systems employ one or more filtering or integrating techniques to 'clean-up' the radar picture (see Volume 11, Chapter 3).
8. In constant false alarm rate (CFAR) receivers a voltage threshold level is set, and returns are only processed as targets if their signal strength exceeds this level. The threshold level is constantly adjusted in line with a running average value of the received signal.
9. Integration may be applied on a pulse-to-pulse or on a scan-to-scan basis. The technique relies on the premise that whereas a target return will be fairly constant in position and strength, clutter returns tend to be more transient. It is therefore possible to set a threshold level so that, for example, a return is only processed as a target if it persists for, say, six pulses out of ten on a single scan pass, and on a minimum number of successive scans. Unfortunately, small targets such as submarine masts and periscopes may often fail to clear these threshold levels since they will often be physically masked in a rough sea.

## Scanning

10. The scan pattern of a maritime radar may be either a forward hemisphere sector scan, or a $360^{\circ}$ scan. However, in the latter case only rarely is a full $360^{\circ}$ achieved since in virtually all installations there will be some screening by the aircraft fuselage or components. In addition where a $360^{\circ}$ scan is available it is normally possible to restrict transmissions to certain sectors, either to increase the data rate from an area of interest, or to deny an enemy any EW information from the transmissions. The radar scanner will usually be stabilized in the horizontal and vertical planes to compensate for aircraft manoeuvre and in some systems the scanner tilt may be controlled automatically to restrict transmissions in accordance with the selected range scale.

## Target Tracking

11. One of the facilities of a maritime radar is the ability to track a number of targets automatically. In a typical system the radar computer divides the search area into a matrix of cells, into which targets are allocated. Once a target has been identified by the operator a computer file is opened, and as the target moves from one cell to another the track and speed are calculated and the file updated. Problems can occur with rapidly manoeuvring targets and with large targets where the radar return occupies more than one cell. In these cases it may be preferable for the operator to allocate a manually assessed track and speed to the target. The efficiency of an auto-tracking system is highly dependent on the suppression of unwanted returns since these can cause the computer to overload as it attempts to associate false targets with established tracks or to generate new, but false, tracks.

## Displays

12. Maritime radars usually employ $360^{\circ}$ or sector PPI displays in their normal operating mode and in the case of a $360^{\circ}$ display it may be heading or north orientated. In addition to the PPI display, some systems will have high resolution displays using an A- or B-scope which enable selected targets to be investigated more thoroughly by scanning through a narrow angle. It may be possible to achieve some degree of target identification using these displays but this is largely dependent on operator skill and experience.


# AP3456 <br> The Central Flying School (CFS) <br> Manual of Flying 

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## CHAPTER 1 - ROTOR AERODYNAMICS AND CONTROL (HELICOPTER)

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## ROTOR AERODYNAMICS

## Introduction

1. The same basic laws govern the flight of both fixed and rotary wing aircraft and, equally, both types of aircraft share the same fundamental problem; namely that the aircraft is heavier than air and must, therefore, produce an aerodynamic lifting force to overcome the weight of the aircraft before it can leave the ground. In both types of aircraft the lifting force is obtained from the aerodynamic reaction resulting from a flow of air over an aerofoil section. The important difference lies in the relationship of the aerofoil to the fuselage. In the fixed-wing aircraft, the aerofoil is fixed
to the fuselage as a wing whilst in the helicopter, the aerofoil has been removed from the fuselage and attached to a centre shaft which, by one means or another, is given a rotational velocity.
2. Helicopters have rotating wings, which are engine-driven in normal flight. The rotor provides both lift and horizontal thrust.

## Rotor Systems

3. Helicopters may be single or multi-rotored, each rotor having several blades, usually varying from two to six in number. The rotor blades are attached by a rotor head to a rotor shaft which extends approximately vertically from the fuselage. They form the rotor, which turns independently through the rotor shaft, see Fig 1.

## 12-1 Fig 1 The Rotor Head Arrangement



The shaft axis is a straight line through the centre of the main drive shaft. The rotor blades are connected to the rotor head, at an angle to the plane of rotation, called the pitch angle, see Fig 2.

12-1 Fig 2 Blade Pitch Angle

4. The axis of rotation is perpendicular to the plane of rotation, and is a line through the rotor head about which the blades rotate. Under ideal conditions the axis of rotation will coincide with the shaft axis. This however is not usually so since the rotor is tilted under most flight conditions, see Fig 3.

## 12-1 Fig 3 The Rotor Disc Tilted


5. The tip path plane, shown in Fig 3, is the path described by the rotor blades during rotation and is at right angles to the axis of rotation and parallel to the plane of rotation. The area contained within this path is known as the rotor disc.

## Forces on an Aerofoil

6. The airflow around the aerofoil gives rise to a pressure distribution. The pressure differences produce a force distribution which can be represented by total reaction, see Fig 4. Total reaction may be resolved into a force perpendicular to the relative airflow (RAF) called lift and a force parallel to the RAF called drag. The angle which the chord line makes with the RAF is the angle of attack.

## 12-1 Fig 4 Total Reaction



The magnitude of lift is given by:

$$
\begin{aligned}
\text { LIFT } & =C_{\mathrm{L}} \times \frac{1}{2} \rho \mathrm{~V}^{2} \mathrm{~S} \\
\text { where } \quad \rho & =\text { Air density } \\
\mathrm{V} & =\text { Velocity of RAF } \\
\mathrm{S} & =\text { Plan area of aerofoil } \\
\mathrm{C}_{\mathrm{L}} & =\text { Coefficient of lift }
\end{aligned}
$$

The magnitude of drag is given by:

$$
\begin{aligned}
\text { DRAG } & =C_{D} \times \frac{1}{2} \rho V^{2} S \\
\text { where } \quad C_{D} & =\text { Coefficient of drag }
\end{aligned}
$$

## Blade Design

7. The design requirements of a rotor blade are complicated:
a. The combined area of the blades is small compared to the wings of an aeroplane of similar weight, so high maximum $C_{L}$ is needed.
b. Power to weight ratio problems can be minimized by use of blades having a good lift to drag ratio.
c. The pitch angle of a blade is held by a control arm and a large pitching moment caused by movement of the centre of pressure would cause excessive stress in this component. A symmetrical aerofoil has a very small pitching moment and is also suitable for relatively high blade tip speeds.
d. Torsional stiffness is required so that pitching moment changes are minimized. A typical blade has an extruded alloy $D$ spar leading edge with a fabricated trailing edge. It is symmetrical, with a thickness ratio of about 1:7, and is rectangular in plan, see Fig 5 . Later designs of blade incorporate torsional stiffness, opposing pitching moments, and aerodynamic and planform balancing to allow cambered and high speed sections to be used to improve the overall performance of the blades.

## 12-1 Fig 5 Typical Rotor Blade Section



## Relative Airflow

8. If a rotor blade is moved horizontally through a column of air, the effect will be to displace some of the air downwards. If a number of rotor blades are travelling along the same path in rapid succession then the column of air will eventually become a column of descending air. This downward motion of air is known as induced flow (IF), see Fig 6. The direction of the airflow relative to the blade (RAF) is the resultant of the blade's horizontal travel through the air and the induced flow, see Fig 7. The angle between the Relative air Flow and the Chord line is the angle of attack.

## 12-1 Fig 6 Induced Airflow



## 12-1 Fig 7 Forces Acting on a Rotor Blade



## Lift and Drag

9. The Total Reaction is the vector resultant of lift, which is produced by the relative air flow passing over the blade at an angle of attack, and drag, which is perpendicular to the lift, or parallel to the RAF. The Total Reaction may be split into components; the Rotor Thrust acting along the axis of rotation, and the Rotor Drag acting parallel to the plane of rotation.

## Total Rotor Thrust

10. The rotor thrusts of each blade are added together and make up the total rotor thrust. The total rotor thrust is defined as the sum of all the blade rotor thrusts and acts along the axis of rotation through the rotor head, see Fig 8.

## 12-1 Fig 8 Total Rotor Thrust



## Equalising Lift

11. The rotational velocity of each part of a rotor blade varies with its radius from the rotor head; the blade tip will always experience a greater velocity of airflow than the root. Lift, and hence rotor thrust, is proportional to $\mathrm{V}^{2}$ and will be much greater at the blade tip than at the root - an unequal distribution of lift which would cause large bending stresses in the rotor blade. There are various methods used by blade manufacturers to equalise lift as follows:
a. Washout. Washout is a designed twist in the blade which reduces blade pitch angle from root to tip giving a more uniform distribution of lift (see Fig 9). The angle of attack, and hence rotor thrust, is decreased with the pitch angle at the tip.

## 12-1 Fig 9 Lift Distribution with Washout


b. Varying Aerofoil Section and Tapering. Varying the aerofoil section, in particular the flattening of the aerofoil section on the outboard, high speed, portion of the blade will reduce the lift produced. Additionally, tapering the outboard section of the rotor thereby reducing the chord and therefore the lifting section can be used to aid equalisation of lift.

## CONTROL

## Introduction

12. For various stages of flight, the total rotor thrust requirements will change. Although rotor rpm $(\mathrm{Nr})$, and hence rotational velocity, can be changed, the reaction time is slow and the range of values is small. The other controllable variable is pitch angle; a change in pitch angle will cause a change in angle of attack and, therefore, total rotor thrust.

## Collective Pitch Changes

13. The pitch angle of a rotor blade is changed by turning it about a sleeve and spindle bearing on its feathering hinge by means of a pitch operating arm connected to a rotating swash plate. The rotating plate may be raised and lowered or have its angle changed by a non rotating swash plate below, which is connected to the collective pitch lever and cyclic control stick in the cockpit by control rods which are usually hydraulically assisted, see Fig 10.

## 12-1 Fig 10 Rotor-Head Detail



The pitch angle is thus increased or decreased collectively by the pilot raising or lowering the collective pitch lever or changed cyclically by movement of the cyclic control stick.

## Control of Rotor RPM (Nr)

14. Changes in total rotor thrust will produce corresponding changes in rotor drag. Engine power must, therefore, be controlled to maintain Nr when altering total rotor thrust.
15. Most helicopters have automatic devices to sense the slightest variation in rotor speed and to compensate by altering the fuel supply to the engine to maintain constant Nr. Such control is usually provided by a fuel computer or a hydro-mechanical governor.

## Flapping

16. Flapping is the angular movement of the blade above and below the plane of the hub. Flapping relieves bending stresses at the root of the blade which might otherwise be caused by cyclic and collective pitch changes or changes in the speed and direction of the airflow relative to the disc. In a rigid rotor system bending stresses are absorbed by designed deformation of the rotor/hub combination. In an articulated rotor, bending stresses are avoided by allowing the blade to flap about the flapping hinge, see Fig 11.

## 12-1 Fig 11 Flapping Hinge



## Coning

17. Rotor thrust will cause the blades to rise about the flapping hinges until they reach a position where their upward movement is balanced by the outward force of centrifugal reaction being produced by the rotation of the blades(see Fig 12). In normal operation the blades are said to be coned upwards, the coning angle being measured between the spanwise length of the blade and the blades tip path plane. The coning angle will vary with combinations of rotor thrust and Nr (see Fig 12). If rotor thrust is increased and Nr remains constant, the blades cone up. If Nr is reduced, centrifugal force decreases and if rotor thrust remains constant, the blades again cone up. The weight of the blade will also have some effect but for any given helicopter this will be constant.

## 12-1 Fig 12 Centrifugal Reaction



## Limits of Rotor RPM

18. Because the area of the rotor disc reduces as the coning angle increases, the coning angle must never be allowed to become too big. As centrifugal force gives a measure of control of the coning angle through Nr , providing the Nr is kept above a laid down minimum, the coning angle will always be within safe operating limits. There will also be an upper limit to Nr due to transmission considerations and blade root loading stresses. Compressibility, due to high blade tip speeds, is also a limiting factor. Nr limits are to be found in the appropriate Aircrew Manual.

## Overtorqueing

19. Overtorqueing can be avoided by careful monitoring of the torque gauge and careful use of the helicopter controls. The condition is described in Volume 12, Chapter 12.

## Overpitching

20. Overpitching is a dangerous condition reached following the application of pitch to the rotor blades without sufficient engine power to compensate for the extra rotor drag. The condition is described fully in Volume 12, Chapter 12.

## CHAPTER 2 - HOVERING AND HORIZONTAL MOVEMENT

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## HOVERING

## Take-off and Climb to a Free Air Hover

1. To lift a helicopter off the ground, a force must be produced greater than the weight which acts vertically downwards through the aircraft's centre of gravity (CG). On the ground with minimum pitch set, the total Rotor Thrust is small, and on some aircraft can even be negative, and the aircraft remains on the ground. As the collective lever is raised blade pitch and the angle of attack are increased and the Total Rotor Thrust (TRT) becomes equal to AUW and the helicopter is resting only lightly on the ground. A further increase in angle of attack causes TRT to exceed the AUW and the helicopter accelerates vertically (in still air conditions) (see Fig 1).

## 12-2 Fig 1 Forces in the Take-off and Climb


2. As the Rate of Climb (ROC) increases there is a relative airflow down through the rotor. This adds to, and increases, the induced airflow. The Angle of Attack and Total Rotor Thrust are automatically reduced by the increased induced flow (IF) and the acceleration decreases until a steady ROC is achieved with TRT = AUW (see Fig 2).

## 12-2 Fig 2 Steady Rate of Climb


3. In the climb, the Total Reaction Vector is tilted away from the axis of rotation because the direction of the Relative Aiflow (RAF) has changed. Rotor drag is increased and more power is required to maintain rotor rpm $(\mathrm{Nr})$.
4. To stop the climb, collective pitch and angle of attack are reduced and the TRT is now less than AUW. The helicopter's ROC decreases, IF reduces, angle of attack re-increases and TRT increases until a steady hover is achieved with TRT equal to AUW. The helicopter is now said to be in a Free Air Hover.

## Vertical Descent

5. At low rates of descent the sequence is the reverse of the vertical climb, that is, due to downward movement, IF will be opposed and angle of attack will increase (see Fig 3). At higher rates of descent, airflow is more complex and is discussed in detail in Volume 12 Chapter 5, paras 4 to 10.

## 12-2 Fig 3 Vertical Descent


6. When climbing or descending there will be some parasite drag from the fuselage but the amount is small, since a ROC or ROD of $1200 \mathrm{ft} / \mathrm{min}$ is barely 12 kt .

## Ground Effect

7. In a free air hover, the airflow through the rotor disc begins at zero velocity some distance above and accelerates through the disc and into the air below. There is little resistance to the downward movement of air. If the helicopter is hovered close to the ground, the downwash meets the ground, is opposed, and escapes horizontally. A divergent duct is produced causing an increase in pressure (see Fig 4).

## 12-2 Fig 4 Hover in Ground Effect



The increased pressure of the air beneath the helicopter opposes and reduces the IF so that angle of attack and hence TRT are increased for a given pitch setting (see Fig 5).

## 12-2 Fig 5 Angle of Attack and Total Rotor Thrust Increase



In order to remain at a constant height the collective pitch must be reduced, to reduce the angle of attack and keep the TRT equal to AUW (see Fig 6). The TR will have moved closer to the axis of rotation, producing a reduction in rotor drag in power required to hover is Ground Effect. Helicopters are said to hover Inside Ground Effect (IGE) or, when in free air hover, Outside Ground Effect (OGE).

## 12-2 Fig 6 Collective Pitch Decreasing


8. Factors affecting Ground Effect. Ground effect is affected by the following factors:
a. Height. The reduction in IF is greater when the rotor is close to the ground. Ground effect reduces with increase in height until it is negligible above $2 / 3$ rotor diameter distance from the ground.
b. Slope. On sloping ground much of the air flows downhill and there is reduced ground effect because there is no development of a divergent duct.
c. Nature of the Ground. Rough ground will tend to disrupt the air flow preventing a divergent duct from being formed.
d. Wind. The ground effect is displaced downwind reducing ground effect. However, as wind speed increases IF is reduced by translational lift which is described in Volume 12, Chapter 3.

## Recirculation

9. Whenever a helicopter is hovering near the ground, some of the air passing through the disc is recirculated and it would appear that the recirculated air increases speed as it passes through the disc a second time (see Fig 7).

## 12-2 Fig 7 Increased IF near the Blade Tips



This local increase in IF near the tips gives rise to a loss of rotor thrust. Some recirculation is always taking place, but over a flat, even surface the loss of rotor thrust due to recirculation is more than compensated for by ground effect. If a helicopter is hovering over tall grass, or similar types of surface, the loss of lift due to recirculation will increase and, in some cases the effect will be greater than ground effect and more power would be required to hover near the ground than in free air (see Fig 8). Heavy helicopters can experience this phenomenon hovering over water.

12-2 Fig 8 Increased Recirculation due to Long Grass

10. Recirculation will increase when any obstruction on the surface or near where the helicopter is hovering prevents the air from flowing evenly away. Hovering close to a building, wire link fencing or cliff face may cause severe recirculation (see Fig 9).

## 12-2 Fig 9 Recirculation near a Building



## HORIZONTAL MOVEMENT

## Cyclic Pitch Changes

11. For a helicopter to move horizontally, the rotor disc must be tilted so that the total rotor thrust vector has a component in the direction required (see Fig 10). To enable the rotor disc to tilt, the swash plates are tilted so that the pitch angle on one side of the disc increases causing the blade to rise, while the pitch angle on the other side of the disc must, at the same time, be decreased by the same amount, causing the blade to descend. The tilting of the swash plates is controlled by the pilot moving the cyclic stick.

## 12-2 Fig 10 Producing Horizontal Movement



## Flapping to Equality

12. A cyclic pitch change does not markedly alter the magnitude of total rotor thrust but simply changes the disc attitude. This is achieved by the blades flapping to equality of rotor thrust. If a blade in a hover has an angle of attack, $\alpha$ (Fig 11a), a cyclic stick movement will decrease the blade pitch and, assuming that initially the direction of the RAF remains unchanged, the reduction in pitch will reduce both the blade's angle of attack ( $\alpha$ ) and rotor thrust (Fig 11b). The blade cannot maintain horizontal flight and will now begin to flap down, causing an automatic increase in the blade's angle of attack. When the angle returns to $\alpha$, the blade thrust will return to its original value and the blade will continue to follow the new path required to keep the angle of attack constant (Fig 11c). Thus, cyclic pitch will alter the plane in which the blade is rotating, but the angle of attack remains unchanged. The reverse takes place when a blade experiences an increase in cyclic pitch. It should be remembered that when a cyclic pitch change is made, the blades continuously flap to equality as they travel through $360^{\circ}$ of movement.

## 12-2 Fig 11 Flapping to Equality



## Control Orbit

13. In its simplest form of operation, movement of the cyclic stick causes a flat plate, or non-rotational swash plate, mounted centrally on the rotor shaft to tilt, the direction being controlled by the direction in which the cyclic stick is moved. Rods of equal length, known as pitch operating arms (POA) or pitch change rods connect the swash plate to the rotor blades. When the swash plate is tilted the pitch operating arms move up or down, increasing or decreasing the pitch on the blades (see Fig 12). The amount by which the pitch changes, and which blades are affected, depends on the amount and direction in which the swash plate is tilted. The swash plate can be more accurately described as a control orbit because it represents the plane in which the pitch operating arms are rotating.

## 12-2 Fig 12 Control Orbit



Pitch Operating Arm Movement
14. Now consider the effect of the movement of the POA when the control orbit has been tilted $2^{\circ}$ (assuming that the control orbit tilts in the same direction as the stick is being moved), see Fig 13a. A Plan view, Fig 13b, shows clearly the amount by which the control orbit has been tilted at four positions, $A, B, C$ and $D$.

## 12-2 Fig 13 Pitch Operating Arm Movement



If the movement of the POA through $360^{\circ}$ of travel is plotted on a simple graph, the result would be as shown in Fig 14.

