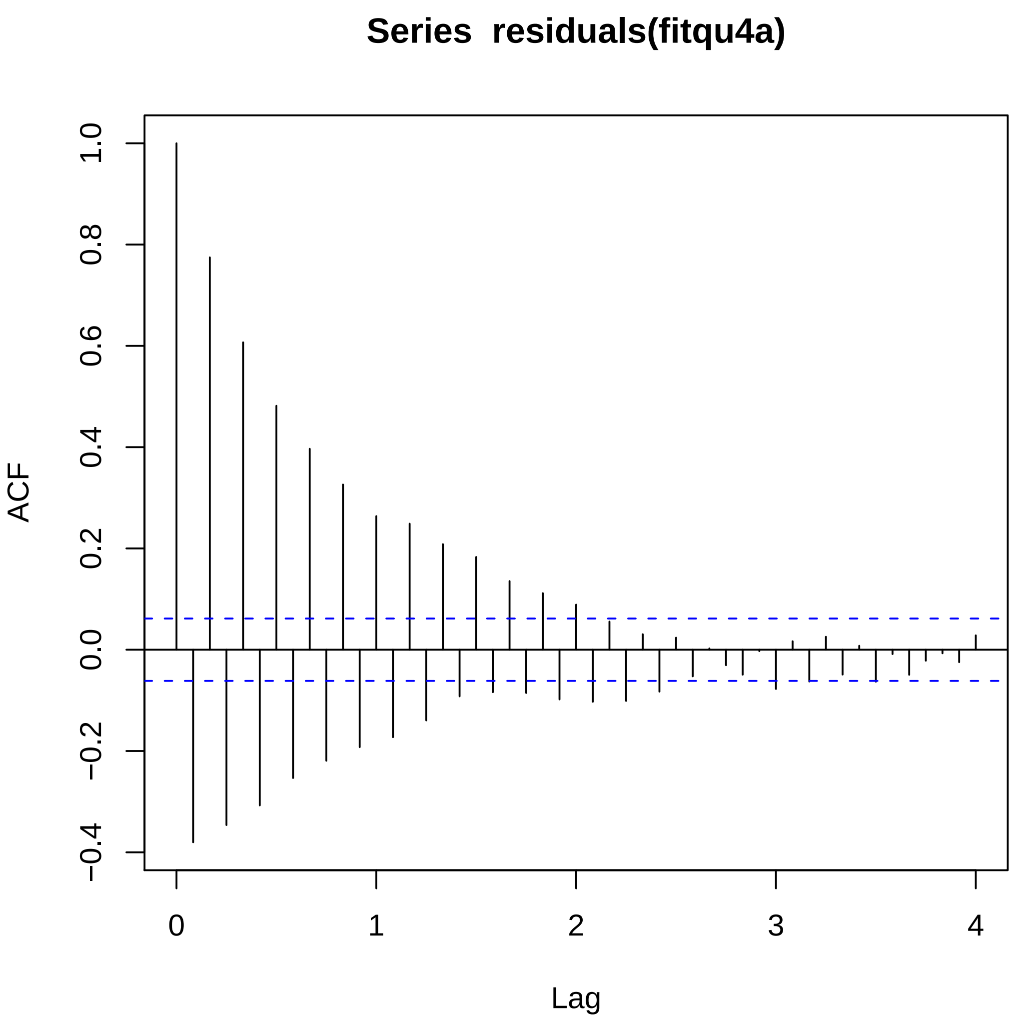
e) Taking into account the plots above, indicate how you would go about trying to improve the fit of the model.

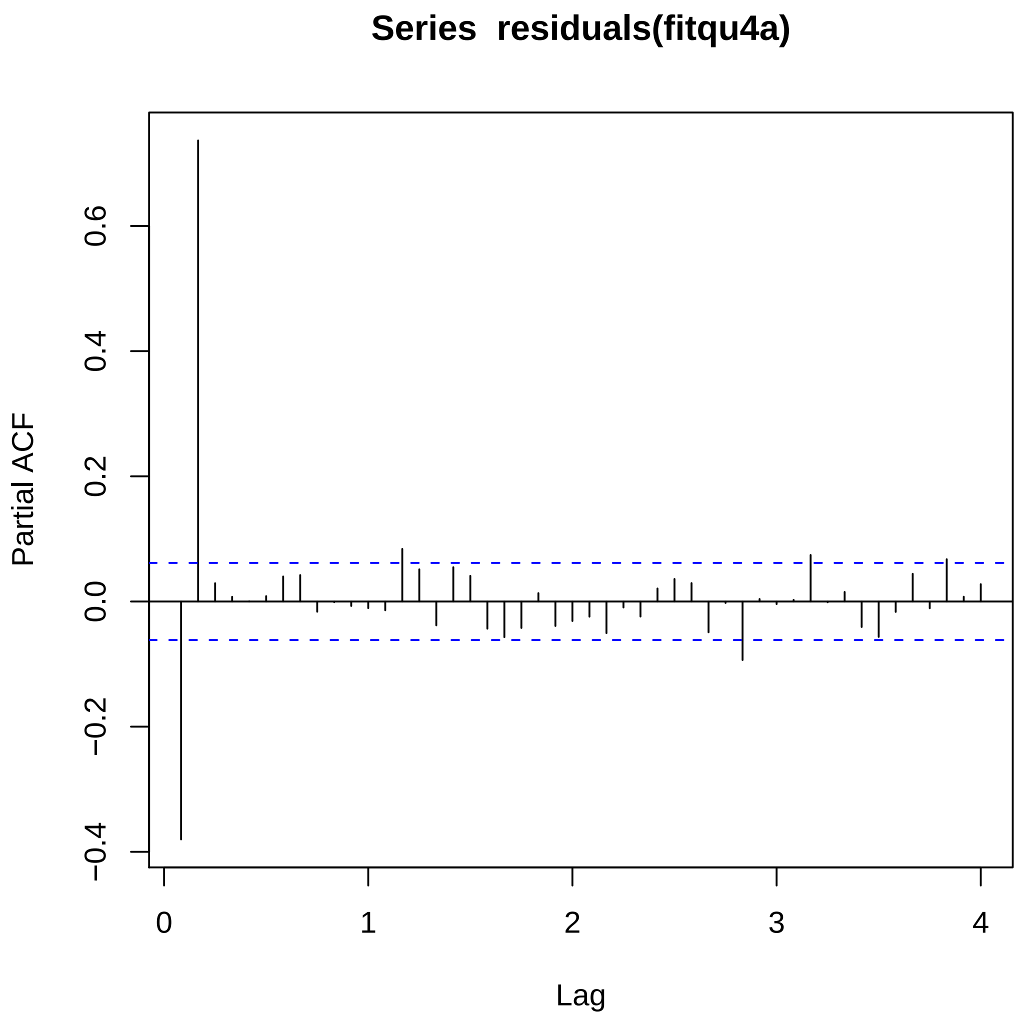
[8 marks]

A4.
A time series, $\{x_t\}$, is analysed with R and a model is fitted with the `arima` function with the following results.

```
Call:
arima(x = tsqu4, order = c(0, 1, 0), seasonal = list(order = c(0, 1, 1), period = 12))

Coefficients:
        sma1
      0.8784
s.e.  0.0128

sigma^2 estimated as 2.499:  log likelihood = -1885.84,  aic = 3775.68
```

a) Here are plots of the autocorrelations and partial autocorrelations of the residuals from the above fit.





Explain what each of these plots say about the quality of the model's fit.

b) On the basis of the above analysis it was decided to fit a ARIMA$(2, 1, 0)(0, 1, 1)_{12}$ model to $\{x_t\}$. Explain why this is a sensible decision.

c) Numerical results for two models fitted to the observed series are shown on the next page. One of them is the model suggested in part (b). For both models the residual diagnostics did not show evidence for bad fit.
What inferences about the quality of the fit of each of the two models can be drawn from these results? Which model would be your choice and why?