## 12-2 Fig 14 Movement of Pitch Operating Arms Through $360^{\circ}$



The rate at which the POA is moving up and down is not uniform. This can be shown more clearly as a comparison is made between the control orbit in plan view and the control orbit in side elevation; and noting how much movement takes place in each $30^{\circ}$ of travel over a range of $90^{\circ}$ (see Fig 15).

## 12-2 Fig 15 Rate of Movement of Pitch Operating Arm


15. Resultant Change in Disc Attitude. In order to determine the resultant change in disc attitude, the movement of each blade is followed through four points $A, B, C$ and $D$ during $360^{\circ}$ of movement.

The control orbit has been tilted by the cyclic stick and hence the pitch operating arms move so that a maximum pitch of $+2^{\circ}$ is applied at point $B$; a minimum pitch, $-2^{\circ}$, at point $D$, and zero pitch at points $A$ and C (see Fig 16).

## 12-2 Fig 16 Relationship of Blade Position to Control Orbit Position



As the blade moves clockwise from A, it will experience an increase in pitch and the blade will begin to flap up. The rate of flapping will vary with the amount of pitch change so the blade will be experiencing its greatest rate of flapping as it passes $B$, the point of maximum pitch change. In its next $90^{\circ}$ of travel the pitch is returned from $+2^{\circ}$ to $0^{\circ}$ at point C and the rate at which the blade is flapping will slowly reduce to reach zero at point $C$. Flapping up, however, will have continued past $B$ and the blade will be at its highest point at C . The exact reverse will take place after C , resulting in the blade being at its lowest at point A. The disc will now be tilted along the axis B-D. This is $90^{\circ}$ out of phase with the maximum and minimum pitch positions, see Fig 17.

## 12-2 Fig 17 High and Low Blade Positions



## Phase Lag

16. When cyclic pitch is applied the blades will automatically flap to equality and, in so doing, the disc attitude will change, the blade reaching its highest and lowest positions $90^{\circ}$ later than the point where
it experiences the maximum increase and decrease of cyclic pitch. The variation between the tilt of the control orbit in producing this cyclic pitch change and subsequent tilt of the rotor is known as phase lag. Phase lag will also occur when the blades experience a cyclic variation resulting from a change in speed or direction of the RAF, as occurs in horizontal flight.

## Advance Angle

17. Phase lag, if uncorrected, would have the effect that movement of the cyclic stick would cause the rotor to tilt in a direction $90^{\circ}$ out of phase with the direction in which the cyclic stick is moved. Thus moving the cyclic stick forward would have the effect of moving the helicopter sideways. This undesirable feature is overcome by arranging for the blade to receive the maximum alteration in cyclic pitch change $90^{\circ}$ before the blade is over the highest and lowest points on the control orbit (see Fig 18). The angular distance that the POA is positioned on the control orbit in advance of the blade to which it relates is known as the advance angle.

## 12-2 Fig 18 Advance Angle



When the control orbit tilts to follow the stick, to compensate fully for phase lag, the advance angle would have to be $90^{\circ}$. If the control orbit is $45^{\circ}$ out of phase with stick movement, then the advance angle needs to be only $45^{\circ}$ to make full compensation for phase lag (see Fig 19).

## 12-2 Fig $1945^{\circ}$ Advance Angle



## Dragging

18. Dragging is the freedom given to each blade to allow it to move in the plane of rotation independently of the other blades. To avoid bending stresses at the root, the blade is allowed to lead or lag about a dragging hinge (see Fig 20), but rate of movement is restricted by some form of drag damper to avoid undesirable oscillations.

## 12-2 Fig 20 Dragging Hinge



Dragging is caused by:
a. Periodic Drag Changes. When the helicopter moves horizontally, the blade's angle of attack is continually changing during each complete revolution to provide symmetry of rotor thrust. The variation in angle of attack results in variation in rotor drag and consequently the blade will lead or lag about the dragging hinge.
b. Conservation of Angular Momentum. If a helicopter is stationary on the ground in still air conditions, rotor running, the radius of the blade's CG relative to the axis of rotation/shaft axis will be constant. If the cyclic stick is now moved the blades will flap to produce a change in disc attitude. The axis of rotation will no longer be coincident with the shaft axis and this results in a continual change of the CG radius relative to the shaft axis through $360^{\circ}$ of travel. The radius variation will cause the blades to speed up or slow down depending on whether the radius is reducing or increasing (see Fig 21).

## 12-2 Fig 21 Variation in Radius of Blade CG Resulting from Flapping


a

b
c. Hooke's Joint Effect. Hooke's joint effect is the movement of a blade to reposition itself relative to the other blades when cyclic stick is applied; its effect is very similar to the movement of the blades CG relative to the hub. If a rotor is hovering in still air (see Figs 22a and 22b), when viewed from above the shaft axis the blades $A, B, C$ and $D$ appear equally spaced relative to the shaft axis. When a cyclic tilt of the disc occurs (Figs 22c and 22d), the cone axis will have tilted but, if still viewed from the shaft axis, which has not tilted, blade A will appear to have increased its radius and blade $C$ decreased its radius. Blades $B$ and $D$ must maintain position as in Fig 22c in order to achieve their true positions on the cone. It follows therefore that they must move in the plane of rotation to position themselves, as in Fig 22d.

## 12-2 Fig 22 Hooke's Joint Effect



## CHAPTER 3 - CONTROL IN FORWARD FLIGHT (HELICOPTER)

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## Introduction

1. When torque is applied to the rotor shaft of a helicopter, there is an equal and opposite torque reaction applied to the helicopter by the rotor shaft. If the torque reaction is not balanced the helicopter fuselage will turn in the opposite direction to the rotor. In this chapter, torque reaction and the solution to it will be discussed. The forces in the hover and in forward flight, and transition from forward flight to the hover will also be discussed.

## Torque Reaction

2. The torque reaction on a single rotor helicopter is shown in Fig 1. A torque compensating force at the tail is the most common method of balancing torque reaction and the force is provided by a tail rotor, shrouded tail rotor (Fenestron), or blown air in the case of No-Tail Rotar (NOTAR) helicopters (Volume 12, Chapter 8).

## 12-3 Fig 1 Torque Reaction


3. The Tail Rotor. The tail rotor is mounted vertically at the rear of the fuselage and clear of the main rotor (see Fig 2). It is driven from the main gearbox by a tail rotor drive shaft and geared such that the shaft revolves at a very high speed compared to the main gearbox and the tail rotor. The reason for an increase in the rpm of the tail rotor drive shaft is to allow the construction of it to be flimsier because the torque, which is directly proportional to rpm, is reduced. It is also easier to balance the shaft if it rotates at high rpm.

12-3 Fig 2 Conventional Tail Rotor

4. The Shrouded Tail Rotor. The shrouded tail rotor, or Fenestron, is a high speed, variable pitch ducted fan mounted in a cambered fin. It has many features in common with a propeller but it has control characteristics similar to a tail rotor (see Fig 3 and Volume 12, Chapter 8).

## 12-3 Fig 3 Shrouded Tail Rotor (Fenestron)


5. Control Mechanism. When the moment of the tail rotor thrust equals the torque reaction couple, then the fuselage will maintain a constant direction. As the torque reaction is not constant some means must be provided to vary the thrust from the tail rotor. This is achieved by the pilot moving yaw pedals which collectively change the pitch and, therefore, the thrust from the tail rotor.
6. Additional Tail Rotor Functions. The tail rotor has the following additional functions:
a. Heading control in the hover is achieved by increasing or decreasing tail rotor thrust so that torque reaction is not balanced and the helicopter is able to turn about the rotor shaft.
b. Balance in forward flight is adjusted by tail rotor thrust in a similar fashion to the rudder control of an aeroplane.
c. In power-off flight (autorotation), there is no torque reaction. The rotor is turning and there is friction in the transmission which tends to turn the helicopter in the same direction as the rotor. The turn is prevented by negative pitch on the tail rotor which produces thrust opposite to that in powered flight.

## Tail Rotor Compensation

7. Tail Rotor Drift. If a fuselage is being turned by a couple $Y Y_{1}$, about a point, the rotation will stop if a couple $Z Z_{1}$, of equal value, acts in the opposite direction (see Fig 4a). The rotation would also stop if a single force $Z Z_{1}$ was used to produce a moment equal to the couple $\mathrm{YY}_{1}$ (Fig 4b), but there would now be a side force $X$ on the pivot point (Fig 4c). This side force is known as tail rotor drift and, unless corrected, it would result in the helicopter moving sideways over the ground.

## 12-3 Fig 4 Tail Rotor Drift


8. Correcting For Tail Rotor Drift. Tail rotor drift can be corrected by tilting the rotor disc away from the direction of the drift. This can be achieved by:
a. The pilot making a movement of the cyclic stick.
b. Rigging the controls so that when the stick is in the centre the disc is actually tilted by the correct amount.
c. By mounting the gearbox so that the drive shaft to the rotor is offset.
9. Tail Rotor Roll. If the tail rotor is mounted on the fuselage below the level of the main rotor the tail rotor drift corrective force being produced by the main rotor will create a rolling couple with the tail rotor thrust, causing the helicopter to hover one wheel or skid low. The amount of roll depends upon the value and angle of the tail rotor thrust and the vertical separation between main and tail rotors. In the hover, the helicopter will roll about the horizontal couple until the movement is balanced by the couple of the vertical component of total rotor thrust and the helicopter all up weight (see Fig 5).

## 12-3 Fig 5 Tail Rotor Roll



A helicopter is usually designed so that the tail rotor is in line with the main rotor head at forward speed. In the hover, tail rotor roll is accepted.

## Rotor Configurations

10. It is possible to counteract torque reaction by using twin main rotors which may be mounted coaxially and revolve in opposite directions, or in a fore and aft configuration, or even side-by-side. In all cases synchronization of the rotors is vital for the maintenance of directional control.

## Forces in the Hover and in Forward Flight

11. Forces in Balance - Hover. In a free air hover the total rotor thrust will be acting vertically upwards through the axis about which the blades are rotating and at right angles to the tip path plane. Weight will be acting vertically downwards through the CG, Fig 6a. If the helicopter is loaded to position the CG immediately below the blades' axis of rotation, and discounting downdraft on any horizontal surfaces, no change in fuselage attitude will occur when the helicopter leaves the ground, Fig 6b. If, however, the CG is not below the axis of rotation, Fig 6c, a couple will exist between total rotor thrust and the weight, and the fuselage will pitch until both forces are in line, Fig 6d. It should be noted that a helicopter in the hover often adopts a nose-up attitude in any case, irrespective of the position of the CG. This happens because downwash from the main rotor exerts a force on the tail stabilizer causing a tail-down moment. In still air conditions the nose-up attitude is quite marked but as wind speed increases the vertical component of rotor downwash is reduced and the helicopter adopts a more level attitude. Hovering attitude is also affected by flapback which is discussed in para 28.

## 12-3 Fig 6 Forces in Balance - Hover

Fig 6a


Fig 6b


Fig 6c


Fig 6d

12. Forces in Balance - Forward Flight. If a helicopter moves from a free air hover into forward flight with no change in the fuselage attitude, the rotor disc will be tilted forward and the disposition of forces will be as shown in Fig 7a. Total rotor thrust is now inclined forward and produces a nosedown pitching moment about the CG. The vertical components of TRT and AUW remain in line but a couple now exists between the horizontal component of TRT and fuselage parasite drag as the aircraft speed increases. The fuselage will pitch forward but the moment will now be opposed by the vertical component of TRT and Wt with the forces resolved as in Fig 7b. The fuselage will only pitch forward until the couples are in balance. This will occur when TRT is in line with the CG. CG therefore controls the position of the fuselage in relation to the disc. This relationship is affected in forward flight by the negative lift effect of the tail stabilizer and the moment exerted by it.

## 12-3 Fig 7 Forces in Balance - Forward Flight

Fig 7a Level Attitude With Pitching Moment


Fig 7b Forces in Balance


## Transition

13. To achieve forward flight the rotor disc is tilted so that TRT produces not only a vertical force to balance the weight but also a horizontal force in the direction of flight. The change of state from the hover to forward flight, or from forward flight to the hover, is known as transition.
14. The Sequence of Events During Transition. As the helicopter moves initially from the hover the disc, and hence TRT, is tilted. The vertical component of TRT is reduced and becomes less than Wt and to prevent the helicopter from descending TRT is increased with more collective pitch. The power required increases (see Fig 8). As the aircraft accelerates the fuselage acts pendulously below the main rotor and pitches nose down (Fig 9).

## 12-3 Fig 8 Forces During Transition



## 12-3 Fig 9 Fuselage Pendularity



## Translational Lift

15. When a helicopter is in a free air hover in still air conditions, for a given rotor rpm (Rrpm) a certain value of collective pitch, say $8^{\circ}$, will be required to support it in the air. A column of air, the induced flow, will be continually moving down towards the rotor disc, and thus downward flow of air must be considered when determining the direction of the airflow in relation to the blades (see Fig 10). It will be noted that the angle of attack, say $4^{\circ}$, is less than the pitch angle. The angle of attack depends on the value of the induced flow; if the induced flow is removed, the angle of attack becomes the same as the pitch angle.

## 12-3 Fig 10 Induced Flow from Vertically Above


16. If the effect of a helicopter facing into a 20 kt wind is considered, and it is assumed that it is possible for it to maintain the hover without tilting the disc, the horizontal flow of air (wind) will blow across the vertically induced column of air and deflect it downwind before it reaches the disc. The column of air, which was flowing down towards the disc, will, therefore, be modified and gradually be replaced by a mass of air which is moving horizontally across the disc. The rotor will act on this air mass to produce an induced flow but the velocity of the induced flow will be greatly reduced (see Fig 11). Therefore, an airflow parallel to the disc must reduce the value of the induced flow, increase the angle of attack and, therefore, rotor thrust.

## 12-3 Fig 11 Induced Flow with Air Moving Horizontally


17. To maintain the hover condition when facing into wind, the disc must be tilted forward. The horizontal flow of air will not now be parallel to the disc, and a component of it can now be considered to be actually passing through the disc at right angles to the plane of rotation, effectively increasing the induced flow (see Fig 12). To consider the extreme case if the rotor disc were tilted $90^{\circ}$ to this horizontal flow of air, then all of it would be passing through the disc at right-angles to the plane of rotation.

12-3 Fig 12 Induced Flow with Disc Tilted Forward

18. As described in para 16, the effect of the horizontal airflow across the disc when hovering into wind is to reduce the induced flow but, because the disc has had to be tilted forward, (para 17) a component of this horizontal airflow will now be passing through the disc, effectively increasing the induced flow; both of these effects must now be taken into consideration to give the total flow towards the disc and to determine the direction of the airflow relative to the blades. Provided the reduction in the induced flow caused by the flow parallel to the disc is greater than the increase caused by the component of horizontal airflow passing through the disc, then the relative airflow will be nearer the plane of rotation than when the helicopter is in the hover, the angle of attack will increase and the aircraft will climb. Therefore, the collective pitch can be decreased to say, $7^{\circ}$, while maintaining an angle of attack of $5^{\circ}$. As the relative airflow moves nearer the plane of rotation, the total reaction must move forward. There will, therefore, be less rotor drag, and rotor rpm can be maintained with less power.
19. The reduction in induced flow, translational lift, first takes effect when air moves towards the disc at approximately 12 kt . The reduction is appreciable at first, and although it continues to reduce as the velocity of the horizontal airflow increases, the rate at which it reduces becomes progressively less because there is less induced flow to be influenced. If induced flow is plotted against forward speed, the graph appears as shown in Fig 13a.

## 12-3 Fig 13 Variation of Induced Flow and Component of Horizontal Airflow passing through the Disc with Forward Speed

Fig 13a


Fig 13b

20. The rotor disc has to be tilted forward to provide a thrust component equal to parasite drag. Parasite drag is low at low forward speed so only a small tilt of the disc is required to provide a balancing amount of thrust and, with only a small tilt of the disc, only a small component of the horizontal airflow will be passing through the disc at right angles to the plane of rotation. Because the parasite drag increases as the square of the speed, the greater must be the amount that the disc must be tilted to provide the necessary increase in thrust and, as the horizontal airflow approaching the disc increases, the greater
will be the component of it passing through the disc at right angles to the plane of rotation (see Fig 13b). If the curves in Fig 13a and Fig 13b are now transferred to one graph it will be seen that the total flow of air at right angles to the plane of rotation at first decreases and then increases again, becoming a minimum when the two airflows have the same value (see Fig 14). As the flow of air through the disc decreases, less collective pitch and power will be required to maintain the required angle of attack. When the flow of air through the disc begins to increase again, then collective pitch and power must be increased if the required angle of attack is to be maintained.

## 12-3 Fig 14 Variation of Total Airflow Through the Disc with Forward Flight



## Summary of Transition

21. The sequence of events as a helicopter moves into forward flight is summarized as follows:
a. The cyclic stick is moved forward to tilt the disc and the TRT forward.
b. The vertical component of TRT is reduced and the collective pitch must be increased to maintain height. More power is required.
c. As airspeed increases the disc flaps back. The disc attitude is maintained by increasing forward cyclic control.
d. As airspeed increases inflow roll tilts the disc to the advancing side. The disc attitude is maintained by cyclic control to the retreating side.
e. As airspeed increases the TRT increases with increased translational lift and the pilot lowers the collective to maintain height. Less power is required.
f. During power changes the changing torque reaction is balanced by movement of the yaw pedals.

## Transition From Forward Flight to Hover

22. In order to decelerate a helicopter from steady level flight to the hover the balance of forces must be changed. The general method of coming to the hover from forward flight is by the pilot executing a flare by tilting the disc in the opposite direction to that in which the helicopter is moving. The handling techniques needed to control the manoeuvre differ from those required for a more gentle transition.
23. The Flare. To execute a flare the cyclic stick is moved in the opposite direction to that in which the helicopter is moving. The harshness of the flare depends upon how far the stick is moved. The flare will produce a number of effects.
24. Flare Effects. The following effects occur during the flare:
a. Thrust Reversal. By tilting the disc away from the direction of flight, the horizontal component of total rotor thrust will now act in the same direction as parasite drag causing the helicopter to slow down very rapidly (see Fig 15a). The fuselage will respond to this rapid deceleration by pitching up because reverse thrust is being maintained whilst parasite drag decreases. If no corrective action is taken the disc will be tilted further still, adding to the deceleration effect (Fig 15b).

## 12-3 Fig 15 The Flare Effect

Fig 15a


Fig 15b

b. Increase in Total Rotor Thrust. Another effect of tilting the disc back whilst the helicopter is moving forward is to change the airflow relative to the disc, Fig 16. As was explained in para 19, translational lift, a component of the horizontal airflow (due to the forward movement of the helicopter) is passing through the disc at right angles to the plane of rotation, opposing the induced flow. The result is an increase in total rotor thrust. To prevent a climb the collective lever must be lowered.

## 12-3 Fig 16 Change in Relative Airflow

Fig 16a

c. Increase in Rotor RPM. Unless power is reduced when collective pitch is reduced, the Rrpm will rise. They will also increase rapidly in the flare for two other reasons, conservation of angular momentum and reduction in rotor drag.
(1) Conservation of Angular Momentum. An increase in total rotor thrust causes the blades to cone up. The radius of the blades' CG from the shaft axis decreases and the rotational velocity will automatically rise.
(2) Reduction in Rotor Drag. Rotor drag is reduced in the flare because the total reaction moves towards the axis of rotation. This results from the changed direction of the relative airflow. The forward movement of the total reaction vector causes the rotor drag component to be reduced, Fig 16b.

As a result of the flare the speed reduces rapidly and the flare effects disappear. Collective pitch and power which had been reduced during the flare must be replaced and, in addition, more collective pitch and power must be used to replace the loss of translational lift caused by the speed reduction, otherwise the aircraft would sink. The cyclic stick must also be moved forward to level the aircraft and to prevent the helicopter moving backwards. The power changes necessary during the flare have an effect on the aircraft in the yawing plane. Therefore, yaw pedals must be used to maintain heading throughout.