Model 1

```
Call:
arima(x = tsqu4, order = c(2, 1, 0), seasonal = list(order = c(0, 1, 1), period = 12))

Coefficients:
          ar1     ar2    sma1
      -0.0960  0.7472  0.8220
s.e.   0.0209  0.0210  0.0203

sigma^2 estimated as 0.9605:  log likelihood = -1406.77,  aic = 2821.54
```

Model 2

```
Call:
arima(x = tsqu4, order = c(3, 1, 1), seasonal = list(order = c(0, 1, 1), period = 12))

Coefficients:
         ar1     ar2      ar3      ma1    sma1
      0.2037  0.7836  -0.2157  -0.3222  0.8215
s.e.  0.4976  0.0546   0.3720   0.5055  0.0204

sigma^2 estimated as 0.9599:  log likelihood = -1406.45,  aic = 2824.91
```

[8 marks]

SECTION B
Answer 2 of the 3 questions

B5. Consider the seasonal model

$$(1 - 0.8\mathbf{B}^{12})X_t = (1 + 0.5\mathbf{B})\varepsilon_t,$$

where $\{\varepsilon_t\}$ is white noise with variance 1, i.e. $\{\varepsilon_t\} \sim \mathrm{WN}(0, 1)$.

a) Write the model in difference equation form.

b) This is an ARIMA$(p, d, q)(p_s, d_s, q_s)_s$ model. Give the values of s, p, d, q, p_s, d_s, and q_s.

c) Show that the model is invertible.

d) Show that the infinite moving average representation of $\{X_t\}$ is

$$X_t = \sum_{k=0}^{\infty} 0.8^k \mathbf{B}^{12k} \varepsilon_t + \sum_{k=0}^{\infty} 0.5 \times 0.8^k \mathbf{B}^{12k+1} \varepsilon_t,$$

e) Write down the coefficients of ε_t, ε_{t-1}, and ε_{t-12}, in the above infinite moving average representation. What are the values of the coefficients of ε_{t-i} for $i = 2, 3, \ldots, 11$?

f) Assuming that at time t the following information is available: ε_t, X_t, and X_{t-i} for $i = 1, 2, \ldots, 11$,

i) write down the predictors of X_{t+h} at time t for horizons $h = 1, 12, 13$;

ii) give the variances of the corresponding prediction errors.

[24 marks]

B6. Let $\{X_t\}$ be a stationary time series with $\mathrm{E}\,X_t = 0$.

a) For any integer $j \geq 1$ consider the best linear predictor of X_t in terms of $X_{t-1}, \ldots, X_{t-j}$. Denote by v_j the variance of the corresponding prediction error, and by $\phi_1^{(j)}, \ldots, \phi_j^{(j)}$ the coefficients of $X_{t-1}, \ldots, X_{t-j}$, respectively.

i) Derive the Yule-Walker equations relating the coefficients $\phi_1^{(j)}, \ldots, \phi_j^{(j)}$ and v_j to the autocovariances $\gamma_0, \gamma_1, \ldots, \gamma_j$.

ii) Write down the Yule-Walker systems for the particular cases $j = 1$ and $j = 2$.

iii) What name is usually used to refer to the coefficients $\phi_1^{(1)}, \phi_2^{(2)}, \ldots, \phi_j^{(j)}, \ldots$?

b) Assume further that the first two theoretical partial autocorrelations of $\{X_t\}$ are $\beta_1 = 0.8$ and $\beta_2 = -0.25$, respectively.

For each of the following models state, giving your reasons, whether or not it is consistent with the provided information about $\{X_t\}$. *Hint: Aim to avoid unnecessary calculations, when possible, by referring to properties we have studied.*

i) $X_t = 0.8X_{t-1} + \varepsilon_t$

ii) $X_t = -0.8X_{t-1} + \varepsilon_t$

iii) $X_t = \varepsilon_t + 0.8\varepsilon_{t-1}$

iv) $X_t = \varepsilon_t - 0.8\varepsilon_{t-1}$

v) $X_t = X_{t-1} - 0.5X_{t-2} + \varepsilon_t$

vi) $X_t = X_{t-1} - 0.25X_{t-2} + \varepsilon_t$

vii) $\{X_t\}$ is a random walk.

[24 marks]

B7. Let $\{X_t\}$ be a stationary process with $\mathrm{E}\,X_t = \mu_x$ and autocovariances γ_k, $k = 0, 1, 2, \ldots$. Define another process, $\{Y_t\}$, by the equation

$$Y_t = \beta_0 + \beta_1 t + \cdots + \beta_q t^q + X_t,$$

where β_0, β_1, $\ldots$, β_q, are constants and $\beta_q \neq 0$.

a) Show that the process $(1 - \mathbf{B})X_t$ is stationary.

b) Show that the process $(1 - \mathbf{B})^m X_t$ is stationary for any positive integer m.

c) Find the mean of Y_t.

d) Show that $\mathrm{Cov}(Y_t, Y_{t-k}) = \gamma_k$, for $k = 0, 1, 2, \ldots$.

e) Show that the process $(1 - \mathbf{B})^m Y_t$ is not stationary for $m < q$ and stationary for $m \geq q$.

[24 marks]

END OF EXAMINATION PAPER

Two Hours

Mathematical formula books and statistical tables are to be provided

THE UNIVERSITY OF MANCHESTER

GENERALISED LINEAR MODELS

30 May 2013

14:00 – 16:00

Answer ALL questions in Section A and TWO questions in Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** of the two questions

A1.

(a) State the definition of the exponential family of distributions.

[4 marks]

(b) Show that the Binomial distribution $\mathrm{B}(n, \pi)$ with probability mass function

$$P(Y = y) = \binom{n}{y} \pi^y (1 - \pi)^{n-y}, \ y = 0, 1, 2, \ldots, n$$

belongs to the exponential family, where $\pi \in [0, 1]$ is an unknown parameter and $n > 0$ is known.

[4 marks]

(c) Find the mean and variance of the Binomial distribution in (b) using Property 1 of the exponential family.

[4 marks]

(d) Derive the canonical link when the response variable follows a Binomial distribution.

[4 marks]

(e) Define the probit link function and express π in terms of the linear predictor $\eta = \underset{\sim}{x}'\underset{\sim}{\beta}$ under this link function.

[4 marks]

[20 marks in total for this question.]

A2.

(a) Write down the Poisson regression model (with canonical link) for $y_i \sim Pois(\lambda_i)$ on a single explanatory variable x with values x_i, $i = 1, \ldots, n$ and state any further assumption on $y_1, y_2, \ldots, y_n$. An intercept or constant term should be present in the model.

[3 marks]

(b) Show that the log-likelihood function for the model in (a) is

$$\ell(\underset{\sim}{\beta};\ \underset{\sim}{y}) = \sum_{i=1}^{n} y_i(\beta_0 + \beta_1 x_i) - \sum_{i=1}^{n} e^{\beta_0 + \beta_1 x_i} - \sum_{i=1}^{n} \log(y_i!).$$

[3 marks]

(c) Derive an equation in matrix form for the maximum likelihood estimates of the parameters in model (a) and write down (i) the iteratively re-weighted least squares algorithm and (ii) the Newton-Raphson/Fisher scoring algorithm for solving this equation.

[6 marks]

(d) Find the (expected/observed) Fisher information matrix for the parameters in model (a).

[3 marks]

(e) Derive the deviance of the model in (a) when fitted to data and state without proof its asymptotic distribution. Can the deviance be used to test the validity of the model?

[5 marks]

[20 marks in total for this question.]