## Landing

25. If collective pitch is reduced slightly in a hover IGE, the helicopter will descend but settle at a height where ground effect has increased total rotor thrust to again equal all up weight. Therefore a progressive lowering of the collective lever is required to achieve a steady descent to touchdown. When the helicopter is close to the ground the tip vortices are larger and unstable causing variation in the thrust around the rotor disc and turbulence around the tail and makes control difficult. For this reason, and to help to prevent ground resonance, the helicopter is normally landed firmly to decrease the chance of drifting when touching down.

## Symmetry and Dissymmetry of Rotor Thrust

26. Symmetry of Rotor Thrust. If a helicopter is stationary on the ground in still air conditions, rotor turning and some collective pitch applied, then the rotor thrust produced by each blade will be uniform. The speed of the relative airflow over each blade will be equal to the speed of rotation of the blade, and if a given section on each blade of a four-bladed rotor is considered, the vector showing the relative airflow will have the same value irrespective of the position of the blade during its $360^{\circ}$ of travel, see Fig 17. As the velocity of this airflow is equal to the blade's speed of rotation, this airflow will be referred to as VR.

12-3 Fig 17 Relative Airflow - Still Air

27. Dissymmetry of Rotor Thrust. If the conditions change and the helicopter now faces into a wind, during the blade's rotation through $360^{\circ}$ half the time it will be moving into wind and the remainder of the time it will be moving with the wind. The disc can therefore be divided in half, one half being the advancing side and the other the retreating side, see Fig 18.

## 12-3 Fig 18 Relative Airflow - Wind Conditions



When the blade is at right angles facing into wind (position $B$ ), the velocity of the relative airflow will be a maximum and if the value of the wind speed is referred to as $V_{w}$, then at position $B$ the velocity of the relative airflow will be $V_{R}+V_{W}$. As the blade continues to rotate, the effect of $V_{W}$ will decrease and when the blade reaches position $D$ the velocity of the relative airflow will have become $V_{R}-V_{w}$. If no change has taken place in the blade's plane of rotation, the rotor thrust being produced by the advancing blade at position $B$ will be greatest and, for the retreating blade at position $D$, least. The
value of rotor thrust across the disc will no longer be uniform and unless some method is employed to provide equality, the helicopter will roll towards the retreating side. This condition, where one side of the disc produces more rotor thrust than the other, is known as dissymmetry of rotor thrust.

## Flapback

28. To maintain control of the helicopter dissymmetry must be prevented; one method of doing this is to decrease the angle of attack of the advancing blade and increase the angle of attack of the retreating blade so that each blade again produces the same value of rotor thrust. With the fully articulated rotor head this change in angle of attack takes place automatically by flapping but, as a result, the disc attitude changes. The manner in which it changes and the reason why this change in attitude prevents dissymmetry can be seen by following the movement of a blade through $360^{\circ}$ of travel.
29. Starting at position A of Fig 19, the blade starts to travel on the advancing side and the relative airflow will increase. Rotor thrust begins to increase and, because it is free to do so, the blade will begin to flap up about the flapping hinge. As the blade flaps up the angle of attack will begin to decrease, rotor thrust decreases and the blade will proceed to follow a path to maintain the same value of rotor thrust as it was producing before it began to flap up. The blade, in fact, is flapping to equality. The further round that the blade progresses on the advancing side, the greater will be the velocity of the relative airflow; therefore, to maintain a constant value of rotor thrust, the rate at which the blade is flapping will steadily increase, with the maximum rate of flapping and, therefore, minimum angle of attack occurring when the blade reaches position $B$. For the next $90^{\circ}$ of travel the velocity of the relative airflow begins to decrease, so the rate of flapping will decrease. When the blade reaches position C , the relative airflow will have the same value as at position $A$, so the rate of flapping dies out completely but, because the blade has been rising all the time from position $A$, the blade will reach its highest position at $C$. The reverse will take place on the retreating side, with the blade having its maximum rate of flapping down and, therefore, its maximum angle of attack at position $D$, reaching its lowest position at $A$. In flapping to equality, the blade will have flapped away from the wind. This change of disc attitude, which has occurred without any control movement by the pilot, is known as flapback.

## 12-3 Fig 19 Disc Tilt Resulting from Blades Flapping to Equality


30. Fig 20a and Fig 20b show that when the helicopter is on the ground and the disc is subject to wind, the disc attitude is altered, although no cyclic stick has been applied. The disc has flapped back
relative to the wind and to the control orbit, and the blades are moving about their flapping hinges. However, the rotor thrust being produced will be the same value as before the disc flapped back, but tilted in direction.

## 12-3 Fig 20 Relationship between Disc, Control Orbit and Stick resulting from Flapback


31. If the pilot now moves the stick forward to return the disc to its original position (Fig 20c) it will be seen that the disc is now flapped back only in relation to the control orbit and not to the wind, and that movement is no longer taking place about the flapping hinges. Thus flapback has been counteracted by cyclic feathering, and, since the cyclic stick only changes the disc attitude, the value of the rotor thrust force remains unchanged. When the helicopter is airborne and moving in any horizontal direction, the effect will be the same as has been described for a helicopter on the ground facing into wind, with flapback being prevented by cyclic feathering. The first movement of the cyclic stick will tilt the disc to initiate horizontal flight, then a second movement will be necessary to prevent the disc from flapping back when the aircraft moves and gains speed. It should be noted however that some movement about the flapping hinges will still take place if the CG of the helicopter is not in the ideal position.

## Inflow Roll

32. The effect of moving air horizontally across the disc causes a reduction in the induced flow. However, this reduction is not uniform because air passing across the top of the disc is being continually pulled down by the action of the rotors. Thus air which is moving horizontally towards the disc will cause the greater reduction in induced flow at the front of the disc, and the smallest reduction at the rear of the disc (see Fig 21).

12-3 Fig 21 Relative Airflow at Front and Rear of Disc


The reduction in induced flow for the disc as a whole will produce an increase in rotor thrust but because the increase in the angle of attack is not uniform, it will also produce a change in the attitude of the disc. Assuming the flapback has been corrected (see Fig 22), the effect of a cyclic variation in angle of attack for a blade starting at position B (Fig 22b) must be considered.

## 12-3 Fig 22 Combined Effect of Flapback and Inflow Roll

Fig 22a

Fig 22b
Fig 22c Flight Direction


Flight Direction


As the blade moves towards position $C$, the increased angle of attack will cause the blade to flap to equality. The rate of flapping up will be a maximum as the blade passes position $C$ because this is the point where there has been the greater reduction in induced flow. In the next $90^{\circ}$ of travel the rate of flapping will slow down, dying out completely when the blade is at position D . Thus the blade will be rising all the time it is travelling from position $B$ to reach its highest position at $D$. The reverse will take place for the next $180^{\circ}$ of travel, with the blade having its maximum rate of flapping down at $A$ and its lowest position at $B$. As a result of the inflow the disc will, therefore, tilt about axis AC towards the advancing side. The combined effect of inflow roll and flapback is, therefore, to tilt the disc about axis $\mathrm{ZZ}_{1}$, Fig 22c. As inflow roll will have its greatest effect at low speed, and flapback its greatest effect at high speed, the axis about which the disc will tilt will vary with forward speed. In general, the cyclic stick has to be positioned forward towards the retreating side to correct these effects in forward flight.

## Factors Affecting Maximum Forward Speed

33. There are several factors which must be taken into account when trying to increase the maximum speed of a helicopter.
34. Cyclic Control Limits. To achieve forward flight the cyclic stick is moved to tilt the disc forward, the disc tilting by the same amount that the control orbit has been moved. As the airspeed increases the rotor disc flaps back relative to the cyclic control position and the attitude of the disc is maintained by moving the cyclic stick forward. There will be a speed at which the cyclic stick is fully forward and no further acceleration is possible. The amount of forward cyclic control is reduced if the helicopter's centre of gravity is aft.
35. Power Available. In level flight $\mathrm{V}_{\text {MAX }}$ is limited by power available (Volume 12 Chapter 6). $A$ higher speed may be possible in a descent.
36. Structural Strength. As speed increases both the forces on the rotor and transmission and the levels of vibration increase. Apart from the limitation of the strength of the airframe and other components against these forces, the combination of stress and vibration causes fatigue. It is impractical to make components so strong that they do not suffer fatigue and, therefore, the level of vibration must be kept below that at which the failure of components may occur. This will set a limit to the maximum speed.
37. Airflow Reversal. The speed of rotation of the retreating blade is high at the tip and low at the root, but the airflow from forward flight will have an equal value for the whole length of the blade and, where the airflow from forward flight is greater than the blade's rotational velocity, eg at the root end, the airflow will be from the trailing edge to the leading edge, causing a loss of rotor thrust. At higher airspeeds the airflow is reversed over a progressively large section of the blade leading to a greater loss of thrust (see Fig 23). The reduction of rotor thrust on the retreating blade by airflow reversal is countered by greater cyclic control and hence the retreating blade operates at an increasingly higher pitch angle and hence angle of attack.

12-3 Fig 23 Airflow Reversal

38. Retreating Blade Stall. As the angle of attack of the retreating blade is steadily increased with increasing forward airspeed there will be a speed at which the airflow breaks away and the blade stalls. The large sudden loss of rotor thrust will cause the blade to flap down, but instead of flapping
to equality the effect will be simply to stall the blade even further. The stall starts at the tip first and spreads inboard as shown in Fig 24.

12-3 Fig 24 Retreating Blade Stall


The reason why the stall commences at the tip is shown in Fig 25. The variation in angle of attack along the blade will be offset to some extent by washout but in all conditions of forward flight the highest angle of attack will be at the tip.

## 12-3 Fig 25 Stall at Tip Before Root


39. Characteristics of Retreating Blade Stall. The approach of the retreating blade stall can be detected by:
a. Rotor roughness.
b. Erratic stick forces.
c. Stick shake.

If these conditions are ignored, a pitch up tendency will develop followed by a roll to the retreating side. There will be a substantial loss of control and if the stall is severe, control may be lost completely.
40. Causes of Retreating Blade Stall. Retreating blade stall can occur as a result of:
a. High forward speed.
b. High G manoeuvres.
c. Rough, abrupt or excessive control movements.
d. Flying in turbulent air.

A high all up weight/high density altitude will also aggravate the situation. Recovery action will depend upon which of the above in-flight conditions are prevailing when the stall symptoms are recognized. Recovery will normally be made by reducing forward speed, reducing collective pitch, reducing the severity of the manoeuvre or by a combination of these recovery actions.
41. Compressibility. As an example, the speed of rotation of the tip of a Gazelle rotor blade is approximately 400 kt . In forward flight at 150 kt , the advancing blade tip has a relative velocity of 550 kt . The velocity of sound at sea level is 660 kt . Compressibility is therefore significant. The main effects of compressibility are:
a. A reduction in the lift/drag ratio, requiring more power for the same total rotor thrust.
b. An increase in the pitching moment on the aerofoil which is normally very small. This requires greater control forces and leads to vibration.
c. The production of shock waves which increase vibration and noise. The effects can be reduced by using a high speed aerofoil section or sweep back at the blade tips. Any such solutions have penalties at low speeds.

## CHAPTER 4 - AUTOROTATION (HELICOPTER)

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## AUTOROTATION IN STILL AIR

## Introduction

1. In powered flight, the rotor drag is overcome with engine power but, when the engine fails or is deliberately disengaged from the rotor system, some other force must be used to maintain the rotor rpm. This is achieved by allowing the helicopter to descend and by lowering the collective lever fully so that the resultant airflow strikes the blades in such a manner that the airflow itself provides the driving force. When the helicopter is descending in this manner, the rate of descent becomes the power equivalent and the helicopter is said to be in a state of autorotation.
2. Although most autorotations are carried out with forward speed, the explanation as to why the blades continue to turn when in rotation can best be seen if it is considered that the helicopter is autorotating vertically downwards in still air. Under these conditions, if the various forces involved are calculated for one blade, the calculations will be valid for all the other blades irrespective of where the blade is positioned in its $360^{\circ}$ of travel. The various airflows and angles which will be referred to are shown in Fig 1

## 12-4 Fig 1 Autorotation - Terms Used


3. It will be noted that the inflow has been determined from the blades' rotational velocity and the airflow arising from rate of descent. This is not strictly true as the action of the blades slows down the rate of descent airflow, producing, in effect, an induced flow, making the inflow angle smaller than has been shown in Fig 1; the fact that it is smaller and how this affects the blade is considered later.

## Autorotative Force/Rotor Drag

4. Consider three sections, $A, B$ and $C$, of a rotor blade (Fig 2). The direction of the relative airflow for each section can be determined from the rotational velocity and the helicopter's rate of descent. The rate of descent will have a common value for each section but the rotational velocity will decrease from the tip towards the root.

## 12-4 Fig 2 Distribution of Rotational Forces



Comparing sections $\mathrm{A}, \mathrm{B}$ and C , the inflow angle must therefore be progressively increasing (Fig 3). Because of the wash-out incorporated in the blade, the pitch angle is also increasing and as the blade's angle of attack is the pitch angle plus the inflow angle, the blade's maximum angle of attack will be at the root.

## 12-4 Fig 3 Autorotation - Position of Total Reaction and Autorotative Forces

Fig 3a Position of Total Reaction


Fig 3b Autorotative Force

5. If the angle of attack for each section of the blade is known, the lift/drag ratio for these angles of attack can be ascertained by referring to the aerofoil data tables, and, by adding lift and drag vectors in the correct ratio, the position of the total reaction can be determined (Fig 3). Relating total reaction position to the axis of rotation (see Fig 3b) at section $A$, the total reaction lies behind the axis; at section $B$ it is on the axis and at section $C$ it is in front of the axis. Having determined the position of the total reaction, it can now be considered in terms of rotor thrust and rotor drag (Fig 3b). At section A, the condition is the same as in powered flight and the component of total reaction in the plane of rotation opposes rotation and is continually trying to decelerate the blade. At section B no part of the total reaction is acting in the plane of rotation and it is all rotor thrust; at section $C$ the component of total reaction in the plane of rotation assists rotation and is continually trying to accelerate the blade. Under these conditions it is no longer referred to as rotor drag, but as the autorotative force.
6. Considering the blade as a whole, the section producing an autorotative force will be accelerating the blade, whilst the section producing rotor drag will endeavour to slow it down. To maintain a constant rotor rpm, the autorotative section must be sufficient to balance the rotor drag section of the blade, plus the drag set up by the ancillary equipment, tail rotor shaft and tail rotor, all of which continue to function in autorotation.
7. In normal conditions with the lever lowered, the blade geometry is such that the autorotative rpm are in the correct operating range, provided an adequate rate of descent exists. If the lever is raised during autorotation the pitch angles increase on all sections (Figs 2 and 3). Section B will tend towards section $A$ and section $C$ will tend towards $B$, thus the autorotative section moves outwards. However, section $D$ at the root becomes stalled and the extra drag generated causes a decrease in the size of the autorotative section and therefore rpm decreases, stabilizing at a lower figure. This continues with further raising of the lever until such time as the blade is no longer able to autorotate.
8. Autorotative descent from high altitudes or at a high all-up weight leads to high rates of descent. Inflow angles will be higher and autorotative sections will be further outboard on the blades; rpm will be higher in autorotation under these conditions. It should be noted, however, that descent into more dense air decreases rate of descent and rpm for a constant lever position.

## Rate of Descent

9. If the engine fails during a hover in still air and the collective pitch is reduced, the helicopter will accelerate downwards until such time as the angle of attack is producing a total reaction to give an autorotative force to maintain the required rotor rpm and a rotor thrust equal to the weight. When this condition has been established, the acceleration will stop and the helicopter will continue downwards at a steady rate of descent. If some outside influence causes the angle of attack to increase, there will be an automatic reduction in the rate of descent, the reverse taking place if the angle of attack is decreased.
10. Compared with a vertical autorotation in still air, the rate of descent will initially decrease with forward speed, but beyond a certain speed the rate of descent will start to increase again. The cause of this variation of rate of descent with forward speed is the changing direction of the relative airflow which occurs throughout the speed range in autorotation.

## Relative Airflow - Vertical Autorotation

11. Consider a helicopter of a given weight requiring a mean angle of attack of $8^{\circ}$ to provide the required rotor thrust and autorotative forces to maintain it in a vertical autorotation, and assume that this angle of attack is obtained when the rate of descent is $2,000 \mathrm{fpm}$. If the inflow angle is determined from rate of descent and a mean rotational velocity, it will be found to have a value of, say $10^{\circ}$ (Fig 4a) but because the action of the blades slows down the airflow coming from below the disc, the actual inflow angle will be less, say, only $6^{\circ}$ (Fig 4b). If the mean pitch value of the blade is $2^{\circ}$, then the angle of attack will be $8^{\circ}$, which is the angle required. So $2,000 \mathrm{fpm}$ rate of descent is required by this particular helicopter to produce an inflow angle of $6^{\circ}$.

## 12-4 Fig 4 Inflow Angle and Rate of Descent Relationship

Fig 4a


Fig 4b


## AUTOROTATION WITH FORWARD SPEED

## Relative Airflow - Forward Autorotation

12. In determining the direction of the relative airflow when the helicopter is in a forward autorotation, three factors must be taken into account. The effect of these factors on the inflow angle will first be considered individually and then collectively.

## 13. Individual Effect.

a. Factor A. To achieve forward autorotation the disc must be tilted forward. If the effective airflow from rate of descent (Fig 4) remains unchanged then the inflow angle must decrease (Fig 5). The angle of attack and therefore the rotor thrust must also decrease, causing an increased rate of descent.

## 12-4 Fig 5 Inflow Angle - Disc Tilted Forward


b. Factor B. When the helicopter is moving forward, the disc will be subjected to not only the descent airflow, but also to a horizontal airflow. Because the disc is tilted to this horizontal airflow, it will further reduce the inflow angle (Fig 6). The angle of attack is further decreased therefore, causing an increased rate of descent.

## 12-4 Fig 6 Inflow Angle - Effect of Horizontal Airflow


c. Factor $\mathbf{C}$. When the helicopter moves forward, the disc is moving into air which has not been slowed down by the action of the blades to the same extent as it is when the helicopter is descending vertically, therefore the effective rate of descent airflow will increase, which will result in the inflow angle increasing (Fig 7). The angle of attack and rotor thrust increases, giving a decreased rate of descent.

## 12-4 Fig 7 Inflow Angle - Effect of Forward Speed


14. Combined Effect. At low forward speed only a small tilt of the disc is required and the effect of factor $C$ will be greater than the combined effects of factors $A$ and $B$, so the inflow angle will increase. Angle of attack, and therefore rotor thrust, will increase and the rate of descent will decrease. As the rate of descent reduces, the inflow angle will decrease and the rate of descent will stabilize again when the angle of attack is such that the value of rotor thrust equals the weight. As forward speed is progressively increased, the effect of factor $C$ will continue to increase the inflow angle, but, similar to the induced flow in powered level flight, its effect is large initially but diminishes with increasing forward speed. Since the disc has to be tilted more and more to overcome the rising parasite drag from the fuselage, the combined effects of factors $A$ and $B$ rapidly increase with forward speed. Therefore, a forward speed is eventually reached where the combined effects of factors $A$ and $B$ equal C and balance out. When this occurs the helicopter will be flying at the speed to give minimum rate of descent. Beyond this speed the effects of factors $A$ and $B$ will be greater than factor $C$, inflow angle will therefore reduce and the required rotor thrust can only be obtained from a higher rate of descent.

## Rate of Descent Requirements in Autorotation

15. In autorotation, a rate of descent will be required to:
a. Produce a rotor thrust equal to the weight.
b. Provide an autorotative force for the selected rotor rpm.
c. Produce a thrust component equal to parasite drag.

If these three components are plotted against forward speed, the graph would be similar to the one showing the power requirements for level flight (Fig 8).


## Autorotation for Endurance and Range in Still Air

16. Autorotating to give the maximum time in the air must be at the speed to give the minimum rate of descent. The speed for endurance will therefore correspond to the lowest part on the rate of descent curve (Fig 9). Maximum range will be achieved when the helicopter is descending along its shallowest flight path. This will be achieved when flying at the best forward speed/rate of descent ratio. Relating this to the rate of descent curve, the optimum ratio will be at the speed where a line drawn from the point of origin of the graph is tangential to the rate of descent curve. For both range and endurance, rotor rpm should be as quoted in the Aircrew Manual.

## 12-4 Fig 9 Range and Endurance



Flare
17. The flare effect in autorotation will be exactly the same as for a flare in powered flight. Rotor rpm will rise because the increased inflow angle will cause the autorotative section to move further out towards the tip, and increased rotor thrust will reduce the rate of descent while flare effects last.

## Avoid Area for Autorotation

18. To establish fully developed autorotation, following power failure, it is vital to lower the lever immediately, probably fully, depending on forward speed and how quickly the lever is lowered after power loss is detected. At low forward speed it may also be necessary to gain forward speed. Lowering the lever and gaining forward speed will require considerable height loss before full autorotation is established at a safe speed to execute an engine-off landing. If power failure occurs above optimum autorotation speed, flare may be used to recover $N_{r}$ and reduce height loss as autorotation is established. At high airspeed and low level there may be insufficient time to reduce speed for a safe landing, despite the use of the lever and flare to maintain $N_{r}$ and reduce height loss as autorotation is established. Avoid areas, determined by test flying, are published in the relevant aircraft Aircrew Manual; Fig 10 shows an example. Power failure when operating inside the avoid areas may result in an unsuccessful engine off landing as the aircraft may be too low and too slow, or too low and too fast, to establish full autorotation at a safe speed for landing. Operation within the avoid areas should be kept to a minimum. The relevant Aircrew manual should be consulted for specific techniques following power failure.

## 12-4 Fig 10 Typical Autorotation Avoid Areas


19. Autorotative Landing. When engine failure occurs at height, the aircraft has potential energy to dissipate and this is converted into kinetic energy during the descent process in autorotation. When near the ground, the kinetic energy stored in the rotor by virtue of its rpm is converted into work, in the form of a large increase in rotor thrust, by use of the collective lever, with a consequent rapid decay in Rrpm as the kinetic energy is used.

## CHAPTER 5 - HAZARDOUS CONDITIONS AND RECOVERY ACTION

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## Ground Resonance

1. Ground resonance can be defined as being a vibration of large amplitude resulting from a forced or self-induced vibration of a helicopter in contact with, or resting on, the ground. The pilot will recognize ground resonance from a rocking motion or oscillation of the fuselage. If early corrective action is not taken, the amplitude of the oscillation can increase to the point where it will be uncontrollable and the helicopter will roll over. The speed of onset of ground resonance can be very fast and result in extremely violent oscillations. Experience has shown that, in extreme cases, there may be less than 2 seconds from onset to rolling over. It is, therefore, important that pilots are aware of the circumstances when ground resonance can occur, and the need to initiate appropriate recovery action immediately.
2. Causes of Ground Resonance. The initial vibration which causes ground resonance can already be present in the rotor head before the helicopter comes into contact with the ground. Ideally
the disc should have its centre of gravity (CG) over the centre of rotation. However, if for any reason its CG is displaced, a 'wobble' will develop; the effect being similar to an unbalanced flywheel rotating at high speed. Ground resonance can also be induced by the undercarriage being in light contact with the ground, particularly if the frequency of the oscillation of the oleos and/or tyres is in sympathy with the rotor head vibration.
a. Rotor Head Vibration. Rotor head vibration can be caused by:
(1) Blades of Unequal Weight or Balance. Blades should be correctly weighed and balanced during manufacture, but blade damage or flight in icing conditions, which can cause imbalance due to the uneven shedding of ice on the rotor blades, can change the balance. Moisture absorption can also be a cause of imbalance.
(2) Faulty Drag Dampers. With a multi-bladed system the blades should be equally spaced and drag dampers are fitted and adjusted to ensure this. If one damper is faulty allowing that blade to assume a dragged position different to the others, the CG of the entire rotor will be displaced from the axis of rotation (see Fig 1).

## 12-5 Fig 1 Effect of Faulty Drag Dampers


(3) Faulty Tracking. Rotor blades are 'tracked' to ensure that the tip path planes of all blades coincide. This is done by making adjustments to the basic pitch settings of the blades. If one blade has excessive basic pitch, its tip path plane will be higher than the others. More important, it will have a higher rotor drag and will, therefore, maintain an excessive dragged position, causing an out-of-balance condition, and a roughness or vibration apparent to the pilot (see Fig 2).

## 12-5 Fig 2 Faulty Tracking


b. Fuselage Vibration. Fuselage vibration can be caused by:
(1) Mishandling of the cyclic stick during landing which causes the aircraft to bounce from side to side.
(2) A taxiing take-off or run-on landing over rough, uneven ground.
(3) Incorrect or unequal tyre or oleo pressures.
(4) A wheel dropping into a hole or rut on landing, or deplaning troops contacting the undercarriage when hovering in light contact with the ground.
3. Susceptibility to Ground Resonance. Helicopters with multi-bladed rotors and fully articulated heads are more susceptible to ground resonance, since there are more areas where vibrations can arise. Conversely, helicopters with teetering-head two-bladed rotors are almost immune. Similarly, aircraft fitted with skids, instead of pneumatic tyres and oleos, are much less prone to ground resonance, since the part of the aircraft in contact with the ground is more rigid and less likely to initiate vibration.
4. Recovery Action. The more appropriate of the following actions should be taken:
a. If the aircraft is serviceable to fly and rotor rpm (Rrpm) are available, take-off immediately. Rrpm should always be maintained in the operating range until the final landing has been completed.
b. If the aircraft is not serviceable to fly or Rrpm are not available, lower the lever, close the throttles and shut down. The rotor brake should be applied as quickly as possible.

## Blade Sailing

5. A condition known as 'blade sailing' can occur when a rotor is starting up or slowing down in strong wind conditions, particularly if the wind is gusting (this is a common problem with helicopters aboard ships). With the helicopter facing into wind, the advancing blade experiences an increase in lift and will flap up excessively due to the low centrifugal force, reaching a maximum height to the front of the helicopter. As the blade progresses on the retreating side, it experiences a sudden loss of lift and will flap down rapidly, flex and reach its lowest position to the rear of the helicopter, over the tail cone. There is a danger that the blade may strike the tail cone or, when the wind is from the rear, the blade may strike the ground. Because of poor stick control and low Rrpm it is almost impossible to control blade sailing. The effects may be minimized by displacing the stick forward and slightly into wind, or by facing the helicopter slightly out of wind so that the lowest point of the blade's path does not occur over the tail cone. It may also be possible to find shelter in the lee of a hangar, but it may be more turbulent there and care is needed. Helicopter pilots should also be aware of the possibility of blade sailing occurring due to the downwash from other helicopters which are hover-taxiing nearby.
6. Since the condition occurs at low Rrpm in strong winds, it is advisable to slow the rotor down as quickly as possible on shut down by using rotor brake. On start up, Rrpm should be increased at a faster rate than normal. Limits for maximum permitted power and rotor brake are published in the Aircrew Manual. Safe wind limits for engaging the rotor are found in trials and are published as polar diagrams in the limitations section of the Aircrew Manual.

## Static and Dynamic Rollover

7. Static Rollover. A simple explanation of static rollover is that it occurs when a helicopter is parked on a slope which is steeper than it can negotiate. A normal helicopter could be expected to withstand slopes in the order of $40^{\circ}$ to the horizontal. Static rollover will occur when the CG moves outside the down-slope skid or undercarriage wheel (see Fig 3).

## 12-5 Fig 3 Static Rollover


8. Static Rollover - Flying Configuration. Once the helicopter is in a flying configuration the factors affecting static rollover become much more complicated. Thus, a more practical definition would be that static rollover occurs when all the forces acting on the airframe, without taking into consideration angular momentum, cause the aircraft to roll over. The following are brief summaries of the factors that determine the point at which static rollover will occur:
a. Tail Rotor Thrust. If the down-slope is in the same direction as the tail rotor thrust, then the tail rotor thrust will tend to roll the aircraft in that direction (see Fig 4).

## 12-5 Fig 4 Effect of Tail Rotor Thrust


b. Landing Point. As explained in para 7, any lateral slope will play a part in static rollover. However, a nose-down slope will often increase the rolling moment due to the tail rotor by effectively increasing its height above the point of rotation (Fig 5).

## 12-5 Fig 5 Effect of Nose-down Slope

Fig 5a Level Ground


Fig 5b Sloping Ground


Note: Extra care must be taken by pilots of Apache aircraft in this particular configuration, when using tail rotor thrust to control fuselage roll.
c. Wind Velocity. A crosswind in the same direction as the tail rotor thrust will tend to roll the fuselage in the same direction as the tail rotor thrust. This effect will be most noticeable on aircraft with large lateral cross-sectional areas, such as the Puma, Sea King and Merlin. A wind from the 12 o'clock position will cause the main rotor blades to flap back. A wind from the upslope direction will cause the rotor disc to flap down-slope with an additional force acting laterally on the fuselage (Fig 6).

## 12-5 Fig 6 Wind Effect - Plan View


d. Centre of Gravity. The majority of helicopters store fuel in the floor section or sponsons. Thus a low fuel state will result in the CG being vertically higher. This will make the aircraft less statically stable. In addition, a CG displaced laterally towards the down-slope will make static rollover more likely (see Fig 7). Asymmetric weapon loads on external pylons can cause extreme lateral movement of the CG.

## 12-5 Fig 7 Centre of Gravity


e. Undercarriage. For numerous reasons, it is possible to compress the main wheels and/or oleos on one side of the aircraft. Normally this would be countered with lateral cyclic; however, if the compression is not countered then the effect on the CG will be to move it towards the 'compressed' undercarriage. In addition, because the axis of rotation of the rotor disc is displaced, any application of power will tend to rotate the fuselage towards the 'compressed' undercarriage side.
9. Dynamic Rollover. Dynamic rollover may cause a helicopter to be irreversibly committed to rolling over at angles of much less than $10^{\circ}$, depending on the rate of roll. The principle contributor to this condition is the build-up of angular velocity of the helicopter mass about the skid or wheel in contact with the ground. The contributory factors are:
a. The degree of slope of the landing point. The deck of a moving ship will have the same effect, possibly magnified.
b. The lateral control authority available to the pilot.

Other factors that affect the likelihood of dynamic rollover are:
c. The total mass of the helicopter.
d. The distance of the CG from the undercarriage.
e. The tail rotor thrust (if it enhances the rolling moment initiated by collective). This will apply to:
(1) Left undercarriage up-slope for aircraft with clockwise rotating blades e.g. Puma and Squirrel.
(2) Right undercarriage up-slope for aircraft with anti-clockwise rotating blades eg Merlin, Lynx and Sea King.
f. Extraneous factors concerning the surface conditions may predispose the aircraft towards rollover, e.g. asymmetric deck lashings or obstacles. Landing gear might also sink through snow, mud or sand, or possibly freeze to the surface.
10. Cyclic Roll Power. In order to understand why dynamic rollover is so dangerous once it has developed, it is important to understand the limitations of cyclic roll power. Aircraft designers calculate the roll power of a helicopter to give it a specific roll rate about the axis of rotation whilst in forward flight. The axis of rotation is roughly concurrent with the longitudinal axis through the CG. The roll power is, therefore, calculated to be sufficient to rotate large masses with very short moment arms and small masses with long moment arms. On sloping ground (or in other circumstances outlined in para 9), when a rate of roll is initiated with the collective, the point of rotation transfers to the up-slope undercarriage. This new pivot point results in the rotation of large masses and large moment arms and therefore, higher magnitudes of angular momentum than the aircraft is designed to control with the cyclic.
11. The Development of Dynamic Rollover. The following sequence describes the build up of dynamic rollover during take-off from sloping ground, although it would apply equally to crosswind or deck take-offs, or to a take-off where one wheel/skid is stuck to the ground. Fig 8 shows a helicopter on sloping ground with the cyclic central. The CG is well within the undercarriage and there is a small tail rotor force acting up the slope.

## 12-5 Fig 8 Helicopter on Sloping Ground



The disc is now levelled by use of the cyclic (Fig 9). Any offset flapping hinges will produce a turning force on the fuselage; semi-rigid and rigid rotor heads produce more force. As the collective lever is applied the aircraft will pivot about the up-slope undercarriage and the CG moves, creating angular momentum. The collective power has now created a rate of roll about the up-slope undercarriage which, because of the long moment arm, creates significant angular momentum. It is possible, therefore, that application of collective can start a rate of roll about the up-slope undercarriage that is impossible to overcome with cyclic.

## 12-5 Fig 9 Helicopter on Sloping Ground with Rotor Disc Parallel to the Horizon


12. Recovery from Static and Dynamic Rollover. The two types of rollover described previously act in opposite senses, in that static rollover will tend to roll the aircraft down the slope and dynamic rollover will roll the aircraft towards the up-slope. From the pilot's point of view, there are a number of similarities that call for similar aircraft handling to prevent rollover and also to recover should it start.
a. Overcontrolling. One handling characteristic, which may lead to rollover of either kind, is overcontrolling on the collective or cyclic. Overcontrolling can lead to high rotational speeds for which the cyclic roll power is insufficient to overcome; thus, a pilot's natural reaction to counter roll with cyclic is ineffective. The initial reaction to undemanded roll should be to rapidly lower the collective (the chances of recovery diminish rapidly with time) and follow with cyclic. Using smooth collective movements to initiate a lift into the hover is the best way to avoid overcontrolling. Care should always be employed when lifting off with AFCS disengaged as this often results in more rapid control responses.
b. Disc and Aircraft Attitude. The pilot must always monitor the disc's attitude in relation to the aircraft's attitude. The disc's attitude can be monitored against the visual horizon and should, whenever possible, be parallel to that horizon, particularly when changing power in contact with sloping ground. The aircraft's attitude is more difficult to monitor using external references and therefore requires frequent cross-reference to the attitude indicator. In some aircraft, it is not possible to pre-position the cyclic, so a smooth application of collective prior to cyclic movement will suffice. Special care must be taken at night (and particularly if using NVD) when it is difficult to monitor disc and aircraft attitudes.
c. Wind Velocity. The pilot should monitor the wind velocity continuously to anticipate and prevent any adverse effects on the airframe and rotor blades.
d. Centre of Gravity. The pilot should always be aware of the aircraft's CG. The fore and aft displacement is normally calculated and, for some aircraft, lateral displacement is measurable. However, as vertical displacement is rarely calculated, pilots must remain alert to those circumstances when vertical movement of CG has taken place eg low fuel states, payload changes.
13. Summary. The dangers of static and dynamic rollover remain ever present. When operating on sloping ground, most pilots are aware of the phenomenon, and adjust their operation accordingly. However, two out of three accidents caused by rollover have occurred on relatively level landing sites. A static, stable helicopter can, after rotor start, quickly become unstable in certain conditions. A running take-off/landing in a crosswind is one of the less obvious circumstances which may also lead to the scenarios described within this section. Finally, pilots should always be careful when operating with AFCS disengaged and be alert to any unusual or rapid roll rates.

## Vortex Ring

14. Although vortices are always present around the periphery of the rotor, under certain airflow conditions the vortices will intensify and, coupled with a stall spreading outwards from the root end of the blade, result in a sudden loss of rotor thrust and a subsequent rapid loss of height. This condition is similar in some ways to stalling in a fixed wing aircraft and when it occurs the helicopter is said to be in a state of vortex ring. This state can be entered from several in-flight manoeuvres but the airflow conditions which give rise to its formation remain substantially the same in all cases. These conditions will only occur when all of the following are present:
a. The helicopter has induced flow passing down through the disc, as occurs in powered flight.
b. There is an external airflow directly opposing the induced flow, as occurs with a high rate of descent.
c. The indicated airspeed is low.

One flight manoeuvre from which vortex ring state can develop is when the helicopter enters a powerassisted descent with low airspeed. Other manoeuvres where vortex ring can develop are:
d. As a result of applying power to recover from a low airspeed autorotation without first increasing the airspeed.
e. Allowing the helicopter to lose height during a harsh flare, such as at the end of a gate approach or a quickstop.
f. Downwind approach.
g. A steep approach.

## Development of Vortex Ring State

15. When the helicopter is hovering in still air (Fig 10), the direction of the relative airflow can be determined from the blade's speed of rotation and the induced flow, both of which will have their greatest value near the tip. Assuming that the ratio of the rotational velocity to induced flow is constant throughout the length of the blade, then the direction of the relative airflow all along the blade will be the same, but, because of the wash-out, the root end of the blade will have the greatest angle of attack (Figs 10b and 10c).

## 12-5 Fig 10 Hover

Fig 10a Airflow


Fig 10b Blade Tip


Fig 10c Blade Root
16. The effect of reducing collective pitch to commence a rate of descent is shown in Fig 11a. When the descent is established, a new airflow component will exist directly opposing the induced flow which, in turn, will alter the direction of the relative airflow along the blade. If, at the root end of the blade, the airflow from rate of descent is equal to the induced flow, then the relative airflow will be in the plane of rotation, causing the angle of attack to increase (Fig 11c).

## 12-5 Fig 11 Slow Descent

Fig 11a Airflow


Fig 11b Blade Tip


Fig 11c Blade Root


In the area of the tip, the conflicting airflow outside and inside the disc will intensify the tip vortices, further increasing the induced flow (Fig 11b). If the increase in induced flow has the same value as the airflow from the rate of descent, a change will take place in the direction of the airflow relative to the blade but, because the collective pitch has been lowered, the angle of attack in the area of the tip will have actually decreased (Fig 11b).
17. If the collective pitch lever is lowered further, the rate of descent will again increase (Fig 12a). The process will be repeated, and eventually a condition will be reached where the root end of the blade will reach its stalling angle (Fig 12c).

## 12-5 Fig 12 Vortex Ring State

Fig 12a Airflow


Fig 12b Blade Tip


Fig 12c Blade Root


At this stage, rotor thrust is decreasing both at the tip of the blade, due to the vortices, and at the root of the blade, because of its stalled condition, leaving an area in between to produce the rotor thrust necessary to balance the weight. Any further increase in rate of descent resulting from lowering the lever will further reduce the area of the blade that is effectively producing rotor thrust. Once a condition is reached where rotor thrust becomes insufficient to balance the weight, then the rate of descent will rapidly increase, being as high as $8,000 \mathrm{fpm}$ on some types of helicopter. Wind-tunnel experiments indicate that vortices form and intensify in a most erratic manner, subjecting each blade inboard from the tip to large and sudden variations in angle of attack. Dissymmetry of rotor thrust occurs and the helicopter will pitch, roll and yaw to no set pattern, making control of the aircraft extremely difficult. In the fully developed vortex ring state, raising the collective pitch lever will only aggravate the condition and, instead of checking the rate of descent, it will cause it to increase. The higher the all-up weight (AUW) of the helicopter for a given Rrpm, the higher the collective pitch setting necessary to maintain the hover at the given Rrpm. Consequently, vortex ring state can occur at an earlier stage in a heavily-laden helicopter than it would in a lightly-laden one, under the same conditions.