SECTION B

Answer **TWO** of the three questions

B1. The inverse Gaussian distribution $IG(\mu, \sigma^2)$ has density function

$$f(y) = \frac{1}{\sqrt{2\pi y^3}} \exp\left\{ -\frac{1}{2\mu^2 y} \left(\frac{y-\mu}{\sigma} \right)^2 \right\}, \; y > 0,$$

where $\mu > 0$ and $\sigma > 0$ are parameters.

(a) Write the density function above in exponential family form with $a(\phi) = \phi = \sigma^2$ on the denominator.

[3 marks]

(b) Show that the mean of the distribution is μ and find the canonical link.

[3 marks]

(c) A generalised linear model with IG response, linear predictor $\eta = \alpha + \beta x + \gamma x^2$ and canonical link was fitted to $n = 30$ observations on (x, y). The deviance was 41.23 for the fitted model.

(i) Can the deviance be used to test the validity of the model? Give a reason for your answer.

[4 marks]

(ii) Estimate σ^2 using the deviance.

[4 marks]

(iii) The deviance changed to 45.61 for the reduced model with $\eta = \alpha + \beta x$. Use this information to decide if the x^2 term is required.

[6 marks]

[20 marks in total for this question.]

B2. For each of three insecticides A, B and C, batches of fifty insects were exposed to six different dosages and the numbers of insects killed after exposure for a week were recorded. Two generalised linear models were fitted to the data, with the natural logarithm of dosage (Ldosage) as a covariate and the insecticide as a factor.

```
> fit1<-glm(y ~ Insecticide * Ldosage, family=binomial)
> anova(fit1,test="Chisq")

  Analysis of Deviance Table
  Terms added sequentially (first to last)

                      Df  Deviance Resid. Df   Resid. Dev  P(>|Chi|)
  NULL                                   17      407.13
  Insecticide          2     169.53      15      237.60    1.541e-37
  Ldosage              1     214.92      14       22.68    1.161e-48
  Insecticide:Ldosage  2       3.39      12       19.29    0.18

> fit2<-glm(y ~ Insecticide + Ldosage, family=binomial)
> summary(fit2)

  Coefficients:
               Estimate  Std. Error  z value  Pr(>|z|)
  (Intercept)  -4.4541     0.3575    -12.460  < 2e-16
   InsecticideB  0.6144    0.1999      3.074  0.00211
   InsecticideC  3.0314    0.2521     12.022  < 2e-16
   Ldosage       2.6938    0.2146     12.551  < 2e-16

   Null deviance:      407.129 on 17 degrees of freedom
   Residual deviance:   22.685 on 14 degrees of freedom
```

(a) Is fit1 the full or saturated model? Write down the saturated model in mathematical/statistical terms and explain your notation.

[2+4 marks]

(b) Use its deviance to decide if the model fit1 is adequate for the data.

[3 marks]

(c) Use 'change in deviance' to determine if the model fit1 can be simplified.

[3 marks]

(d) Based on the summary of fit2, for each insecticide compute the fitted probability of an insect being killed by a 4mg dosage (you are given that $\log 4 = 1.3863$).

[4 marks]

(e) Based on the summary of fit2, give the median lethal dose (LD50) for each insecticide.

[4 marks]

[20 marks in total for this question.]

B3. A random sample of 227 males were classified according to hair colour (black, brown or fair) and eye colour (brown, grey or blue). A generalised linear model was fitted in R with the following output, where Hair and Eye are factors.

```
> fit<-glm(y ~ Hair * Eye, family=poisson)
> summary(fit)
      Coefficients:
                             Estimate   Std. Error  z value    Pr(>|z|)
     (Intercept)              2.30259    0.31623     7.281     3.30e-13
      HairBrown               0.47000    0.40311     1.166     0.244
      HairFair               -0.69315    0.54772    -1.266     0.206
      EyeGrey                 0.87547    0.37639     2.326     0.020
      EyeBlue                -0.22314    0.47434    -0.470     0.638
      HairBrown:EyeGrey       0.06551    0.47808     0.137     0.891
      HairBrown:EyeBlue       0.70865    0.57093     1.241     0.215
      HairFair:EyeGrey        0.98083    0.61067     1.606     0.108
      HairFair:EyeBlue        2.78809    0.66361     4.201     2.65e-05

      Null deviance:      1.0971e+02 on 8 degrees of freedom
      Residual deviance: -3.5527e-15 on 0 degrees of freedom

> anova(fit,test="Chisq")
  Analysis of Deviance Table
  Terms added sequentially (first to last)
                Df  Deviance Resid.  Df    Resid. Dev    P(>|Chi|)
        NULL                         8      109.711
        Hair     2       26.829      6       82.882      1.493e-06
        Eye      2       46.078      4       36.804      9.872e-11
        Hair:Eye 4       36.804      0     -3.553e-15    1.977e-07
```

(a) Let y_{ij} be the number of people in cell (i, j), i=1,2,3 (hair), j=1,2,3 (eye). What is the model for $\{y_{ij}\}$ under the sampling scheme described above?

[4 marks]

(b) What is the model when people arrive at one of the 3x3 categories independently (and there happened to be 227 in total)?

[3 marks]

(c) Give three reasons why model (b) is fitted when the data follow model (a).

[3 marks]

(d) What conclusions can be made from the analysis of deviance table and what further information does the HairFair:EyeBlue line in the summary provide?

[4 marks]

(e) Calculate the odds of having blue eyes and the odds ratio of having blue eyes between people with fair hair and those with other coloured hair. Comment on your answers.

[6 marks]

[20 marks in total for this question.]

END OF EXAMINATION PAPER

Statistical Tables are provided

Two hours

The University of Manchester
School of Mathematics

Design and Analysis of Experiments

Date: 4 June 2013

Time: 14:00 – 16:00

Answer the question in Section A (30 marks) and TWO of the three questions in Section B (30 marks each). If all three questions in Section B are attempted then credit will be given for the two best answers.

The total number of marks in this paper is 90. A further 10 marks are available from coursework during the semester making a total of 100.

Electronic calculators may be used, provided that they cannot store text.

Section A

Question 1.

(a) A scientist wants to conduct an experiment in order to find out whether any of the factors x_i, $i = 1,\ldots,7$, affects a response variable of interest, y. How is such an experiment called? Write down an experimental design with the smallest possible number of observations where the variables act at 2 levels. Give its alias structure. Describe its positive and negative features.

(9 marks)

(b) Having conducted the experiment described in (a), the scientist would like to conduct a confirmatory experiment with only 5 of the factors studied initially. Write down a 2-level fractional factorial design with the smallest possible number of observations that can be used to estimate all main effects and 2 two-factor interactions. State its resolution, write down the alias structure, and hence, what model can be estimated. Write down the least squares estimator for one of the parameters of this model.

(9 marks)

(c) The scientist realizes that the observations have to be collected in 4 blocks. Therefore, instead of using the design in (b), the scientist requests a complete 2-level factorial design for the 5 studied variables with observations divided in 4 blocks with equal number of observations. Describe how such a design can be constructed.

(6 marks)

(d) A scientist wants to conduct an experiment where 3 factors will be studied using a Central Composite design. Construct this design so that it has 12 observations. State what model can be estimated. How should the experiment be organized?

(6 marks)
Total: 30 marks

Section B

Question 2.

An experiment has been carried out in order to assess the variability in the lifetime Y (in hundreds of miles) of car tires due to driver and type of road used. Three drivers were selected at random, and three randomly chosen routes of equal length were used in this study. Each driver used each route 4 times. Hence, the total number of observation is 36.

(a) What type of experimental design has been used in this study? Describe its main features.

(5 marks)

(b) Write down an appropriate model for the data.

(6 marks)

(c) Complete the ANOVA table for the data by calculating the missing entries noted by question marks. Interpret the results.