## Effects of Vortex Ring State and Recovery Action

18. Effects of Vortex Ring State. The onset of vortex ring state can be identified from the following effects:
a. Significant vibration caused by vortices forming and breaking away, and the stall at the root increasing the pitch control forces.
b. Random pitching and rolling due to the complex airflow causing the rotor blades to flap without control inputs.
c. Fluctuating power demands and torque indications due to large changes in rotor drag.
d. Random yawing caused by the tail rotor being in the unstable airflow from the tip vortex region.
e. Slow control response caused by the reduced length of rotor blade which is producing thrust and, therefore, able to respond to control inputs.
f. Rapid increase in rate of descent.
19. Recovery From Vortex Ring. If the vortex ring is allowed to develop, a very high rate of descent will occur. An incipient stage can be identified by an increasing rate of descent with power on. This will be accompanied by an increase in vibration and random pitch, roll and yaw. Recovery action must be taken at this stage because recovery from a fully developed vortex ring state may be impossible because control response is so restricted. The recovery requires the following:
a. Increasing airspeed by a large nose-down attitude change, then applying power
b. If sufficient height is available, entering auto-rotation by reducing power to zero and then gaining airspeed; however, it may be impossible to prevent the rotor overspeeding.
20. Avoidance of Vortex Ring. The actions described in sub-paras 19a and 19b both entail a considerable loss of height, but the conditions in which vortex ring may occur are those close to the ground. Therefore, the following stages of flight should be carried out with great care:
a. Vertical descent. When descending vertically into a confined area from above the level of the obstacles, it is difficult to judge height and a high rate of descent can develop (Fig 13).

## 12-5 Fig 13 Vertical Descent


b. Steep Approach. In light winds, a misjudged steep approach can cause the conditions for vortex ring (see Fig 14).

12-5 Fig 14 Steep Approach

c. Downwind Manoeuvres. Downwind manoeuvres result in low or negative airspeeds (Fig 15).

## 12-5 Fig 15 Low Airspeed During Downwind Manoeuvre


d. Quick Stop Flares. When a helicopter is flared in a quick stop, the horizontal airflow past the rotor comes more nearly from below as the disc is tilted back. If a rate of descent develops, the airflow directly opposes the induced airflow (Fig 16).

## 12-5 Fig 16 Quick Stop Flare



If a recovery is made from a practice autorotation by increasing power in the flare, before levelling the helicopter, the situation is similar to the quick stop. This is not so when carrying out an engine-off landing, when the rotor is autorotating until the lever is raised to cushion the touchdown. If recovery from a slow speed autorotation is made with low airspeed, the situation is similar to descending into a clearing.

## Tail Rotor Failure

21. The primary function of the tail rotor is to produce a variable thrust to counter torque reaction from the main rotor system and to change aircraft heading in the hover, so that a balanced condition can be maintained throughout the flight envelope (see Volume 12, Chapter 3). A failure or malfunction will cause the tail rotor thrust to become fixed anywhere between zero (e.g. drive failure) and maximum (e.g. pitch control failure). Successful recovery to a safe landing will depend upon the pilot's speedy and accurate assessment of the nature of the problem. The following paragraphs are intended as general guidelines for conventional helicopters, detailed procedures for specific types will be found in Flight Reference Cards .
22. Structural Failure. In powered flight, structural failure of the tail rotor will result in an immediate yaw in the direction opposite to main rotor rotation, with severe vibration. If part of the assembly has become detached, there is likely to be a sharp nose-down trim change. Imbalance in a large rotating mass would impose unacceptable stress loadings, and the immediate response must be to lower the collective fully, shut down the engines, and carry out an engine-off landing (EOL), maintaining positive airspeed.
23. Drive Failure in Forward Flight. Tail rotor drive failure in forward flight results in a sharp yaw in the direction opposite to main rotor rotation, with the loss of pedal control. The immediate action is to lower the collective fully to reduce yaw, regain control of the aircraft, and assess height. Pilots should be aware that ASI readings will be unreliable due to sideslip, and that loss of airspeed at this stage may lead to loss of control. If height is insufficient for a reasonable period in autorotation, shut down the engines and carry out a progressive flare EOL, ideally retaining some forward speed for a run-on landing to provide directional stability. The tailwheel or back end of the aircraft can be dragged along the ground in the flare attitude to prevent excessive yaw as the airspeed decreases during the landing run. If an immediate EOL is inadvisable, and height permits, establish autorotation at about 65 to 70 kt , then apply power gradually to find a power/speed combination at which directional stability can be maintained in level flight. Cyclic pitch changes will be required to counter yaw as power is increased, and bank in excess of $20^{\circ}$ will probably be necessary to hold a heading with level flight power applied. Manoeuvres under power will require extreme caution. Turns should be made in the direction of main rotor rotation if possible; turns in the opposite direction (which are probably achieved merely by reducing bank) may induce excessive side-slip, leading to loss of airspeed, an increased rate of yaw, and the risk of entry into an uncontrollable spiral descent. Finally, although an EOL is normally the better choice, if aircraft performance, landing site, or weather considerations dictate the need for a powered landing, the first priority is to determine the minimum speed at which control and heading can be maintained. Minimum approach and touchdown speeds should then be at least 15 kt and 5 kt higher, respectively. The landing site must be wide enough to accept the inevitably curved landing run. A slightly nose-up flare (not more than $5^{\circ}$ ) is recommended a few feet above the ground, whilst gently lowering the collective to maintain the rate of descent. The aircraft is then levelled laterally, and as the wheels/skids touch, the collective is lowered fully and the engines shut down immediately. Asymmetric wheel braking can be applied to keep straight.
24. Drive Failure in the Hover. The consequences of tail rotor drive failure are most severe in the hover since, without forward speed, both the facility to apply cyclic turn against the yaw, and the directional stability derived from the fuselage/tail boom, are lost. Because of the high power setting in the hover, the rotation forces are strong; the yaw at failure will be very rapid, and may be accompanied by violent attitude changes in pitch and roll which, if allowed to develop, could cause pilot disorientation. From a low hover the recommended action is to shut down the engines immediately, attempt to level the aircraft, and cushion the touchdown with collective. In a high hover the problem may be complicated by a deployed winch cable or an underslung load; if the engines are shut down, the aircraft is likely to reach the ground with too high a rate of descent and insufficient main rotor rpm. Clearly the collective will probably need to be lowered to reduce yaw but subsequent actions will depend on aircraft performance, AUW, wind and terrain. In any case, the engines must be stopped before the collective is raised for touchdown. Power-on landings have been made in the past but some aircraft types do not lend themselves to this course of action.
25. Pitch Control Failure. Control failure can leave the tail rotor pitch fixed anywhere between maximum and minimum, and the pedals could be free, stiff, or jammed. The first action must be to control the aircraft and attempt to climb to a safe height to carry out systems checks. If pitch control cannot be regained it is essential to ascertain broadly the setting at which it has become fixed, since the effects at the opposite extremes of the range will be reversed. Thus, with a fixed high value of tail rotor pitch, the selection of mid-power cruise settings will result in a tendency to yaw in the direction of turn of the main rotor. Lowering the collective will increase the yaw, and a power increase will reduce it. Clearly this is not a safe condition for autorotation or an EOL. Conversely, the selection of mid-power cruise settings with tail rotor pitch fixed in the negative range will result in a yaw opposite to the direction of turn of the main rotor; raising the collective will increase the yaw and a power reduction will lessen it.
a. Control Failure at High Pitch Settings in Forward Flight. With the tail rotor fixed in high pitch, the main rotor thrust will need to be in the upper range to maintain balance, and even small power reductions must be made with great caution. If the tail rotor pitch has 'thrown-on' to its normal top limit or beyond, some degree of yaw might not be containable at the lower power settings for a descent. Before attempting a descent, it is important to establish the lowest collective position at which yaw in the direction of main rotor rotation is acceptable, and it must not be set below this position until after the aircraft has landed. The descent must then be a compromise between the need for a low airspeed with a high power setting, and the requirement to convert speed into cyclic turn. Uniquely in this case, a low speed spiral descent in the direction opposite to main rotor rotation would appear to be the best course of action. Attempt to level the aircraft using cyclic and collective immediately prior to reaching the ground, and when firmly in contact, shut down the engines before fully lowering the collective.
b. Control Failure at High Pitch Settings in the Hover. Without forward speed, neither cyclic turn nor the airframe's inherent directional stability can counter yaw. However, if tail rotor control failure occurs in the hover, the power settings are likely to be high, and the rate of yaw ought not to be excessive. In the low hover the recommended recovery is to lower the collective and get the aircraft onto the ground quickly, before the tail rotor induced yaw has time to develop, stopping the engines immediately on touchdown. From the high hover, the best course of action is probably to apply power, gain airspeed and then proceed as for failure in forward flight. Alternatively, it may be possible to inch down vertically, but the aircraft is likely to be rotating under the influence of tail rotor torque throughout the descent.
c. Control Failure at Low Pitch Settings. The symptoms of control failure at low pitch settings are obviously similar to, but less severe than, those of drive failure, and the recommended actions are also similar in general terms. However, in this case a powered landing is considered preferable to the EOL alternative.

## Loss of Tail Rotor Effectiveness (LTE)

26. LTE is a phenomenon that can happen to any single main rotor helicopter during low speed flight ( $\leq 40 \mathrm{kt}$ ). It manifests itself as an uncommanded yaw rate that, if not corrected promptly, can rapidly increase and lead to loss of directional control. The direction of uncommanded yaw is dependent on the direction of rotation of the main rotor and the consequent direction of thrust of the tail rotor. Helicopters with a clockwise-rotating main rotor would experience uncommanded yaw to the left with the opposite being true of helicopters with an anti-clockwise-rotating main rotor. LTE is dependent on features such as the size of the helicopter and the geometric and aerodynamic relationship between the main and tail rotors, thereby varying the susceptibility of different helicopter types. LTE is an aerodynamic effect and not a result of mechanical malfunction. The following paragraphs highlight scenarios and conditions with reference to a clockwise rotating main rotor.

## 27. Factors Affecting Control of Yaw.

a. Tail Rotor Thrust Margin. The tail rotor is designed to produce sufficient thrust such that heading control can be maintained at all points within the flight envelope. When operating at the edge of the low speed area of the flight envelope, there will be an amount of tail rotor authority remaining and therefore a tail rotor thrust margin. This may be indicated by the amount of residual
pedal travel. Helicopters with a small tail rotor thrust margin are more susceptible to LTE, are likely to experience a greater rate of uncommanded yaw, and will be more difficult to recover.
b. Relative Wind. Flight trials and wind tunnel tests have shown that there are three relative wind regions that create an environment in which LTE can occur. The size of these regions is dependent on helicopter type, so the figures indicate typical regions only. Two of the three regions are dependent on main rotor rotation direction and, for simplicity, these areas are described with reference to a clockwise rotating main rotor. Where appropriate, the corresponding region for an anti-clockwise rotating main rotor is noted at the end of each subpara. The regions are:
(1) Main Rotor Vortex Interference ( $030^{\circ}$ to $\mathbf{0 8 0 ^ { \circ }}$ ). Relative winds in this region, at velocities in the 10 to 30 kt range, can cause the main rotor vortex to be blown into the tail rotor, thereby creating a turbulent environment (see Fig 17). As the vortex passes through the tail rotor, the angle of attack can change markedly with consequent large changes in tail rotor thrust. Pilots need to react positively to prevent an excessive rate of yaw to the left from developing. (Anti-clockwise region is $280^{\circ}$ to $330^{\circ}$ ).

## 12-5 Fig 17 Main Rotor Disc Vortex Interference


(2) Downwind $\left(12 \mathbf{0}^{\circ}\right.$ to $\left.\mathbf{2 4 0}{ }^{\circ}\right)$. Relative winds in this region will attempt to weathercock the helicopter into wind (see Fig 18). Again, pilots need to react positively to control the heading or rate of turn.

## 12-5 Fig 18 Downwind


(3) Right Crosswind $\left(03 \mathbf{0}^{\circ}\right.$ to $\left.\mathbf{1 5 0}^{\circ}\right)$. Relative winds in this region can result in the development of a vortex ring state of the tail rotor, as its induced flow will be opposed by the crosswind (see Fig 19). As vortex ring develops, the thrust produced by the tail rotor will vary markedly making control of heading or yaw rate difficult. This may induce an element of overcontrolling by the pilot, which may lead to LTE. (Anti-clockwise region is $210^{\circ}$ to $330^{\circ}$ ),

## 12-5 Fig 19 Right Crosswind



Fig 20 shows a combined azimuth diagram with the areas of overlap highlighted. Slow reaction to uncommanded yaw in one region could lead to a rapidly accelerating rate of yaw as an overlap area is entered.

## 12-5 Fig 20 Areas of Overlap


c. Manoeuvring. During low speed manoeuvres, power requirements, and therefore tail rotor thrust, can vary markedly, especially if translational lift is lost. If a low speed left hand turn is being flown, any reduction in wind awareness may lead to a reduction in IAS and a consequent increased power requirement in order to maintain height. A large power demand will require a large increase in tail rotor thrust thereby increasing the potential for LTE.
d. Wind Velocity. A steady wind velocity can make assessment of the onset of LTE easier. However, there are many environments in which wind velocity can change markedly. For example, during mountain flying, rapid and unexpected changes of both wind direction and strength can occur. These changes could lead to LTE as a combination of factors mentioned previously. Additionally, inadvertent operation outside the published flight envelope could occur making recovery from LTE impossible.
e. Ground Effect. LTE is possible in both Inside Ground Effect (IGE) and Outside Ground Effect (OGE) flight but is more likely in OGE due to the greater collective pitch settings.
f. Other Factors. Any increase in power requirement needs more collective pitch and therefore greater tail rotor thrust. The following factors should be considered:
(1) Increased Aircraft Mass.
(2) Higher Density Altitude.
(3) Main Rotor Droop.
(4) Overcontrolling.

Pilots should be aware that LTE is possible if they enter a flight profile where one or a combination of the above factors is present. They should recognise the onset and react quickly before it develops.
28. LTE Conducive Flight Profiles. The factors described in para 27 will vary according to helicopter type. However, there are certain flight profiles where LTE is more likely to occur, regardless of type. Note that this list is not exhaustive.
a. Reconnaissance.
b. Weapons Firing.
c. Searching.
d. Winching.
e. Underslung Loads (USLs).
f. Confined Areas.
g. Mountain Flying.
29. Recovery Action. Having encountered LTE, recovery action will vary according to the circumstances, but the suggested actions should be:
a. Maintain full yaw-opposing pedal.
b. Increase forward airspeed.
c. If height above the ground allows, lower the lever.

Carrying out the first two actions simultaneously will allow yaw control to be regained in the shortest time. Knowledge of helicopter performance and handling characteristics (especially any stated critical azimuths) and maintenance of situational awareness are the best tools for assisting with prevention of LTE. The conditions for, and onset of, LTE can be nebulous; early detection followed by positive corrective action is essential to prevent prolonged loss of control.
30. Warning. A natural pilot reaction, particularly if close to the ground, might be to raise the lever. This should only be done in the knowledge that it will increase the rate of uncommanded yaw.

## CHAPTER 6 - HELICOPTER POWER REQUIREMENTS

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## Introduction

1. The power required to maintain level flight in a helicopter will vary from the hover to maximum forward speed. This chapter considers, in detail, how and why these requirements vary.

## Work

2. If a body is to be moved from one position to another, then a force must be applied to overcome the resistance to movement. When the body is moved, work is said to have been done, and it is calculated by multiplying the force used by the distance that the body has been moved. The resistance set up by the rotor blades to be turned, or the resistance caused by moving the fuselage through the air, is termed drag. Since, in any state of equilibrium, force equals drag, then work must equal drag $\times$ distance.

## Power

3. Power is defined as the rate of doing work, or the ratio of the work done to the time taken. Therefore:

$$
\begin{aligned}
\text { Power } & =\frac{\text { Work }}{\text { Time }} \\
& =\frac{\text { Drag } \times \text { Distance }}{\text { Time }} \\
\text { But } \quad & \frac{\text { Distance }}{\text { Time }}=\text { Velocity, and therefore } \\
\text { Power } & =\text { Drag } \times \text { Velocity } \\
& =\text { Drag } \times \text { TAS }
\end{aligned}
$$

The equation for calculating drag is:

Drag $=C_{D}{ }^{1} / 2 \rho V^{2} S$
therefore, Power $=C_{D} 1 / 2 \rho V^{2} S \times$ Velocity.

Assuming $C_{D} 1 / 2 \rho S$ is constant, $(K)$,

Power $=\mathrm{KV}^{2} \times \mathrm{V}=\mathrm{KV}^{3}$

The resistance, or drag, of a body moving through the air will vary as the square of the speed, but the power required to balance the drag will vary as the cube of the speed. Power is normally expressed in terms of kilowatts ( 1 kw is equal to 737.6 foot pounds force/sec).

## POWER REQUIRED

## Introduction

4. The power required by the rotor to maintain level flight throughout the helicopter's speed range can be considered under three headings:
a. Rotor Profile Power.
b. Induced Power.
c. Parasite Power.

## Rotor Profile Power

5. Rotor Profile power is the power required to drive the rotor at minimum pitch at a constant Nr , plus the power required to drive the tail rotor and all ancillary equipment. With minimum pitch applied, there is drag on the blades as they rotate. As the speed of the airflow past the rotor increases, the profile drag (Zero Lift Drag, see Volume 1, Chapter 5) of the advancing blade is increased, and that of the retreating blade is reduced. There is, however, an imbalance because the amount by which the drag is increased on the advancing blade is greater than the amount by which the drag is reduced on the retreating blade and so, as airspeed increases, the power required to maintain Nr will also increase. Furthermore, since power increases in proportion to speed cubed, the graph representing rotor profile power might be expected to rise very steeply. This is not the case, however, because in the early stages of the increase in airspeed, the tail rotor experiences translational lift and therefore less pitch and less power are required to keep the aircraft straight. The conventional tail rotor also flaps back and so obtains flare effect leading to a small further saving in power. As forward speed increases, the rotor profile power curve rises only slowly at first but rises more rapidly in the higher speed range as the beneficial effects of the conventional tail rotor are over-ridden by the increasing drag, see Fig 1. The fenestron is different in that the aerofoil section of the cambered fin provides thrust in the required direction as airspeed increases, and hence the tail rotor requires less power as airspeed increases. This applies up to about 120 kt.

## 12-6 Fig 1 Rotor Profile Power



## Induced Power

6. When the collective pitch is minimum, there is virtually no rotor thrust being produced. In order to increase rotor thrust, it is necessary to increase blade pitch and this leads to an increase in rotor drag. To maintain Nr, the power must be increased to overcome the rising drag. This increase in power is known as the induced power because it is the extra power required to overcome the rise in drag when
the blades are inducing air to flow down through the rotor. As explained in Volume 12, Chapter 3, induced flow diminishes with forward speed, and less collective pitch is needed to produce the required angle of attack. The curve of induced power will start at a position on the vertical axis of the graph at Fig 2 and will fall rapidly at first due to the onset of translational lift, and then fall more slowly as forward speed increases. The ground effect, shown by the dotted line, will also reduce power required to hover. Induced power accounts for approximately $60 \%$ of the power required to hover.

## 12-6 Fig 2 Induced Power



## Parasite Power

7. As the helicopter speed increases, so does fuselage parasite drag and the rotor disc needs to be tilted progressively further forward to provide an increasing horizontal component of total rotor thrust to balance the parasite drag. The further forward the rotor disc is tilted, the greater the horizontal airflow through the disc becomes. This component adds to, and increases, the induced airflow hence increasing rotor drag. Parasite power is the power required to overcome this increasing rotor drag. Parasite power increases as $\mathrm{V}^{3}$, see Fig 3.

## 12-6 Fig 3 Parasite Power



## Power Required

8. The power required to maintain the helicopter in steady straight and level flight at any given forward speed will be the combination of rotor profile power, induced power and parasite power for that speed, see Fig 4.

## 12-6 Fig 4 Power Required



## Power Available

9. For a helicopter, the power available is considered to be the power which is available to the rotor and not that which is available from the rotor. For any given altitude, this power will remain more or less constant and it, therefore, appears on the power graph as a straight line, see Fig 5.
10. Performance. The performance of a helicopter will lie in the relationship between the power available and the power required - the greater the difference between them the greater the margin of power. From Fig 5, it can be seen that a surplus of power available over the power required exists over the greater part of the speed range. The greater the power margin the more power can be used for manoeuvring or for climbing.
11. Significant Speeds. Significant speeds are marked on Fig 5.
a. The best rate of climb speed is at point 1 , the maximum power margin.
b. $\quad \mathrm{V}_{\max }$ occurs at point 2, where there is no longer power available to accelerate the helicopter in level flight.
c. Minimum power required, and also minimum fuel consumption, occur at point 1. This is the endurance speed and, for most helicopters, is around 60 kt to 70 kt .
d. The range speed occurs at point 3 where a tangent from the origin to the curve indicates the best ratio of power required to airspeed.

## 12-6 Fig 5 Power Available/Power Required



## Effect of Limited Power

12. With changes in air density, weight and altitude, the power available and power required curves will move closer together, and power available may eventually be sufficient to hover only with the assistance of ground effect; in extreme conditions, there may be insufficient power to hover at all. Under these conditions, there will be a minimum speed below which, even with ground effect, the helicopter cannot maintain height (see Fig 6).

## 12-6 Fig 6 The Effect of Reduced Power Available on Helicopter Performance



## Power Checks

13. Conditions at the take-off and landing areas may differ, and in order that the pilot may make an airborne assessment of the power available before committing himself to a landing, a simple power check can be carried out. When flying straight and level at a predetermined speed and with landing Rrpm, the torque required to maintain that speed is noted. The difference between this torque and the maximum available represents the power margin which, by reference to prepared data, can be used to determine the slow speed capabilities of the helicopter. A similar check can be carried out while hovering and before moving into forward flight, in order to assess the take-off capabilities.
14. In making the check of power available, some allowance must be made if the helicopter is operating above the altitude where the rotor is most efficient. Information on this is available from the Operating Data Manual for the aircraft.

## Best Climbing Angle

15. When operating with limited power, the helicopter must be moving forward in order to climb. To assess the steepest climbing angle, it is necessary to find the best rate of climb/forward speed ratio. This can be determined by drawing a line from the point where the power available line cuts the vertical axis of the graph, tangential to the power required curve (see Fig 6). The point of tangency indicates the speed for maximum climbing angle, and this will always be less than the speed for maximum rate of climb.

## Turning

16. In addition to providing a component to balance the weight and a thrust force to maintain speed, the total rotor thrust must supply a further component to change the direction of the helicopter in a balanced turn, and the greater the angle of bank, the greater this force must be. Its effect is similar to an increase in weight; with $30^{\circ}$ of bank, the apparent weight increases by $15 \%$, with $60^{\circ}$ of bank, the apparent weight will increase by $100 \%$. More collective pitch and, therefore, more power will be required to maintain height in the turn, and the effect on the power required curve is to cause it to move up the graph. The maximum angle of bank to maintain a level turn is reached when full power is applied and best climbing speed is maintained. If bank is increased beyond this point, any attempt to maintain height by use of lever will result in loss of Rrpm, due to overpitching, see Volume 12, Chapter 1 and Volume 12, Chapter 12.

## CHAPTER 7 - HELICOPTER STABILITY

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## Introduction

1. Stability is discussed in some detail in Volume 1, Chapter 17, with particular reference to fixed wing aircraft. For the sake of completeness the following paragraph summarizes stability generally and the rest of the chapter is devoted specifically to stability in helicopters.
2. Stability can be simply classified as static stability or dynamic stability.
a. Static Stability. If an object is disturbed from a given position and following this disturbance it tends to return to this position of its own accord, it is said to be statically stable. If, following the disturbance, it continues to move further and further away from its original position, it is said to be statically unstable; if it remains in the disturbed position, it is said to be statically neutrally stable. Figs 1a, 1b and 1c illustrate this.

## 12-7 Fig 1 Static Stability

Fig 1a Stable


Fig 1b Unstable


Fig 1c Neutrally Stable

b. Dynamic Stability. If an object is statically stable it will return to its original position, but in doing so, it may initially overshoot. If the amplitude of the oscillations decreases and dies out, it is said to be dynamically stable. If the amplitude of the oscillations increases, then it is said to be dynamically unstable, and if the oscillations continue, but at a constant amplitude, it is said to be dynamically neutrally stable. Figs $2 a, 2 b$ and $2 c$ illustrate this.

## 12-7 Fig 2 Dynamic Stability

Fig 2a Stable


Fig 2b Unstable


Fig 2c Neutrally Stable


## Stability in the Hover

3. Consider a helicopter hovering in still air when a gust of wind affects the rotor disc from the side. The disc will flap away from the wind and, if no corrective action is taken by the pilot, the helicopter will move away from the gust. After a short while, the gust of wind dies out but, because the helicopter is moving sideways, it will now experience an airflow coming from the opposite direction. The helicopter will now slow down as the disc begins to flap away from this new airflow; in addition, the fuselage will tend to follow through as an overswing, thereby tilting the disc further than it was tilted before, and the helicopter will move sideways back towards its original position faster than it originally moved away. The movement of the helicopter will result in it experiencing continual sideways changes in the airflow affecting the disc and, although it will be statically stable, because the amplitude of the oscillations will be continually increasing, it will be dynamically unstable. The effect of a gust of wind from any direction will produce the same effect on the disc, therefore the helicopter is dynamically unstable in the pitching and rolling planes.
4. A gust of wind will also affect the tail rotor. If, for example, the helicopter has a starboard mounted rotor and is struck by a gust from the starboard side, the tail rotor's angle of attack will decrease. Assisted by the weathercock action of the fuselage, the helicopter will then yaw into the gust, i.e. to starboard. The aircraft will also move away from the gust and in so doing it will reduce the effect of the gust on the tail rotor. The aircraft will then experience an airflow from its own sideways movement and the aircraft will yaw to port. Following the movement of the helicopter as in para 3, it can be seen that the fuselage will be alternately yawing to port and starboard with each successive sideways movement of the helicopter. Therefore, when hovering, the helicopter is statically stable but dynamically unstable in the yawing plane.

## Stability in Forward Flight

5. If a gust of wind from the starboard side strikes the fuselage of a helicopter with a starboard mounted tail rotor in forward flight, the immediate effect is for the tail rotor's angle of attack to decrease and the helicopter to yaw to starboard. But the inertia of the helicopter will continue to keep it on its original flight path; weathercock action will then return the fuselage to its original position. In forward flight, therefore, the helicopter is both statically and dynamically stable in the yawing plane.
6. If a gust of wind affects the disc from ahead, the disc will flap back, and forward thrust will reduce and the aircraft will decelerate. Because the centre of gravity is below the thrust line, the inertia of the fuselage will cause the aircraft to pitch nose up, taking the disc back further and thus decreasing speed even more. When the speed has stabilized at a lower figure, the fuselage will start to pitch down below its original position (pendulosity): at the same time the disc will flap forward relative to the fuselage (reduced flap back due to lower speed). Now the speed will start to increase with the helicopter descending in a shallow dive and, as the speed increases, the disc will begin to flap back again and the cycle will be repeated, but with increasing amplitude. The helicopter will finally be pitching outside control limits unless cyclic correction is applied early in the cycle. The helicopter is, therefore, statically stable because each oscillation will take it through its original position, but is dynamically unstable because the amplitude of the oscillations progressively increases.

## Stability Aids

7. Tail Stabilizer. One method of improving stability in forward flight is by fitting a stabilizer at the tail of the fuselage. Its purpose is to help prevent the fuselage from following through when a gust of wind causes the disc to flap back. As the fuselage begins to pitch up, the increasing angle of attack on the stabilizer will damp down the movement and the rearward tilt of the disc will be greatly reduced; the reverse effect takes place when the fuselage pitches down. It should be noted, however, that the stabilizer will produce adverse effects if the helicopter is moving backwards: following a gust of wind which causes the disc to flap forward, the fuselage will pitch nose down and the tail will pitch up; this will increase the lifting force on the stabilizer, thereby increasing the pitch-up movement of the tail to a dangerous degree.
8. The Autostabilizer. The autostabilizer, is the simplest form of control system. The autostabilizer is a damping device without the ability to hold a given datum, hence a helicopter autopilot often consists of an autostabilizer to which long term datum holding is added. There are two types of autopilot which may be fitted to helicopters:
a. Basic Autopilot. A basic autopilot provides long-term datum holding of one or more variables but does not permit the pilot to introduce demands through his controls. Trimmers may be used to make limited adjustments.
b. Directed Autopilot. A directed autopilot provides long-term datum holding of one or more variables and also permits the pilot to introduce demands through his controls. Such an autopilot is also described as an Attitude Manoeuvre Demand System and may also be called Automatic Stabilization Equipment.
9. Flight Control System. When a basic autopilot receives signals other than those required to hold a simple datum e.g. height and heading, it is generally known as a Flight Control System. Such signals come from a variety of sources and their purpose is to control the helicopter in relation to some
fixed or moving exterior reference, e.g. ILS. Some of the more common command options available in a flight control system are given below, together with the manner in which control is carried out:
a. Barometric Altitude Hold. Barometric altitude hold is conventionally applied to the collective lever for height hold at low speed but can be applied through the cyclic stick where it is normally required for cruising flight.
b. Radio/Radar Altitude Hold. Radio altitude information is applied through the collective lever as it is often used during programmed manoeuvres at low speed.
c. Airspeed Hold. Airspeed hold is applied through the cyclic stick.
d. Co-ordinated/Programmed Turns. Two types of turning mode may be incorporated. Fully automatic turns onto preselected headings, or a balanced turn resulting from the application of bank.
e. Programmed Manoeuvres. If doppler and radio height information is available programmed transition to and from the hover may be provided. Such transitions are of two types; those that have a constant transition time irrespective of entry/exit conditions, and those that use constant acceleration/deceleration and thus have variable transition times.
f. Coupled Manoeuvres. The flight path of the helicopter may be coupled to information from outside sensors, examples being the cable hovering mode of ASW and automatic following of VOR and ILS information.

## Control Power

10. Control power can be defined as the effectiveness of the cyclic control in achieving changes in fuselage attitude. The main factor determining the degree of control power is the distance from the main rotor shaft at which a cyclic force is effective. This in turn depends upon which of the three basic types of rotor is being considered. The three types of rotor systems, described further in Volume 12, Chapter 8, are:
a. The teetering head
b. The fully articulated head
c. The semi-rigid rotor
11. The Teetering Head. If a cyclic change is made on a teetering head, the plane of the disc alters and total rotor thrust, acting through the shaft, is tilted. This produces a moment about the CG and causes the attitude to change, Fig 3.

12-7 Fig 3 Teetering Head

12. The Fully Articulated Head. With a fully articulated head, a cyclic change alters the plane of the disc and tilts total rotor thrust. However, the point at which cyclic force acts in causing a change in fuselage attitude is not only the shaft as in para 10. The plus and minus applications of cyclic pitch, as well as changing the plane of the disc, are felt at the flapping hinges. A couple is set up which is additional to the single force of the total rotor thrust in the teetering head; it is therefore more effective (Fig 4). There is still a lag in fuselage response to cyclic changes. The further the flapping hinges are from the centre of the hub, the greater is the effect of the couple set up at these points in producing attitude changes with application of cyclic pitch.

## 12-7 Fig 4 Articulated Head


13. The Semi-rigid Rotor. In the semi-rigid rotor case (see Volume 12, Chapter 8, Fig 9), cyclic pitch changes set up a powerful aerodynamic couple which alters the fuselage attitude almost instantaneously. The couple is estimated to be the equivalent of placing flapping hinges on an articulated head at $17 \%$ rotor radius from the shaft. Flexing properties of the blade account for the insignificant lag that does exist.
14. Comparison of Control Forces. If the same cyclic force were applied to the three rotor systems, the semi-rigid rotor would be the most effective in changing the aircraft attitude, the fully articulated rotor less effective and the teetering head rotor least effective in terms of control power. Therefore, control power determines the aircraft manoeuvrability and, to some degree, speed range.

## CHAPTER 8 - HELICOPTER DESIGN

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## Introduction

1. The helicopter is generally a low speed aircraft, but, because of its ability to hover and to take off and land vertically, it is particularly suitable for many military roles. This chapter deals generally with the design of helicopters in common use with the Services and with developments suited to higher performance helicopters.

## Types and Configurations

2. There are four main types of rotorcraft which may be categorized according to the methods used to provide lift and propulsion.
a. Gyroplane. The gyroplane or 'autogyro' has a freely rotating wing supplying the aerodynamic force for lift; all forward thrust is supplied by a separate propeller as in a conventional aircraft.
b. Pure Helicopter. The pure helicopter has powered rotating wings supplying all necessary aerodynamic forces for lift and propulsion.
c. Compound Helicopter. The major part of the lift of a compound helicopter is supplied at all times by the rotor, but supplemented by power units or stub wings, mainly at high speed.
d. Convertible Helicopter. The convertible helicopter is capable of modifying its configuration during flight so that lift is transferred from rotating wings to other fixed wings, and vice versa.
3. Most current helicopters have the power unit mounted on top of the fuselage, with a mechanical transmission system to drive the rotors. The most common configuration is a single main rotor with a separate tail rotor to balance torque reaction. Sometimes however, two main rotors are used which
contrarotate to balance torque reaction. These are normally arranged in tandem, but can be arranged coaxially with some loss of efficiency, or side-by-side. Fig 1 shows some configurations employed by helicopter designers.

## 12-8 Fig 1 Helicopter Configurations

Fig 1a
Single Rotor with Tail Rotor


Fig 1d
Tandem


Fig 1b
Torqueless Single Rotor


Fig 1e
Co-axial Contra-rotation


Fig 1c Side-by-side Intermeshing


Fig 1 f
Tandem Overlapping


Pilot's Controls
4. The helicopter pilot's main controls are shown in Fig 2 and consist of:
a. Collective Pitch Lever. This is usually operated by the pilot's left hand and controls the total lift produced by the rotor. Movement of the collective pitch lever simultaneously alters the pitch of all the blades by the same amount.
b. Cyclic Pitch Control Column. This is usually operated by the pilot's right hand and varies the pitch of each blade cyclically, so tilting the rotor disc and enabling the helicopter to move horizontally. In forward flight, the effect of moving the cyclic pitch control is similar to that of a fixed wing aircraft control column.
c. Yaw Pedals. The yaw pedals are operated by the pilot's feet and vary the force produced by the tail rotor to oppose torque reaction, thus controlling the movement of the helicopter about the vertical axis. The sense of movement is identical to that of the rudder pedals of a conventional aircraft.
d. Throttle. Most helicopters do not have a pilot-operated throttle. The engine speed is controlled by the variation in pilot demands of the rotor. Where a throttle is used, it is mounted on the end of the collective pitch lever and usually takes the form of a twist-grip.

## 12-8 Fig 2 Helicopter Pilot's Controls


5. Movement of the collective pitch lever will require changes of power because of the variation of lift and, therefore, the induced drag on the rotor blades. An interconnecting linkage is normally fitted between the collective pitch lever and the power unit so that when collective pitch is varied the power setting is varied by a corresponding amount to keep the speed of rotation of the rotor essentially constant. In most free turbine-engine helicopters, rotor rpm is maintained by centrifugal governors or by computer control of the fuel flow.

## POWER UNITS

## Piston Engines

6. Early helicopters had piston engines but, except for a few small types, most current helicopters have free turbine engines. Although the piston engine is economical on fuel and cheaper to produce and service, these advantages are more than outweighed by the much higher power to weight ratio and greater reliability of the free turbine engine. Cooling is a problem with the piston engine due to the low air speeds at which the engine may have to give full power. A cooling fan can absorb up to $10 \%$ of the engine power output.

## Free Turbine Engines

7. The power to weight ratio of the free turbine engine is so superior to the piston engine that it has become the popular choice for a helicopter engine despite its relatively high cost. The free turbine engine has a gas generator of one or two spools, and both axial and centrifugal compressors are used. Power take-off is effected by a one or two-stage free power turbine, which is connected to the transmission through a reduction gearbox. A clutch is not required on a free turbine engine as there is no mechanical connection between the gas generator and the free turbine. Early free turbine engines were mounted in the nose of the helicopter as a legacy of earlier piston engine types, but more recent helicopters have the engines mounted high in the airframe near the main rotor gearbox. In the latter position, there is less chance of foreign object damage, a weight saving due to a smaller transmission system, and a design advantage of a better cabin/cockpit with a corresponding increase in cabin space.
8. Free turbine engines are compact, lightweight, highly reliable, easily maintainable and have a relatively low specific fuel consumption. They are supplied complete with a built-in torque meter and engine reduction gearbox, and the power take-off can be from the front or rear of the engine. An important feature is the use of the modular concept; this permits the replacement of major assemblies without recourse to special equipment, expertise or performance checking. From the maintenance and servicing
aspects, the adoption of condition monitoring is widespread. In addition to normal flight instruments, engines have provision for accelerometers to measure incipient vibrations and provision for internal inspection by introscopes. Provision is also made for self-sealing magnetic chip detectors and oil sampling for spectrometric oil analysis. These facilities permit regular monitoring and enable incipient defects to be recognized and rectified before damage occurs.
9. All engine control systems are either mechanical or hydro-mechanical and provide the engine and rotor speed governing with mechanical and electrical overrides for system protection. However, some engines have an automatic fuel computer; during normal operation the engine is automatically controlled to effect constant speeding of the engine/rotor system. The optimum engine/rotor speed is selected by a speed select lever, and the varying power demands occasioned by change of rotor pitch are met thereafter by the automatic fuel computer; the computer varies the rate of fuel flow to suit the engine power demands. The computer works in conjunction with a collective pitch anticipator unit and a throttle actuator. Provision is made in some systems for rapid change-over to manual control in the event of the failure of electrical or computer systems.

## Engine Anti-Icing

10. A helicopter may be required to operate in adverse ambient conditions and therefore engine intake anti-icing is a necessity. The main methods of engine anti-icing used on helicopters are:
a. Hot Air Bleed. Hot air is bled from one of the later stages of the engine compressor and fed to the engine intake system.
b. Electrical. Electrical elements are fitted to the intake cowls and electrical mats are fitted to the fuselage forward of the intakes.
c. Oil. Oil heated within the engine is passed through the lower compressor support struts.
d. Momentum Separation. An air dam placed in front of the intake forces the air stream to make a sharp change of direction and therefore velocity. During the change of direction the higher momentum of water particles causes them to separate from the main air stream. The rearward-facing intake effectively uses this system because to enter the intake air must turn through 180 degrees, throwing any ice or water droplets clear. The multi-purpose intake is able to close off the forward facing air path and open side intakes containing swirl vanes which spin out any water or ice in the air.
11. A typical example of the hot air bleed method of engine anti-icing uses hot air bled from the tenth stage of the engine compressors and fed to the majority of the forward locating engine frames located inside the intakes. In addition, hot air is supplied to the leading edges of the inlet guide vanes and through the nose of the starter bullet. The engine cowling intake flares are also heated by the circulation of hot engine oil, taken from the compressor main bearings.
12. Some credence has been given to the idea of mounting engines facing rearwards so that there is a much reduced icing problem. Unlike a high-speed conventional aircraft, a helicopter jet engine does not rely on ram air effect to increase performance. Also, with the engines mounted facing rearwards, there is no danger of a flameout due to airframe ice becoming dislodged and entering the engine intake.

## TRANSMISSION SYSTEMS

## Typical Layout

13. The transmission system is required to transfer power from the power unit to the rotors. The relative position of the power unit and rotors largely determines the layout of the transmission; in the majority of helicopters, the engines are mounted above the fuselage adjacent to the main rotor and this allows the use of a minimum number of gear-boxes. A typical layout for this type of transmission is shown in Fig 3.