Source	SS	DF	MS	F
Driver	?	?	?	?
Route	39118.72	?	?	?
Interaction	9613.79	?	?	?
Error	18230.75	?	?	
Total	77646.97	35		

(13 marks)

(d) Using the formulas for the expected means squares for driver (*A*), route (*B*) and interaction (*AB*) provided below, calculate the estimates of the relevant variances.

(6 marks)

Formulae (in standard notation):

- $E[MSA] = rt_B\sigma_A^2 + r\sigma_{AB}^2 + \sigma^2$
- $E[MSB] = rt_A\sigma_B^2 + r\sigma_{AB}^2 + \sigma^2$
- $E[MSAB] = r\sigma_{AB}^2 + \sigma^2$

Total: 30 marks

Question 3.

An engineering company is concerned about the consistence of the mean of a quality characteristic and its variability in their manufacturing divisions in the 2 countries where manufacturing takes place. An experiment has been set up to find out whether or not there are statistically significant differences in these respects in the production. In each of the countries 3 machines were used in order to manufacture 4 items each. Hence, there were 24 observations in total.

(a) Describe why the experimental design used in this study is a nested design. State an appropriate model for the data.

(6 marks)

(b) The data (not provided here) were used to calculate some entries in the ANOVA table below. Complete the table. State the assumptions that you have made. Estimate the variance components affecting the results. You can use some formulas provided at the end of the question.

Source	SS	df	MS	F-ratio
A (Country)	0.375	?	?	?
B(A) (Machine within country)	?	?	?	?
Residual	?	?	0.011	
Total	0.798	?		

(12 marks)

(c) What conclusions can be made based on the results obtained in (b)? Use 5% significance level for all statistical tests that you do and state the relevant critical values for the test statistics that you used to reach your conclusions.

(8 marks)

(d) State two reasons why it would be wrong to analyse the data as being collected with a Randomized Block Design.

(4 marks)

Total: 30 marks

Formulae (in standard notation):

- $E[MSB(A)] = r\sigma_B^2 + \sigma^2$ if factor B(A) is treated random.
- $\sigma_B^2 = \frac{1}{r}(MSB(A) - MSE)$

Question 4.

An experimental study has been set up to compare a property (response, y) of interest of 5 formulations (A, B, C, D and E). Samples are taken from 5 batches of the formulations. Also, 5 operators measured the response as shown in the table below. For example, the response measured by Operator 2 using Batch 3 for formulation D is y = 38.

Batch	Operator				
	1	2	3	4	5
1	A=24	B=20	C=19	D=24	E=24
2	B=17	C=24	D=30	E=27	A=36
3	C=18	D=38	E=26	A=27	B=21
4	D=26	E=31	A=26	B=23	C=22
5	E=22	A=30	B=20	C=29	D=31

Answer the following questions using the Fact Sheet provided on the following page and by doing your own calculations.

(a) Identify the experimental design that has been used. Write down a suitable model for the data.

(8 marks)

(b) Carry out statistical analysis of the data. Write down an ANOVA table and carry out appropriate tests for the statistical significance of the differences in the response due to the 3 studied explanatory variables. Interpret the results. State any assumptions you have made in order to draw your conclusions.

(14 marks)

(c) Compare the mean response of interest for the formulations using a least significant difference (LSD) at 5% significance level.

(6 marks)

(d) Obtain a 95% confidence interval for the difference of the mean response for the first and the second formulations.

(2 marks)

Total: 30 marks

(**The question continues on the next page**)

Fact sheet for Question 4:

The total of the observation is 635.
The total of the squared observations is 16805.

Formulation	A	B	C	D	E
Total	111	134	130	128	132

Operator	1	2	3	4	5
Total	107	143	121	130	134

Batch	1	2	3	4	5
Total	143	101	112	149	130

END OF EXAMINATION PAPER

UNIVERSITY OF MANCHESTER: MATH38152

Social Statistics

Two Hours

Electronic calculators may be used provided that they cannot store text

Answer **ALL** five questions in SECTION A (40 Marks)
Answer **TWO** of the three questions in SECTION B (20 marks each)

The total number of marks on the paper is 80.
A further 20 marks are available from coursework during the semester making a total of 100.

21 May 2013

14:00 – 16:00

Statistical Tables and Formula Tables to be provided

SECTION A

Answer ALL five questions

A1.

A researcher interested in fisheries governance wants to know how much fish each country extracts each year. He gets data from the FAO branch of the UN. For each country i the extraction y_i is reported to the UN by the country itself. Some of the figures are given in tonnage and some in the number of fish.

(a) The researcher considers collecting the data through direct measurement instead by taking samples of catches (counting fish) in selected fishing ports. Give a couple (number) of examples of what may differ between y_i as reported by the country itself and for y_i estimated from direct fish counting and why?(4 marks)

(b) What advantages might the secondary data from FAO have over numbers based on samples in fishing ports? Give two examples (4 marks)

[8 marks total]

A2.

A cluster sample was taken to collect information from residents of different suburbs of Manchester on their attitudes to the tram network.

(a) Give one reason why cluster sampling would be preferable to Simple Random Sampling (SRS) in this case (2 marks)

(b) Give one example of a Primary Sampling Unit (PSU) that could be used for this cluster sample (2 marks)

(c) Give one disadvantage of cluster sampling, compared with Simple Random Sampling (SRS). (2 marks)

(d) If the design effect, due to the cluster sample design, for the estimator of the total number of people that were satisfied with the existing tram network was 1.5, what does this indicate? (2 marks)

[8 marks total]

A3.

When conducting an opinion poll on Scottish Independence, it is important that the sample contains information for men and women and for a range of age groups: 16-18; 19-25; 26-35; 36-50; 51 and over.

(a) Write down the two stratification variables for this example and write down the total number of strata in this sample. (2 marks)

(b) Explain how proportional allocation would be used in the sampling for the stratum, aged 51 and over, female if 10% of the population were in this stratum. (2 marks)

(c) What is the key difference in optimal allocation and proportional allocation? (2 marks)

(d) In general, under what conditions does a stratified sample of size n lead to a more precise estimate of a total than a Simple Random Sample (SRS) of size n? (2 marks)

[8 marks total]

A4.

For individuals $i = 1, \ldots, n$ we take measurements $Y_{i,t}$ at two time-points $t = 0, 1$. Assume that we have observations on all $i = 1, \ldots, n$ at $t = 0$ but that $m_i = 0$ for $i = 1, \ldots, r$ and $m_i = 1$ for $i = r+1, \ldots, n$, where m_i is equal to one or zero according to wether $Y_{i,t}$ is observed or not, respectively. Assume that $E(Y_{i,1}) = \mu$ and $V(Y_{i,1}) = \sigma^2$ for all $i = 1, \ldots, n$.