## 12-8 Fig 3 Typical Transmission Layout of a Roof-mounted Engine



## Transmission System Components

14. The following transmission components are normally found in helicopters: main rotor gearbox, free-wheel unit, rotor brake, intermediate driveshaft, intermediate and tail rotor gearboxes.
15. Free-wheel Unit. The free-wheel unit automatically disengages the drive in the event of a power failure, thus preventing engine drag on the rotor. One such system consists of an outer cylindrical drive head and an inner driven cam containing a number of cam lobes. A retainer ring containing rollers is spring-loaded to bias the rollers in the direction of rotation. Under normal drive conditions this will cause the rollers to become wedged between the drive head and the lobes of the driven cams. If power failure occurs the reduction of momentum of the drive head allows the driven cam to overrun, thus destroying the wedge action of the rollers and cams. Another system uses a coiled spring which increases in diameter in the direction of drive and vice versa, and provides a drive against a sleeve; this drive is released in the overrun condition.
16. Main Rotor Gearbox. The main gearbox transmits the drive through epicyclic gearing to the main rotor shaft. The gearbox also provides drive for the tail rotor, generators, alternators, hydraulic and oil pumps, oil coolers and tachometers. The gearbox housing is normally light alloy castings with steel liners in the bearing recesses and steel inserts for the threads. The gearbox is usually bolted to the airframe at the base and is supported by A-frames or struts. The main gearbox is pressure lubricated by a spur gear oil pump and the oil is cooled in a radiator matrix through which air is forced by the oil cooler fan.
17. Main Rotor Brake. The main rotor brake is used to keep the rotor stationary while the power unit is being run up to normal operating speed and to stop the rotor quickly on shut down. The brake may be of
the disc or drum type and it is usually operated hydraulically. The brake is usually located close to the main gearbox; in some installations the brake is mounted on the intermediate drive shaft to the tail rotor.
18. Intermediate Drive Shaft. The intermediate drive shaft transmits the drive from the main gearbox along the rear fuselage to the intermediate and tail gearboxes. The shaft is carried on ball bearings housed in anti-vibration mountings. A limited amount of axial and lateral movement of the shafting is permitted by flexible couplings; these allow for vibration and flexure of the rear fuselage. On helicopters that fold the tail pylon for aircraft carrier stowage, a disconnect coupling is fitted at the tail pylon hinge line.
19. Intermediate and Tail Rotor Gearboxes. The intermediate gearbox changes, by means of bevel gears, the angle of the drive from the intermediate drive shaft to the tail rotor drive shaft. The tail rotor gearbox reduces the speed and changes the direction of the drive to the tail rotor by means of a pair of spiral bevel gears. The tail rotor gearbox also contains the tail rotor pitch control mechanism. The pilot's control movement is transmitted from the yaw pedals to the pitch control shaft, which passes through the centre of the gearbox (Fig 4). Axial movement of the pitch control shaft takes place through the centre of the output gear shaft to the pitch control beam (but the pitch control shaft is splined onto the output shaft and, therefore, rotates with it). Both gearboxes are immersion and splash lubricated.

## 12-8 Fig 4 Tail Rotor Gearbox


20. Torquemeter Transducer. On most twin engine installation helicopters each engine alone can give sufficient power for safe flight, therefore the power output of the engines has to be limited. Transducers of the strain gauge type are often installed in the transmission system so that the level of torque being transmitted can be displayed on a torquemeter in the cockpit and, thus, kept within the limitation of the transmission system.

## Gearbox Condition Monitoring

21. The condition of gearboxes is monitored in a similar manner to the methods used on engines. A magnetic chip detector is mounted in the gearbox, usually in the sump, and the ferrous wear debris collected by the chip detector is analysed periodically. Also, oil samples are taken periodically for spectrometric oil analysis. This analysis has the advantage of measuring both the ferrous and nonferrous wear particles in suspension in the oil. These methods enable the detection of incipient failures and also economically allow the extension of a healthy gearbox beyond its normal overhaul life.

## Conformal Gears

22. The advantage of the conformal mesh over the involute mesh is that the involute mesh only has a line contact between the two teeth of meshing gears, whereas the conformal mesh has an area contact such that higher loads can be taken by conformal gear teeth (Fig 5). For the same load, a conformal gear can have a reduction in the number of teeth on pinions, thus giving a greater gear reduction per stage so that fewer stages are necessary in a main rotor gearbox. This leads to a reduction in the size and weight of the gearbox and an increase in transmission efficiency, as there are fewer gears and bearings to cause friction. There is also a corresponding increase in reliability and maintainability with a more simple and compact gearbox. Conformal gears need high standards of manufacture, and depend on the centre between gears remaining constant. As gearboxes distort under load, the gearboxes containing conformal gears are constructed more rigidly, therefore some of the weight advantage of conformal gears is counterbalanced by heavier gearbox castings.

## 12-8 Fig 5 Gearbox using Conformal Gear



## ROTOR HEADS

## Feathering Hinge

23. The rotor head must contain a feathering hinge for application of collective pitch and in order that cyclic pitch changes can be applied for horizontal flight. The velocity of the air over the rotor due to forward flight produces asymmetric aerodynamic conditions. The helicopter can only be prevented from rolling over by equalizing the lift moment on the advancing and retreating blades. This is effected either by hinging the blades to the hub, or it can be equalized deliberately by cyclic feathering of the blades.

## Gimbal-mounted Teetering Rotor

24. In the simple teetering rotor, the two blades are rigidly connected to each other with a built-in coning angle and gimbal-mounted to the rotor shaft (Fig 6).

## 12-8 Fig 6 Gimbal-mounted Vectoring Rotor


25. The weighted bar attached below the rotor is an aid to stability. The bar rotates with the rotor and, like a gyroscope, tends to maintain a given plane. Control levers from the cyclic and collective pitch mechanisms are linked to the bar. Any tilt of the rotor disc tends to be corrected automatically by a system of mixing levers leading from the bar to the cyclic pitch mechanism of the blades.

## Fully Articulated Rotor Head

26. The fully articulated rotor head (Fig 7) allows the rotor blade to move about three hinges. The blade is allowed to flap vertically about a horizontal hinge (flapping hinge) and to move in the plane of rotation about a vertical hinge (drag hinge). These hinges consist of trunnions mounted in bearings. The blade is also allowed to change pitch about the feathering hinge which is usually outboard of the flapping and drag hinges.

## 12-8 Fig 7 Fully Articulated Rotor Head


27. Blade Flapping Constraints. A fully articulated rotor head is usually fitted with two bladeflapping constraints, viz:
a. Flapping Restrainer. The flapping restrainer prevents the blade from flapping violently in gusty conditions when the rotor head is at low rpm or stationary. When the rpm increases, the centrifugal force is sufficient to overcome a spring causing the flapping restrainers to break a geometric lock and swing outwards, thus permitting the full range of blade flapping for control purposes.
b. Droop Restrainer. The droop restrainer limits the droop of the blade when the blade is rotating below normal speed or is at rest. As the rotor speed increases, the centrifugal force overcomes a spring and carries a cam arm outwards so that a flap pad drops and allows the blade full freedom to flap downwards for control purposes.
28. Drag Dampers. If the rotor blades could swing excessively about the drag hinges, the rotor would be unbalanced and severe vibrations would develop. A drag damper is attached between each blade and the rotor hub and limits the rate and extent of the movement of the blades about the drag hinge. It also absorbs any shocks which might otherwise be transmitted from the blade to the rotor head during acceleration or deceleration. Damping can be carried out using either friction or hydraulic dampers.
29. Hydraulic Damping. Hydraulic damping is achieved by allowing hydraulic fluid to pass from one side of the piston to the other. A differential check valve controls the speed and flow of the fluid through passages in the damper cylinder wall. Two relief valves in the damper piston, which operate in opposite directions, allow rapid transfers of fluid during rapid changes of rotor speed. Rubber shock absorbers are used to limit the travel of the piston and each damper has a fluid replenishment system.
30. Delta-three Hinges. The delta-three hinge is designed to improve stability of the rotor head. When the flapping hinge is mounted at right angles to the span of the blade, the blade does not change pitch during flapping. Instead, the flapping hinge can be set at an angle, thus when the blade flaps up its pitch angle is reduced and the blade tends to reduce its angle of attack (Fig 8a). The stability of the helicopter is improved, as dissymmetry of lift will not cause such a large inclination of the disc due to flapping. Setting the flapping hinge at an angle is not practical because the pitch change mechanism would be affected. However, the delta-three effect can be achieved by offsetting the pitch control horn, as shown in Fig 8b.

## 12-8 Fig 8 Blade Configuration

Fig 8a
Comparison of Normal and Delta-three Theoretical


When a Blade flaps up the Pitch Angle remains the same


When a Blade flaps up the Pitch Angle is reduced

Fig 8b
Achievement of Delta-three Configuration by Position of Pitch Control Arm

31. Lubrication. The rotor head hinges and components are subjected to high centrifugal loads and constant oscillatory movement during normal flight, therefore a reasonable degree of lubrication is required at all times. This is obtained either by periodic lubrication on the ground through nipples
adjacent to the rotor head bearings, or by an automatic flight system which uses centrifugal force and air pressure from an accumulator to force grease or oil into the bearings during flight.
32. Blade Folding. On the ground most helicopters have the ability to fold the rotor blades along the fuselage for picketing or stowage. This is normally achieved by removing all but one of the blade securing pins and swinging the blade on to the fuselage. Each blade is then secured to the fuselage by straps or frames. On aircraft carriers, where there is an operational requirement to fold and spread the blades rapidly, an automatic system can be incorporated which is controlled from the cockpit and operated by hydraulic pressure. The necessary sequencing during blade folding is electrical.

## Semi-rigid Rotor

33. Better performance, improved handling, simplicity and less maintenance can be achieved by replacing the flapping and drag hinges of the articulated rotor with flexible portions at the root of the blade and hub (Fig 9). The flexible portions allow the blade to move in the flapping and dragging planes but are obviously more rigid than hinges. Cyclic feathering is used to equalize the rolling moment during forward flight. The unstable pitching moment due to vertical gusts and pitching motions are greater, and cross coupling effects cannot be avoided. However, very powerful control movements can be generated and the pitching and rolling moments are heavily damped using automatic stabilization to improve handling qualities. A tendency to become unstable at high speed due to incidence instability can be coped with by design of an autostabilizer. Titanium and high performance and non-metallic composite materials have the necessary high strength and high flexibility required for the construction of a semi-rigid rotor hub.

## 12-8 Fig 9 Semi-rigid Rotor


34. The problems of air and ground resonance are different with the semi-rigid rotor due to the flexibility of the blades and the hub. These can be eliminated by either incorporating damping within the blade structure, or fitting hydraulic dampers.

## FLYING CONTROLS

## Swash Plate System

35. The swash plate or azimuth star is divided into two sections. The upper section (rotating star) is connected to the rotor shaft by a scissor link so that it rotates at the same speed as the rotor (see Fig 10). It is also mounted on a ball joint so that it can be tilted in any direction. Tilting of the rotating star alters the blade pitch angles cyclically through the pitch control arms on the rotor blade sleeves. The lower section (non-rotating star) is mounted on the rotating star by bearings and is kept stationary by a
scissor link connecting it to the main gearbox housing. The push/pull rods from the pilot's cyclic and collective pitch control rods are connected to the non-rotating star. The ball joint on which the stars are mounted is a sliding fit on the main rotor shaft. Collective pitch changes are made by moving the whole swash plate bodily up and down while maintaining the tilt constant.

## 12-8 Fig 10 Swash Plate System


36. On swash plate systems, changes of both cyclic and collective pitch are made by moving the star assemblies. Operation of the collective pitch lever and the cyclic pitch control column is combined in a mixing unit which transmits the resultant compound movements to the star assemblies.

## Phase Lag and Advance Angle

37. Due to inertia effects the blades give the desired flapping (up to down) approximately $90^{\circ}$ after the blade pitch has been altered by cyclic changes. To achieve correct tilting of the rotor disc the pitch is altered $90^{\circ}$ before the point at which the desired flapping is required. This effect is called phase lag. In practice, pitch control operating arms are attached at points ahead of the blades they control, the angular distance being known as the advance angle. To correct for the full effect of phase lag, the angular displacement of the fore and aft operating rod or servo-jack from the centre line of the helicopter is, therefore, the difference between the phase lag and the advance angle. The port and starboard lateral operating arms and servo jacks are $90^{\circ}$ disposed to the fore and aft arms.

## Spider System

38. In the spider system of pitch control (Fig 11) the arms of the spider are connected to the leading edge of the blades by control rods, the spider spindle being situated inside the rotor shaft. A ball joint mounting allows the spider to tilt when cyclic pitch changes are made. Collective pitch changes are made by raising or lowering the whole spider.

## 12-8 Fig 11 Spider System of Pitch Control



## Powered Flying Controls

39. The pitching moments arising from aerodynamic and centrifugal forces give resistance to the application of collective pitch. Also, there is a lateral cyclic stick force which increases with forward speed. This means that powered flying controls are necessary to provide sufficient force to operate the controls satisfactorily.
40. A considerable force is required to change the pitch of rotating rotor blades and, apart from the smaller types, most helicopters incorporate some means of assisting the pilot's control effort. This usually takes the form of hydraulically powered servo-jacks fitted to the control system at its input to the spider or, as shown in Fig 12, to the non rotating star. Both main and tail rotor controls may be power assisted and there is provision for reverting to manual control if a hydraulic system failure occurs. On the larger helicopters, the control forces are too great for manual control and an emergency hydraulic system is activated automatically if the normal system fails.

## 12-8 Fig 12 Powered Control Arrangement


41. After a failure of one system, control is satisfactory but prolonged operating is not recommended. The relevant Aircrew Manual will advise on the action to be taken following a hydraulic failure that affects control.
42. Artificial Feel and Trim Control. Artificial feel requirements for the helicopter are simple because of the very limited speed range. Artificial feel is normally only fitted to the cyclic pitch controls and is only a constant rate system as provided by spring altering the datum position of the control column in relation to the spring feel. This can be done either by releasing a clutch and repositioning the control column, or the feel can be trimmed slowly by an electrical actuator unit. The collective pitch normally has only a friction device to maintain the lever in the required position.

## Automatic Control

43. Unlike fixed-wing aircraft, the helicopter is basically unstable in flight. The addition of autostabilization improves the handling of most helicopters and, at the same time, makes them less tiring to fly. More comprehensive automatic systems can be programmed to perform transitions to and from the hover, and to hover at a pre-selected height.

## ROTOR BLADES

## Blade Construction

44. The latest rotor blades are of composite construction using lightweight materials of great strength and resilience. A typical blade and the materials used in its construction are shown in Fig 13.

12-8 Fig 13 Composite Main Rotor Blade

45. When combined with the semi-rigid rotor, which was described in para 33, composite blades considerably improve the forward speed and manoeuvrability of a helicopter.
46. Some blades are constructed of glass reinforced plastic and stainless steel in preference to aluminium alloys which have a lower fatigue life. The trailing edges are normally stiffened with a light
honeycomb structure. The stainless steel/glass fibre blade is more resilient to erosion and as the blades are fabricated and not machined then a non-linear twist and non-parallel planform can be incorporated into its design. The blade is attached to the rotor head assembly by a steel fitting which is attached to the blade root by two bolts. An integral arm facilitates attachment for drag link dampers (Fig 14).

## 12-8 Fig 14 Blade Root End



## Balancing

47. Rotor blades are balanced chordwise to minimize the couple between the inertia axis of the blade and the aerodynamic centre at which the lift can be considered to act. Without chordwise mass balance, the inertia axis is well behind the aerodynamic centre. With upward acceleration the blade would tend to twist, increasing the angle of attack and hence increasing lift still more. This could be catastrophic as a violent type of blade oscillation or flutter could result. The counter-balance weight is either secured or bonded into the leading edge of the blade spar.
48. Spanwise balance is adjusted during manufacture by balancing individual trailing edge skin sections and by balance weights fitted at the outboard end of the spar. The strict weight control and static and dynamic balancing which the blades receive during manufacture permit interchangeability of individual blades.

## Blade Development for Higher Forward Speed

49. The British Experimental Rotor Programme (BERP), a co-operative effort between the UK Government and Westland Aircraft, has produced a blade design that improves rotor forward speed performance by delaying both retreating rotor blade stall and compressibility effects.
50. Blade Camber. To improve blade $\mathrm{C}_{\llcorner\max }$ and, therefore, the stalling limit, cambered blade aerofoil sections are required. Traditional blades are uncambered to avoid the pitching moments and blade twisting associated with cambered sections. The BERP design uses a cambered section for $15 \%$ of the blade's span, just inboard of the tip where high lift capability is mainly required. Inboard of this, a reflex trailing edge cambered section is used to counteract the pitching moment of the cambered section. The slight $\mathrm{C}_{\mathrm{Lmax}}$ penalty imposed by this inboard section is more than offset by the increase in $\mathrm{C}_{\mathrm{Lmax}}$ achieved by the cambered section of the blade.
51. Blade Tip Design. To improve the critical mach number of the tip the BERP blade tip leading edge is progressively swept to a maximum of $30^{\circ}$. To maintain the tip CG coincident with the blade CG, the complete swept tip section is moved forward, and to locate its Centre of Pressure (CP) on the
blade pitch axis, the tip chord and area distribution is adjusted. This design also improves the thickness/chord ratio and gives the tip its distinctive appearance (see Fig 15).

## 12-8 Fig 15 The 'BERP' Blade Tip


52. Blade Tip Vortex. The outermost part of the blade tip is sharp edged and highly swept. At any significant angle of attack, this extremity, which is effectively a delta wing, forms a powerful vortex which moves inboard along the curved leading edge until eventually the entire tip is in the stable vortex flow (see Fig 16).

## 12-8 Fig 16 Vortex Behaviour


53. Beneficial Effect of Vortex. The BERP tip itself does not increase blade $\mathrm{C}_{\text {Lmax }}$, but the stable flow it produces allows the cambered part of the blade to reach its high $\mathrm{C}_{\mathrm{Lmax}}$ without the tip stalling. When the blade does finally stall, the vortex, formed at the leading edge notch where the tip meets the blade, restricts the outward flow of the boundary layer and reduces the severity of the stall.

## Blade Inspection Method

54. The extended spars of a rotor blade can suffer from fatigue or damage. As the failure of a rotor blade would obviously be catastrophic, a system has been developed for checking the integrity of the blade spars. The system is known as Blade Inspection Method (BIM) and consists of a cylindrical indicator situated at the blade root. The blade spars are permanently charged with pressurized nitrogen and the BIM indicator compares the blade spar nitrogen pressure with its own datum pressure. When the spar pressure is within prescribed limits, the indicator shows a series of coloured stripes (usually yellow or white), but any cracks developing in the spar will cause a loss of pressure which will be shown by the exposure of different coloured stripes (usually red or black) on the indicator. The BIM indicator normally has a test facility to check its serviceability.

## TAIL ROTORS

## Tail Rotor Hub

55. The tail rotor hub (Fig 17) is similar in construction to a fully articulated rotor hub, but only flapping and feathering hinges are necessary. The hub is splined and secured to the horizontal drive shaft of the tail rotor gearbox, and pitch changes are accomplished through the pitch change beam and pitch control shaft which is located in the centre of the tail rotor gearbox. The blades are allowed to flap, and the differential thrust of the advancing and retreating blade can be alleviated by the blades flapping independently in conjunction with Delta-three hinges. Each blade is counter-balanced by weights attached to the hub, to assist the pilot to increase pitch.

## 12-8 Fig 17 Tail-Rotor Hub



## Tail Rotor Blade

56. The tail rotor blade is normally of all-metal construction. The leading edge and spar section is formed of a light alloy extruded section. The light alloy sheet skin is reinforced internally by a honeycomb core and bonded to the spar. A polyurethane or stainless steel strip is bonded along the leading edge of the blade to prevent erosion. The blade is also balanced chordwise and spanwise.