(a) What is the complete-case estimate of μ (2 marks)

(b) If missing values at $t = 1$ are filled in (or imputed) using "last observation carried forward" and observations are MCAR, what is the bias when estimating μ by the sample mean $\sum_i^n \frac{y_{i,1}}{n}$ (simplify the expression

as far as possible) (4 marks)
(c) If missing values at $t = 1$ are imputed using i.i.d. normal variates $N\left(\sum_i^n \frac{y_{i,0}}{n}, \sigma^2\right)$. What is the variance of the sample mean $\sum_i^n \frac{y_{i,1}}{n}$ conditional on $m_i = 0$ for $i \leq r$ and $m_i = 1$ for $i > r$?(2 marks)
[8 marks total]

A5.
For a number of local areas $i = 1, \ldots, n$ in the UK, the crime-rate is recorded as Y_i and the proportion of residents that belong to an ethnic minority is recorded as X_i. A criminologist regresses Y_i on X_i and obtains a significant and positive slope coefficient for X_i. The criminologist claims that this proves that ethnic minorities commit significantly more crimes.
(a) Explicate the implicit causal model (provide statistical model and causal diagram) (1 mark)
(b) Can the criminologist reasonably make this claim about people belonging to ethnic minorities? Give a brief explanation (2 marks)
(c) Another researcher with access to the data set finds that there is positive correlation between the area deprivation score Z_i (an index based on income levels, employment rate and benefits; high values mean more deprived area) and both Y_i and X_i. How would you modify the causal model? (3 marks)
(d) Give one example of how the (final) causal model (of your choice in c) would be affected if it turns out that the error terms $\hat{\epsilon}_i$ and $\hat{\epsilon}_j$ are correlated for neighbouring areas i and j? (2 marks)
[8 marks total]

SECTION B

Answer TWO of the three questions

B1.
Discuss the particular challenges faced by social statistics and the differences between the typical statistics applied in social science and statistics used in randomized trials, experiments and physics. Give examples in terms of how the data and measurements relate to research questions (4 marks); how you identify where and how to collect data (4 marks); how you may measure data (4 marks); data quality issues and editing (4 marks); the type of conclusions you may draw (4 marks). It may help to think of a concrete and specific question, phenomenon and context in each case and common data collection strategies such as surveys and consider key concepts such as observational studies and populations.
[20 marks total]

B2.
Answer both parts (i) & (ii)
(i) A school contains 450 boys and 550 girls who each take a reading test. A sample, stratified by gender, of size $n = 100$ is required. From previous data, the variance in the reading test for boys is $\hat{\sigma}^2_{boys} = 36$ and for girls is $\hat{\sigma}^2_{girls} = 16$.
(a) Calculate the proportional allocation for a sample of size n (2 marks)
(b) Calculate the optimal allocation for a sample of size n (3 marks)
(c) Calculate the probability of selection within each of the two strata for the optimal allocation sample (3 marks)
(ii) In a large university with 20,000 students, a sample of 200 students were asked how satisfied they were with their learning experience by scoring it on a scale of 0-100, where 0=not at all satisfied and 100 very

satisfied. This score is the variable of interest, y_i. The sample comprises 140 students aged 24 and under and 60 students aged 25 and over.

- Aged 24 and under: $N_1 = 12,000,\ n_1 = 140,\ \bar{y}_1 = 70,\ s_1^2 = 50$.
- Aged 25 and over: $N_2 = 8,000,\ n_2 = 60,\ \bar{y}_2 = 80,\ s_2^2 = 40$.

(d) Explain the principle of post-stratification in words (3 marks)
(e) Calculate the post-stratified mean (3 marks)
(f) Calculate the post-stratified variance of the mean (3 marks)
(g) Hence calculate a 95% confidence interval for the post-stratified mean. *Assume normality.* (3 marks)
[20 marks total]

B3.

A researcher is interested in the effect of unemployment on mental health. Self-reported mental health Y_{it} is measured for n individuals $i = 1, \ldots, n$ at two points in time $t = 0, 1$. Furthermore, their employment is measured as X_{it}, which is equal to 1 if i is employed at time t and 0 otherwise.
(a) Write up the assumed model for a fixed effect model and explain how this relates to the estimation of the causal effect (7 marks)
(b) We learn that measurement error c_i has been added to Y_{it} at time 0 and 1 for all i. What is the effect on our results? (3 marks)
(c) Give an example of a variable that is correlated with self-reported mental health but that does not affect the conclusions drawn from the estimated model (2 marks)
(d) Give an example of a variable that is correlated with self-reported mental health that would change our conclusions (3 marks)
(e) Assuming we have accounted for unobserved heterogeneity and confounders, does a significant (negative) causal effect prove conclusively that unemployment causes decrease in mental well-being? (5 marks)
[20 marks total]

END OF EXAMINATION PAPER

TWO hours

Electronic calculators may be used, provided that they cannot store text.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL PROGRAMMING

29 May 2013

14.00 – 16.00

Answer **THREE** of the **FOUR** questions. If more than **THREE** questions are attempted, then credit will be given for the best **THREE** answers.

Electronic calculators may be used, provided that they cannot store text.

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1. Explain the purpose of solving a Phase I problem for a given LP instance. State and prove a result linking the solution to Phase I to feasibility of the LP instance, clearly indicating when it is possible to proceed to Phase II.

How is Phase II started in that case?

[6 marks]

Given the following LP

$$\begin{array}{ll} \text{Minimize} & x_1 + x_2 + x_3 \\ \text{subject to} & -x_1 + 2x_2 + x_3 \leq 1 \\ & -x_1 \qquad + 2x_3 \geq 4 \\ & x_1 - x_2 + 2x_3 = 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(a) Formulate the Phase I problem in standard form.

[4 marks]

(b) Solve the Phase I problem through simplex iterations. Hence, either solve the above LP or establish that it has no solutions.

[10 marks]

2. A boat manufacturer produces two types of boats, a family rowboat (R) and a sports canoe (C). The boats are manufactured from aluminium by means of a large press machine and finished by hand labour. A rowboat requires 50 kg of aluminium, 6 min of machine time and 3 hr of finishing time; a canoe requires 30 kg of aluminium, 5 min of machine time and 5 hr of finishing time. The profits on the sale of a rowboat and a canoe are £50 and £60 respectively.

During a monthly production period, there are available a total of 2000 kg of aluminium, 300 min of machine time and 200 hr finishing time.

(a) Formulate the boat manufacturer's problem, to find the optimal production quantities of R and C in order to maximize his profit. Find these quantities and the maximum profit achieved.

[10 marks]

(b) Given an additional supply of 160 kg of aluminium, find the new optimal number of R and C to manufacture, and the new optimal profit.

[5 marks]

(c) Write down the dual problem and solve this problem by using complementary slackness conditions. Show that the increased profit in part (b) can be predicted through the shadow price of aluminium.

[5 marks]

3. Given that the following integer program (IP)

$$\begin{array}{ll} \max & -x_1 + 2x_2 \\ \text{s.t.} & \quad 2x_1 + x_2 \leq 5 \\ & -4x_1 + 4x_2 \leq 5 \\ & x_1, x_2 \geq 0 \text{ and integer.} \end{array}$$

has an optimal solution $\mathbf{x}_{IP}$ with value z_{IP};

(a) Define the *LP relaxation* of this problem and explain briefly the relationship between z_{IP} and the optimal value of the relaxation z_{LP}.

[3 marks]

(b) Given that the optimal solution to the relaxation is $x_1 = \frac{5}{4}, x_2 = \frac{10}{4}$, construct or otherwise obtain the final simplex tableau.

[4 marks]

(c) Find the optimal solution $\mathbf{x}_{IP}$ by use of (two) cutting planes based (in the usual notation) on the rule $\max_i \{f_i\}$.

[8 marks]

(d) Illustrate your solution with the cutting planes graphically.

[5 marks]

4. Define the dual problem for the primal LP given in standard form:

$$\min\left\{\mathbf{c}^T\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\right\}$$

where $\mathbf{x}\,(n \times 1)\,, \mathbf{A}\,(m \times n)\,, \mathbf{b}\,(m \times 1)\,, \mathbf{c}\,(n \times 1)\,,$ and state the duality thorem.

[5 marks]

(a) By converting to standard form, apply the definition to obtain the dual to the modified problem

$$\min\left\{\mathbf{c}^T\mathbf{x} \mid \mathbf{A}_1\mathbf{x} = \mathbf{b}_1, \mathbf{A}_2\mathbf{x} \geq \mathbf{b}_2, \mathbf{x} \geq \mathbf{0}\right\}$$

where $\mathbf{x}\,(n \times 1)\,, \mathbf{A}_1\,(m \times n)\,, \mathbf{A}_2\,(p \times n)\,, \mathbf{b}_1\,(m \times 1)\,, \mathbf{b}_2\,(p \times 1)\,, \mathbf{c}\,(n \times 1)$.

[5 marks]

(b) Obtain the dual of the following LP:

$$\begin{array}{ll} \text{Maximize} & 3x_1 + 5x_2 + 9x_3 \\ \\ \text{subject to} & 4x_1 + 12x_2 + 15x_3 \leq 900 \\ & -x_1 + 2x_2 + 3x_3 = 120 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

[5 marks]

(c) The values

$$x_1 = \frac{100}{3},\ x_2 = 0,\ x_3 = \frac{460}{9}$$

have been proposed as an optimal solution to the LP in part (b).

Test this assertion by using the theorem of duality.

[8 marks]

END OF EXAMINATION PAPER

2 hours

Tables of the cumulative distribution function are provided.

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL MODELLING OF FINANCE

22 May 2013

2:00 pm – 4:00 pm

Answer **all** 4 questions in **Section A** (60 marks in all) and
2 of the 3 questions in **Section B** (20 marks each). If all three questions from Section B are attempted then credit will be given for the two best answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** 4 questions

A1.

(i) If $F(S,t) = Ke^{\lambda t}S^{\alpha}$ where K, λ and α are constants, determine λ (in terms of α) if F satisfies the Black Scholes PDE,

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF = 0,$$

where the volatility σ and interest rate r are constants.

(ii) Consider a financial instrument with price $f(S,t)$ that satisfies the above PDE and provides a single payoff at time $t = T > 0$ amounting to S^{α}; determine $f(S, t < T)$.

(iii) Distinguish the behaviour of $f(S,t)$ with time for $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$.

[15 marks]

A2.

(i) Consider a portfolio comprising European call options on the same underlying asset and same expiration date comprising long two options C_{10} with exercise price 10, short three options C_{20} with exercise price 20 and long one option C_{40} with strike 40. In addition, consider P_{40}, the corresponding put option with exercise price 40. Show that

$$P_{40} \geq 2C_{10} - 3C_{20} + C_{40}.$$

(ii) Show that for a forward contract, if the delivery price F is such that $F > Se^{rT}$ (where S is the current value of the underlying, r is the risk-free rate and T is the time to expiry) then there exists an arbitrage opportunity.

[15 marks]

A3.

(i) Sketch the expiry payoff diagram for the following portfolio (all options are vanilla and have the same expiry date): short two puts both with exercise price X_1 and long four calls all with exercise price X_2. Compare the cases $X_1 > X_2$, $X_1 = X_2$, $X_1 < X_2$.

(ii) A portfolio Π has the following payoff at time T:

$$\begin{aligned} \Pi(S,T) &= X_1 - S, \quad S < X_1 && (1) \\ &= 0, \quad X_1 < S < X_2 && (2) \\ &= S - X_2, \quad S > X_2. && (3) \end{aligned}$$

Replace this portfolio at time t by buying or selling vanilla calls or puts, all with the same expiry date, but with differing exercise prices.

[15 marks]

A4.

Prove the following bounds on the price of European call and put options, denoted by $C(S,t)$ and $P(S,t)$ respectively on an underlying asset S which pays no dividends and where X is the exercise price, r is the constant risk-free rate, t is the time since the purchase of the option, and T is the expiry date of the option:

(i)

$$P \geq Xe^{-r(T-t)} - S,$$

(ii)

$$P \leq Xe^{-r(T-t)},$$

(iii)

$$C \geq S - Xe^{-r(T-t)}.$$

(iv) Use the result in part (iii) to show that it is never optimal to exercise an American call option before expiry when the underlying asset pays no dividends.

You may assume the put-call parity relationship for European options, namely

$$S + P - C = Xe^{-r(T-t)}.$$

[15 marks]

SECTION B

Answer **2** of the 3 questions

B5. The change in the value of an asset S in a (small) time dt satisfies the stochastic differential equation

$$dS = \mu S dt + \sigma S dW$$

where μ and σ are constant and W is a standard Brownian motion.

(i) If $S = \exp F$, show that the stochastic differential equation for F may be written in the form

$$dF = a dt + b dW$$

where you are to determine a and b. Interpret this equation. (Ito's Lemma may be used without proof.)

(ii) Starting with the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where the volatility σ and interest rate r are constants, show that $V(F,t)$ satisfies

$$\frac{\partial V}{\partial t} + c\frac{\partial^2 V}{\partial F^2} + d\frac{\partial V}{\partial F} - rV = 0,$$

where you are to determine c and d.

(iii) By writing $V = g_1(t) + g_2(t)F$, determine the value of an option whose payoff at time $t = T$ is $V(S,T) = \log S$.

[20 marks]

B6.

(i) Suppose that an asset of value S pays out a dividend $DSdt$ in time dt (in other words, D is the dividend yield). Given that the price $V(S,t)$ of an option on this asset satisfies the modified Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0,$$

show that the substitution

$$V(S,t) = e^{-D(T-t)}V_1(S,t)$$

results in the standard Black-Scholes equation, but with a modified interest rate (which you are to determine).

(ii) Given that the solution for a vanilla (non-dividend-paying) European call option is

$$C(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log(S/X) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\log(S/X) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and $N(x)$ is the cumulative distribution function, write down the corresponding result for a call option with a dividend yield of D.

(iii) Consider a stock whose price today (9 months from the expiration of an option) is £20, the exercise price of a European call option on this stock is £19, the risk-free interest rate is 3% per annum (fixed) and the volatility (constant) is 10% per $(\text{annum})^{\frac{1}{2}}$. The stock pays a continuous dividend at a rate 2% of the value of the stock (per annum).

What is today's value of a call option on this asset? Values of the cumulative distribution function may be determined by interpolation of the provided tables.

(iv) Given that the put-call relationship for vanilla European options in the case of a non-dividend-paying stock is

$$C + Xe^{-r(T-t)} - P - S = 0,$$

write down the corresponding relationship between call and put (C_d and P_d, respectively) options on a continuous-dividend-paying asset, with a yield D.

[20 marks]

B7.

Consider the Black-Scholes equation for a put option, P, on a stock S which pays no dividends, namely

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0,$$

in the usual notation, where r and σ are both constants.

(i) By neglecting the time derivative in the above equation, seek solutions of the form

$$P(S) = A_1 S^{\alpha_1} + A_2 S^{\alpha_2},$$

where A_1 and A_2 are constants, and $\alpha_1 > \alpha_2$. Determine the values of α_1 and α_2.

(ii) Consider a perpetual American put option $P(S)$, i.e. an option with no expiry date but where it is possible at any point in time to exercise, which has a (non-standard) exercise value of Ke^{-S}, where K is a constant. This can be valued using the time independent solutions in (i) above. The exercise boundary is denoted by S_f. State and justify the two conditions at $S = S_f$ and the other appropriate boundary condition.

(iii) Determine S_f and the values of A_1 and A_2.

(iv) Describe the behaviour of $P(S)$ and S_f as $r \to 0$.

[20 marks]

END OF EXAMINATION PAPER

Two Hours

The University of Manchester

Contingencies 2

29 May 2013

09:45 – 11:45

Answer ALL questions

Electronic calculators can be used provided they cannot store text

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A copy of *Formulae and Tables for Actuarial Examinations* will be provided

Question 1

Let T_{xy} be a continuous random variable representing the joint future lifetime of lives aged x and y.

Define another random variable $\bar{Z}_{xy} = v^{T_{xy}}$ where $v = \dfrac{1}{1+i}$, and i is the rate of interest.
Describe in words the quantity represented by $\bar{Z}_{xy}$ and derive an expression for its variance in terms of standard actuarial functions. [6]

Question 2

Describe the structure of a unit-linked endowment assurance policy, stating how the payments made by the policy holder are treated and how the benefits payable on death, surrender, or maturity may be determined. (Formulae are not necessary and a discussion of pricing is not required.) [7]

Question 3

A sickness benefit is modelled by a three-state transition model with constant transition probabilities between them.

The states are: Healthy, Sick, Dead.

Transition rates are:

Healthy to Sick, 0.045 per annum;
Healthy to Dead, 0.02 per annum
Sick to Healthy, 0.05 per annum;
Sick to Dead, 0.04 per annum.

(i) Sketch a diagram representing this model. [2]

(ii) Calculate the present value of a sickness benefit of £3,000 p.a. paid continuously to a
life now aged 55 exact and sick, during only this first period of sickness, discounted at an interest rate of 4% p.a. and payable to a maximum age of 65 exact. [6]

[Total 8]

Question 4

You are given the following data referring to the mortality rates of elephants in a game reserve. “Indigenous” refers to the elephants that are part of the original population of the reserve. “Immigrants” refers to elephants in a group recently transported to the reserve.

	Indigenous		Immigrants	
Age band	Exposed to risk	Observed mortality rate	Exposed to risk	Observed mortality rate
5 – 14	103	0.038835	87	0.011494
15 – 24	221	0.009050	233	0.008584
25 – 34	124	0.008065	204	0.009804
35 – 44	86	0.011628	211	0.014218

(i) Calculate the directly and indirectly standardised mortality rates, and the standardised mortality ratio for the immigrant lives, using the total elephant population as the standard population. [11]

(ii) Describe four factors that could account for the difference in mortality between the two populations. [6]

[Total 17]

Question 5

(i) Describe three different types of selection in the membership of a pension scheme. [6]

A pension scheme provides a pension on retirement for any reason of 1/60th of final pensionable salary for each year of service, with fractions counting proportionately. Final pensionable salary is defined as the average salary over the three years prior to retirement.

A member of a pension scheme joined at age 32 exact. He is now aged 57 exact and in the 12 months preceding the scheme valuation date his salary was £55,000.

(ii) Calculate the expected present value now of this member's past service benefits. [4]

(iii) Show that the expected present value of his future service benefits is £59,848. [4]

(iv) Calculate the percentage of salary required to fund the future service element of the pension. [3]

Basis: Example Pension Scheme Table in the Formulae and Tables with Normal retirement Age 65.

[Total 17]

Question 6

A man aged 50 exact wants to buy the following benefits:

(a) an annuity of £10,000 per annum payable monthly in arrear to his wife currently aged 52 exact commencing on his death and for the rest of her life, and

(b) an annuity of £1,000 per annum payable monthly in arrear to his granddaughter currently aged 9 exact payable while at least one of he and his wife are still alive; and

(c) a lump sum of £5,000 payable to his granddaughter at the end of the year of death of the last survivor of him and his wife provided the granddaughter is at least 21.

Calculate the combined single premium. For the annuities (a) and (b) use the basis below and for the assurance (c) use the values given in the table below. [17]

Basis:
Mortality Male life – PMA92C20
Wife – PFA92C20
Granddaughter – ignore
Interest 4% per annum

You are given

Male	Female	Joint
$A_{62} = 0.42815$	$A_{64} = 0.41377$	$A_{\overline{62:64}} = 0.33762$
		$v^{12}\,{}_{12}p_{50} \cdot {}_{12}p_{52} = 0.60171$
		$v^{12}\,{}_{12}p_{50}(1-{}_{12}p_{52}) = 0.012279$
		$v^{12}(1-{}_{12}p_{50})\,{}_{12}p_{52} = 0.010395$

[Total 17]

Question 7

An insurance company sells a non profit endowment assurance with sum assured payable at the end of the year of death or on survival to the end of the three year term. Level premiums are payable annually in advance during the life of the policyholder.

The policy can be surrendered on each policy anniversary for a benefit equal to the premiums paid with interest at 2% per annum, less a £50 administration charge.

Premiums and net premium reserves are calculated with the following basis:

Mortality	AM92 ultimate
Surrenders	None
Interest	4%
Initial expenses	£100
Renewal expenses (in years 2 & 3)	£50 per annum plus 2% of level annual premium

(i) Show that the level annual premium for a policy with sum assured of £10,000 sold to a man aged 62 exactly is £3,225. [8]

The company uses profit testing to calculate the profit margin of this contract. The profit testing basis is the same as above except that it assumes 7% of policies will be surrendered on the first policy anniversary and 4% will be surrendered on the second. It assumes it earns 5% per annum on funds invested and uses a risk discount rate of 8% per annum to calculate the profit margin.

(ii) Calculate the expected profit margin. [20]

[Total 28]

END OF EXAMINATION

Two Hours

The University of Manchester

Contingencies 2

29 May 2013

09:45 – 11:45

Answer ALL questions

Electronic calculators can be used provided they cannot store text

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A copy of *Formulae and Tables for Actuarial Examinations* will be provided

Question 1

Let T_{xy} be a continuous random variable representing the joint future lifetime of lives aged x and y.

Define another random variable $\bar{Z}_{xy} = v^{T_{xy}}$ where $v = \dfrac{1}{1+i}$, and i is the rate of interest.
Describe in words the quantity represented by $\bar{Z}_{xy}$ and derive an expression for its variance in terms of standard actuarial functions. [6]

Question 2

Describe the structure of a unit-linked endowment assurance policy, stating how the payments made by the policy holder are treated and how the benefits payable on death, surrender, or maturity may be determined. (Formulae are not necessary and a discussion of pricing is not required.) [7]

Question 3

A sickness benefit is modelled by a three-state transition model with constant transition probabilities between them.

The states are: Healthy, Sick, Dead.

Transition rates are:

Healthy to Sick, 0.045 per annum;
Healthy to Dead, 0.02 per annum
Sick to Healthy, 0.05 per annum;
Sick to Dead, 0.04 per annum.

(i) Sketch a diagram representing this model. [2]

(ii) Calculate the present value of a sickness benefit of £3,000 p.a. paid continuously to a
life now aged 55 exact and sick, during only this first period of sickness, discounted at an interest rate of 4% p.a. and payable to a maximum age of 65 exact. [6]

[Total 8]

Question 4

You are given the following data referring to the mortality rates of elephants in a game reserve. “Indigenous” refers to the elephants that are part of the original population of the reserve. “Immigrants” refers to elephants in a group recently transported to the reserve.

	Indigenous		Immigrants	
Age band	Exposed to risk	Observed mortality rate	Exposed to risk	Observed mortality rate
5 – 14	103	0.038835	87	0.011494
15 – 24	221	0.009050	233	0.008584
25 – 34	124	0.008065	204	0.009804
35 – 44	86	0.011628	211	0.014218

(i) Calculate the directly and indirectly standardised mortality rates, and the standardised mortality ratio for the immigrant lives, using the total elephant population as the standard population. [11]

(ii) Describe four factors that could account for the difference in mortality between the two populations. [6]

[Total 17]

Question 5

(i) Describe three different types of selection in the membership of a pension scheme. [6]

A pension scheme provides a pension on retirement for any reason of 1/60th of final pensionable salary for each year of service, with fractions counting proportionately. Final pensionable salary is defined as the average salary over the three years prior to retirement.

A member of a pension scheme joined at age 32 exact. He is now aged 57 exact and in the 12 months preceding the scheme valuation date his salary was £55,000.

(ii) Calculate the expected present value now of this member's past service benefits. [4]

(iii) Show that the expected present value of his future service benefits is £59,848. [4]

(iv) Calculate the percentage of salary required to fund the future service element of the pension. [3]

Basis: Example Pension Scheme Table in the Formulae and Tables with Normal retirement Age 65.

[Total 17]

Question 6

A man aged 50 exact wants to buy the following benefits:

(a) an annuity of £10,000 per annum payable monthly in arrear to his wife currently aged 52 exact commencing on his death and for the rest of her life, and

(b) an annuity of £1,000 per annum payable monthly in arrear to his granddaughter currently aged 9 exact payable while at least one of he and his wife are still alive; and

(c) a lump sum of £5,000 payable to his granddaughter at the end of the year of death of the last survivor of him and his wife provided the granddaughter is at least 21.

Calculate the combined single premium. For the annuities (a) and (b) use the basis below and for the assurance (c) use the values given in the table below. [17]

Basis:
Mortality Male life – PMA92C20
Wife – PFA92C20
Granddaughter – ignore
Interest 4% per annum

You are given

Male	Female	Joint
$A_{62} = 0.42815$	$A_{64} = 0.41377$	$A_{\overline{62:64}} = 0.33762$
		$v^{12}\,{}_{12}p_{50} \cdot {}_{12}p_{52} = 0.60171$
		$v^{12}\,{}_{12}p_{50}(1 - {}_{12}p_{52}) = 0.012279$
		$v^{12}(1 - {}_{12}p_{50})\,{}_{12}p_{52} = 0.010395$

[Total 17]

Question 7

An insurance company sells a non profit endowment assurance with sum assured payable at the end of the year of death or on survival to the end of the three year term. Level premiums are payable annually in advance during the life of the policyholder.

The policy can be surrendered on each policy anniversary for a benefit equal to the premiums paid with interest at 2% per annum, less a £50 administration charge.

Premiums and net premium reserves are calculated with the following basis:

Mortality	AM92 ultimate
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(i) Show that the level annual premium for a policy with sum assured of £10,000 sold to a man aged 62 exactly is £3,225. [8]

The company uses profit testing to calculate the profit margin of this contract. The profit testing basis is the same as above except that it assumes 7% of policies will be surrendered on the first policy anniversary and 4% will be surrendered on the second. It assumes it earns 5% per annum on funds invested and uses a risk discount rate of 8% per annum to calculate the profit margin.

(ii) Calculate the expected profit margin. [20]

[Total 28]

END OF EXAMINATION

Two hours

THE UNIVERSITY OF MANCHESTER

RISK THEORY

16 May 2013

09:45-11:45

Answer ALL FOUR questions.
The total number of marks in the paper is 90.

Electronic calculators may be used, provided that they cannot store text.

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1. Consider the Cramér-Lundberg model where the risk process $U = \{U_t : t \geq 0\}$ of an insurance company is given by

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i, \quad t \geq 0,$$

where t is time in years, $u \geq 0$ is the initial capital, $c > 0$ is the premium rate, $N = \{N_t : t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$ that counts the number of claims and the claim sizes $X_1, X_2, \ldots$ are independent, identically distributed and strictly positive random variables that are independent of the Poisson process N. Further, the probability density function of X_1 is given by

$$f_{X_1}(x) = \frac{1}{2}\left(\alpha_1 \mathrm{e}^{-\alpha_1 x} + \alpha_2 \mathrm{e}^{-\alpha_2 x}\right), \quad x > 0,$$

where $\alpha_1, \alpha_2 > 0$.

(a) Determine the probability that the insurance company receives exactly three claims in the first year.

[3 marks]

(b) The insurance company sets the premium rate c by using the expected value principle with loading factor $\beta > 0$. Show that the premium rate is given by

$$c = \frac{1+\beta}{2}\lambda\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right).$$

[3 marks]

(c) Does the ruin probability of the insurance company become higher or lower when the loading factor β is increased? Justify your answer.

[3 marks]

(d) Determine the Laplace exponent ξ of the risk process U which is defined by

$$\mathrm{e}^{\xi(\theta)} = \mathbb{E}\left[\mathrm{e}^{\theta U_1} \middle| U_0 = 0\right], \quad \theta \geq 0.$$

[4 marks]

(e) Assume now $\alpha_1 = \alpha_2$. By using the Laplace exponent ξ, determine the Lundberg coefficient.

[4 marks]

(f) Assume $\alpha_1 = \alpha_2 = 1/3$, $\beta = 0.4$, $\lambda = 1$ and $u = 6$. Find the probability that the first claim leads to ruin.

[6 marks]

(g) Suppose now, in addition to the assumptions made in part (f), that the insurance company has the option to buy excess of loss reinsurance with retention level $M \geq 0$. It is assumed that the insurance company chooses the retention level at time $t = 0$ and cannot alter it in the future. The premium rate that the reinsurer charges to the insurance company is determined via the expected value principle with loading factor equal to 0.6. Denote by U^M the risk process of the insurance company under the excess of loss reinsurance scheme with retention level M.

(i) Determine the reinsurance premium rate as a function of the retention level M.

(ii) For which values of the retention level M, is the net profit condition satisfied for the risk process U^M?

[8 marks]

[Total 31 marks]

2.

(a) What is the variance premium principle?

[3 marks]

(b) When does a premium principle satisfy the no rip-off property?

[3 marks]

(c) Jensen's inequality says that for a positive random variable X with finite mean and a twice differentiable function f whose second derivative is everywhere positive, the following inequality holds:

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$

Use Jensen's inequality to show that the exponential premium satisfies the positive loading property.

[6 marks]

(d) Let X be a risk whose probability density function f_X is given by

$$f_X(x) = \begin{cases} \frac{1}{x+1} & \text{if } 0 \leq x \leq \mathrm{e}^1 - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the Value-at-Risk and the Tail-Value-at-Risk for any confidence level $p \in (0, 1)$.

[7 marks]

[Total 19 marks]

3. An observable random variable X depends on an unobservable parameter $\theta \in [0, 1]$ in such a way that the probability density function of X given θ, is given by

$$f(x|\theta) = \begin{cases} (1-\theta)^2 & \text{if } x = 0 \\ 2\theta(1-\theta) & \text{if } x = 1 \\ \theta^2 & \text{if } x = 2. \end{cases}$$

Assume a Bayesian framework in which the unknown parameter θ is interpreted as a random variable Θ whose prior probability density function is given by

$$f_\Theta(\theta) = 30(1-\theta)^2\theta^2, \quad 0 \leq \theta \leq 1.$$

It is observed that $X = 2$.

(a) What is the posterior distribution of Θ?

[4 marks]

(b) What is the Bayesian estimate of Θ under the quadratic (or squared error) loss function?

[4 marks]

(c) Suppose a second independent observation is made which yields $X = 1$.

(i) What is the posterior distribution of Θ after the second observation?

[4 marks]

(ii) What is the maximum likelihood estimate of θ after the second observation?

[6 marks]

[Total 18 marks]

4. An insurance company ranks policyholders in three different categories: type 1, type 2 and type 3. In each year, a policyholder produces independent claim amounts that are exponentially distributed with parameter 0.01 if he/she is of type 1, with parameter 0.02 if he/she is of type 2 and with parameter 0.04 if he/she is of type 3. The company believes 15% of its policyholders are of type 1, 60% are of type 2 and the rest is of type 3.

(a) A certain policyholder produced claim amounts of 40 in year 1, 70 in year two and 50 in year 3. Determine the Bühlmann credibility estimate of the expected value of the claim amounts in year 4 of this particular policyholder.

[13 marks]

(b) The insurer has grouped together policyholders who are living in the same postcode area. The group corresponding to a particular postcode area produced total claim amounts of 1100 in year 1, 950 in year 2 and 600 in year 3. The total number of policyholders in the group was 12 in year 1, 10 in year 2 and 9 in year 3. In year 4, the total number of policyholders in this group is 10. Assuming that the policyholders in this group are independent, use the Bühlmann-Straub model in order to determine the Bühlmann credibility estimate of the expected value of the total claim amounts in year 4 corresponding to this particular group.

[6 marks]

(c) Name an advantage and a disadvantage of using the Bühlmann credibility estimate instead of the Bayesian credibility estimate.

[3 marks]

[Total 22 marks]

END OF EXAMINATION PAPER