## Shrouded Tail Rotor

57. The conventional tail rotor operates in difficult vibratory and aerodynamic conditions due to its position at the rear of the fuselage and the very severe interference with the main rotor stream, the fuselage wake and the fin. Due to these severe operating conditions, the conventional tail rotor is subjected to considerable stresses which impose a limit to the service life of its components and also generally demands a rugged design. Further disadvantages are its susceptibility to foreign object damage and its danger to ground personnel.
58. One solution to the disadvantages of the conventional tail rotor is the shrouded tail rotor or 'Fenestron'. It consists of a rotor with several small blades hinged about the feathering axis only and rotating within a shroud provided in the tail boom or fin of the helicopter (Fig 18). It is light and less vulnerable to damage by either loose objects or obstructions and is less of a hazard to personnel on the ground in the vicinity of the helicopter. However, a servo-unit is required for pitch control because of the high and variable aerodynamic forces encountered in the hover.

## 12-8 Fig 18 Shrouded Tail-Rotor (Fenestron)



## DESIGN DEVELOPMENTS

## Speed Limitations

59. There are four main factors which affect the maximum forward speed of a helicopter:
a. Compressibility effect on the advancing blade.
b. Retreating blade stall.
c. Reverse flow on the retreating blade.
d. Design limitation of the cyclic pitch control.

The limit on the forward speed of a helicopter is dependent upon the amount of lift force and propulsion force that the rotor is required to generate per unit area of rotor blade; by reducing the airframe drag and reducing the rotor loading the higher speeds can be exploited. The following paragraphs briefly examine some of the designs for increasing the speed range of modern and future helicopters.

## Streamlining

60. More attention has been given to streamlining helicopters and many helicopters now have retractable undercarriages. However, there is still room for improvement, particularly in the reduction of rotor hub drag. A fully articulated hub can account for half the total drag if mounted on a clean airframe. Some hubs have been partially covered by fibreglass fairings (Fig 19) but for the fairing of rotor hubs to be effective, the fairings must be completely sealed otherwise the faired drag can exceed the drag of an unfaired hub.

## 12-8 Fig 19 Faired Rotor Hub and Blade Roots



## Compound and Convertible Helicopters

61. The fully compounded helicopter is one provided with both wings and a forward propulsion system which is independent of the main rotor (Fig 20). In forward flight, the rotor is unloaded to varying degrees, depending on the particular design, and, in some aircraft, the rotor is in a state of autorotation.

12-8 Fig 20 A Compound Helicopter Configuration


Studies are at present directed towards stopping the rotor in flight, and thus further decreasing the drag. A further development of this idea is the folding of the stopped rotor blades into a low drag configuration, or even stowing the folded rotor in the fuselage during conventional wing borne flight (Fig 21). The design problems to be overcome include the aeroelastic difficulties of stopping a rotor at fairly high speeds, and the mechanical, structural weight and stowage volume penalties incurred. Accordingly, there are no flying examples using this technique at present.

## 12-8 Fig 21 Stowed Rotor Concept



## Partially Compounded Helicopters

62. Partial compounding can be achieved by the addition of either a wing, or an independent forward propulsion system. Although they permit speed increases, both systems introduce problems. Where a wing is used in addition to the rotor, the main problem is sharing the lift between the two whilst remaining within the rotor's rpm and flapping limits. The retention of the rotor to provide forward propulsion may also incur unacceptable nose-down attitudes of the fuselage at high speeds, or an excessive range of cyclic stick movement. The addition of a forward propulsion system allows the rotor to approach autorotation, ie the helicopter becomes an autogyro in forward flight.

## Advancing Blade Concept

63. The airflow velocity over the retreating blade of a helicopter is so reduced at high speed that the lift that it is able to generate is very small. In the conventional helicopter the blades are allowed to flap in such a manner that the effective angle of attack of the advancing blade is reduced, and thus the lift it gives is small and balances the lift on the retreating blade in the lateral sense. Therefore, the advancing rotor blade is inefficient, as it is working at low angles of attack and low lift/drag ratios. The Advancing Blade Concept (ABC) utilizes rotor blades that are rigid in the flapping sense so that a sensibly fixed aerodynamic incidence is maintained all around the rotor disc, thus generating high lift from the advancing blade at an efficient lift/drag ratio. Two rotors must be used co-axially to balance the tendencies of the overturning movement towards the retreating blade. One disadvantage of this concept is the probability of high interference drag and vibration between the two rotors.

## Tilt-rotor and Tilt-wing Helicopters

64. Tilt-rotor and tilt-wing designs offer similar solutions for overcoming the limitations of cyclic pitch control at high forward speeds. In the tilt-rotor design (Fig 22a), the rotors are driven by engines housed in nacelles or pylons at the wing tips. These nacelles can be swivelled from the horizontal position for forward flight, through to the vertical position for rotor-borne flights. Although the diameter of the rotors would preclude landing or taking-off with the nacelles in the fully forward position, an intermediate tilt angle might be used for STOL operations when the aircraft auw is above the maximum for VTOL. The tilt-wing design (Fig 22b) operates on the same principle as already described, with the difference that the whole wing tilts with the engine nacelles.

## 12-8 Fig 22 Tilt-rotor and Tilt-wing Aircraft

Fig 22a Tilt-rotor Design


Fig 22b Tilt-wing Design

65. The main problems to be overcome in these two configurations are vibration and stability of the rotor, pylon and wing combination, and the provision of suitable controls for the various phases of flight.

## Payload Increases

66. Early helicopters had a disposable load (payload plus fuel) of about $25 \%$ of the gross weight, whereas a typical modern machine has a disposable load of $50 \%$ or more. This change has been largely brought about by the introduction and development of the free turbine engine. On initial consideration it may appear that the trend of increasing disposable loads could not continue, but the indications are that it can and will do so. Considerable advances in turbine engine technology have been incorporated in the turboshaft engine. The high performance triple-shaft engines now under development will give specific fuel consumption in the order of 0.4 and less, compared with the best of 0.4 to 0.6 in current engines. The smaller volume of these engines and the use of new materials will also achieve useful weight reductions.
67. The adoption of new material, the use of super-critical shafting and new speed reduction methods, such as harmonic drive gearboxes, new gear tooth forms and new bearing technology, could all offer considerable reduction in transmission weights. It is more likely that the steady rate of improvement will be maintained and the weight advantage used to improve the life and reliability of components. Only marginal weight saving is envisaged in rotor blade design as, although new materials such as carbon fibre and boron filament are being applied, the blades of the future must withstand the greater loads imposed by high speed flight, and achieve an improvement in their lives and reliability. This may be compensated for by a useful reduction in hub weight as elastomeric or flexible member designs continue to replace traditional articulated design.
68. New structural methods of analysis using computer techniques should continue to produce somewhat lighter airframe structures, aided by the introduction of new materials. The introduction of fixed wings can be shown not to increase the basic weight of the aircraft, as the wing doubles as, and replaces, the undercarriage support structure and also provides space for fuel tanks. Avionics and communications equipment weights may continue to increase in spite of miniaturization because of the more advanced and comprehensive systems being adopted.

## NOTAR Anti-torque System

69. The NOTAR (NO TAil Rotor) system, as its name implies, does away with the conventional tail rotor and is instrumental in reducing noise. It uses an engine-driven fan to force air under pressure through the hollow tail boom and out of a grille so that it deflects the main rotor downwash and opposes its torque. A side force is generated sufficient to provide some $75 \%$ of the thrust required to counteract torque in the hover. A rotating thruster, also driven by air from the fan, supplies the remaining $25 \%$ of side force. Tailplane fixed aerofoil configuration varies between experimental types and may consist of twin fins (see Fig 23a) or offset aerofoil surfaces (see Fig 23b) which provide much of the anti-torque reaction in forward flight when the rotor downwash is less affected by the NOTAR side force. Collective lever and pilot's pedal movement is transmitted to the fan, thruster and fins automatically to adjust for changes in torque and manoeuvre.

## 12-8 Fig 23 The NOTAR System

Fig 23a Twin-fin Configuration


Fig 23b Offset Aerofoil Arrangement

## Smart Material Actuated Rotor Technology (SMART).

70. One of the fundamental problems in rotor design is how to produce a blade which can alter its twist distribution in flight, an attribute which would markedly reduce rotor vibration. Blade twist design tends to rely on a compromise between requirements in the hover and those best suited for forward flight. The most efficient designs for hover require high power but cause excessive vibration in high-speed cruise. The highly-twisted blade, desirable for the hover, is difficult to trim in forward flight because the advancing blade produces more lift than is capable of being balanced by the retreating blade.
71. The Smart Material Actuated Rotor Technology (SMART) rotor system holds the promise of actively altering blade twist in flight. It uses so-called 'smart' materials embedded in the rotor blade to produce a twisting moment which can be controlled by altering the electrical voltage applied to the material. The system for twisting the blade uses piezo-fibre composites. These are embedded in an epoxy matrix along with glass-fibre reinforced plastic inserts and are activated by electrodes which may be excited appropriately according to the flight regime.
72. One advantage of the system is that no alteration to the rotor drive-train is needed. Furthermore, an increase in range of some $15 \%$ is envisaged. However, the technology brings with it a weight penalty of some $10 \%$, which may counter any increase in payload.

## CHAPTER 9 - TANDEM ROTOR HELICOPTERS

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## Introduction

1. Tandem rotor helicopters are not a new design concept. The first successful designs were built and flown in the 1930s. Over the years there have been many variants and in the 1960s the Royal Air Force gained considerable experience in tandem rotor operation with the Bristol Belvedere. This chapter sets out the advantages and disadvantages of tandem rotor helicopters and considers those aspects of control which differ from single rotor types. The following text is broadly based on the Chinook helicopter.
2. The major advantages of a tandem rotor helicopter compared with the single main rotor helicopter are:
a. Contra-rotating rotors dispense with the need for an anti-torque tail rotor, thereby making more power available for lift with the advantage of greater load carrying potential, but see para 4.
b. A large range of fore and aft centre of gravity positions are permitted, since it is possible to generate larger longitudinal control moments than conventional helicopters by use of differential collective pitch (DCP).
c. The internal cabin space has a large volume area in relation to total fuselage size.
3. The major disadvantages of the tandem rotor helicopter, compared with the single rotor helicopter, are:
a. Transmission weight is higher. The transmission system is complex in order to achieve intermeshing of the main rotors and provide single engine capability.
b. Vibration levels tend to be higher than single main rotor helicopters because of the aerodynamic interference between the rotors.
c. Blade folding may be required due to the large overall dimensions of the rotors.
d. There are stability problems in pitch and yaw.

## Control of Tandem Rotor Helicopters

4. The operation of two rotors in close proximity will modify the airflow of each, hence the performance of the rotor system will not be the same as for the isolated main rotor. Lift is produced conventionally but, since contra-rotating rotors cancel inherent rotor torque, an anti-torque rotor is not required. The Chinook suffers an interference power loss of the same order as the power required to drive a conventional tail rotor, but to a large extent this can be negated if the wind is positioned on the left side so that the non-interlaced portions of the rotor system experience advancing blade conditions. Savings of up to $10 \%$ torque can be made. Longitudinal control is achieved by DCP; moving the cyclic stick forward decreases the pitch of the forward rotor and increases that of the aft rotor and vice versa. A differential airspeed hold (DASH) system ensures that a positive stick gradient is maintained
throughout the speed range. Roll control is achieved by tilting the rotors laterally by an equal amount in the same direction using the cyclic stick. Yaw control is achieved in the natural sense by tilting the rotors laterally in opposition by an equal amount using the yaw pedals. Longitudinal cyclic trim is incorporated to enable the aircraft to be flown throughout the speed range in a substantially level attitude, thereby relieving stress on the rotor shafts and reducing drag.
5. Longitudinal Control. Longitudinal control is achieved on the Chinook by DCP. Moving the cyclic stick forward decreases the pitch of the forward rotor and increases that of the aft rotor and vice versa. Since the forward rotor mast has a greater tilt forward than the aft mast, there is a need for comparatively less collective pitch on the aft rotor at higher speed, ie a negative control gradient. This has to be counteracted by the use of a differential airspeed hold actuator which lengthens the longitudinal control runs with variations in speed, thereby establishing an artificial positive stick gradient. There is another pitch stability problem with tandem configurations caused when the nose pitches up about the CG. The rear rotor has a decrease in angle of attack and hence lift, while the front rotor senses an increase in both angle of attack and lift. This is destabilizing, and as it is aggravated by the rear rotor operating in the downwash of the front rotor, could lead to the rear rotor stalling.
6. Directional Control. Directional control is achieved by application of the rudder pedals which tilts both rotors laterally in opposition. There is a low residual side force in comparison to conventional helicopters, and very little weather-cocking tendency in low speed flight due to the nearly equal keel area ahead of and behind the CG. In forward flight the large rear pylon contributes to the directional stability. However, the forward pylon can act in an adverse sense with any sideslip. In the Chinook it was found necessary to add stall strips to the front pylon to reduce its destabilizing effect.
7. Lateral Control. Roll control is achieved by tilting the rotors laterally by an equal amount in the same direction using the cyclic stick. This produces the desired rolling moment and sideforce.
8. Vertical Control. Lift distribution between the two rotors may not be identical. For example on the Chinook at mid CG it is approximately $45 \%$ on the front rotor, $55 \%$ on the rear rotor. Application of collective pitch is similar to that of conventional helicopters.

## Control Cross-coupling

9. Power changes will cause some change in pitching due to the unequal lift distribution. There may be some slight longitudinal acceleration or deceleration with power changes due to the tilt of the rotors.
10. There may be some control cross-coupling in roll and yaw due to different mast heights and distances from the CG.
11. On the Chinook with the Automatic Flight Control System engaged, there is very little noticeable cross-coupling with power changes.

## CHAPTER 10 - RANGE AND ENDURANCE

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## Introduction

1. The principles of flying for range or endurance in a helicopter are basically the same as for fixed wing aircraft. However, the speed and height range of helicopters is normally less than that of fixed wing aircraft and, in addition, the helicopter pilot has a smaller choice of engine settings available to him than does a fixed-wing pilot.

## Definitions

2. Range and endurance are defined as:
a. Range. The distance that can be covered for a given quantity of fuel.
b. Endurance. The period of time that an aircraft can remain airborne for a given quantity of fuel.
3. For both range and endurance the criterion is fuel consumption. For range flying the best ratio of distance covered to fuel consumed must be achieved, ie the aircraft must be operated at maximum efficiency. In the case of flying for endurance, the minimum fuel consumption for straight and level flight must be achieved.
4. Specific Fuel Consumption (SFC). Specific fuel consumption is the relationship between the power output and the fuel consumption of an engine. SFC is expressed as kg of fuel per hr per kW of power.

## Range

5. Maximum range in a helicopter is achieved by operating at the best speed and also at the best height for range. In both cases, the efficiency of the engine and rotor must be taken into account.
6. Range Speed. When considering range speed it is necessary to take into account the efficiencies of both engine and rotor.
a. Engine Efficiency.
(1) Fixed Spool Engine. The compressor of a fixed spool engine produces a fixed mass of air to the combustion chamber. When a small amount of power is required from the
engine only a small amount of air is required to achieve the correct fuel/air ratio and much of the energy used in producing the compressed air is wasted. As the airspeed is increased more power is required from the engine and hence a greater proportion of the compressed air is used in combustion. Hence, as airspeed increases the engine becomes more efficient and SFC decreases.
(2) Free Power Turbine Engine. As airspeed is increased there is a requirement for increased fuel and air. The engine compressor speeds up to provide the correct mass flow of air and as engine speed increases the engine becomes more efficient and SFC decreases.

Thus with both engine types, although fuel consumption increases with an increase in airspeed, it can be seen that SFC decreases.
b. Rotor Efficiency. In Volume 12, Chapter 6, Para 7 it was shown that parasite drag increased with airspeed. Progressively increasing drag leads to a decrease in rotor efficiency. The rotor is most efficient at the helicopter's minimum drag speed.
c. Combined Engine and Rotor Efficiency. Since the engine is most efficient at high airspeed and the rotor is most efficient at minimum drag speed, allowances must be made for each separate factor and a compromise is necessary in order to ensure the best overall efficiency. The compromise is achieved when the helicopter is flown at the best TAS/Drag ratio. The best TAS/Drag ratio occurs at the point of maximum increase of TAS for minimum increase of drag. This relationship is found by drawing a tangent to the drag curve from the origin of the graph, Fig 1.

## 12-10 Fig 1 Best TAS/Drag Ratio


7. Range Height. The consideration of the best height for range flying must also take account of both engine and rotor.
a. Engine Considerations.
(1) Fixed Spool Engine. The fixed spool engine is designed to provide sufficient air for combustion at high density altitudes. At low density altitude the mass of air that is compressed is greater than that required for combustion and much of the energy used in generating the air is wasted. As the density altitude is increased, air density decreases and more compressed air is needed. In addition as air density decreases, drag on the compressor decreases.
(2) Free Power Turbine Engines. As density altitude increases the compressor speeds up to compensate for the decreased density and to provide the correct mass flow of air required for combustion. As the compressor speed increases its efficiency improves and, in addition, as air density decreases the drag on the compressor also decreases.

Thus in both cases the efficiency of the engine increases and SFC decreases with an increase in density altitude.
b. Rotor Efficiency. For a given airspeed, as density altitude is increased collective pitch must be increased to maintain total rotor thrust. When collective pitch is increased induced power increases but rotor profile power reduces. There will be an optimum altitude where the total power required from the rotor is at a minimum. This occurs when rotor profile power has reduced more than the induced power has increased. This is the altitude at which the rotor is most efficient and can be obtained from the Operating Data Manual (ODM) for the aircraft. Any further increase in height above the optimum will decrease rotor efficiency.
c. Combined Engine and Rotor Efficiency. Maximum range is obtained at a compromise height for engine and rotor efficiency. Range flying information is obtained from the aircraft ODM which should be consulted to find the correct operating height and speed for the ambient conditions.

## Endurance

8. Maximum endurance in a helicopter is achieved by flying at the speed and height for minimum fuel consumption.
a. Endurance Speed. The aim of flying for endurance is to achieve the lowest possible fuel consumption. Since fuel flow varies with power output it follows that for maximum endurance the helicopter should be flown at minimum power speed for level flight, Fig 2. For most helicopters the minimum power speed is between 60 kt and 70 kt .

## 12-10 Fig 2 Endurance Speed


b. Best Height for Endurance. As explained in paragraph 7a, there is less drag on the engine compressor as density altitude increases and the engine becomes more efficient. It, therefore, follows that overall fuel consumption falls and endurance increases as density altitude increases.
c. Combined Effects of Speed and Height on Endurance. Speed for best endurance will always be the minimum power speed. Endurance will increase as density altitude increases. Specific calculations can be made with reference to the ODM.

## Effect of Wind

9. Because of the relatively low speed of helicopters, wind more often than not has a great effect on range and in a majority of situations it will be the overriding factor when selecting the height and speed at which to fly. Flying at a height in excess of that recommended in the relevant ODM for maximum range may be advantageous in the case of a strong tail wind. Conversely, it may sometimes be better to fly lower than the recommended height if strong headwinds are encountered or to increase speed at the expense of fuel consumption in order to achieve a satisfactory ground speed. It may also be advantageous to reduce speed, and therefore fuel consumption, when flying with a strong tailwind since an excellent ground speed, and hence range, will be obtainable at a reduced fuel consumption.

## Effect of Changes in All-up Weight

10. An increase in weight increases the power required and hence fuel consumption. Both range and endurance will be adversely affected. The carriage of external stores and weapons will increase parasite drag which will, in turn, decrease range and endurance.

## Summary

11. For best range, a helicopter should be flown at a speed which is a compromise between engine and rotor efficiency requirements which occurs at the best TAS/Drag ratio. The accurate speed can be determined from graphs in the ODM and corrected for wind as necessary.
12. The best height to fly at, for maximum range, is a compromise between the engine requirement for a high density altitude and the requirement for low rotor profile power and can be found in the ODM.
13. The best helicopter endurance is achieved by flying at minimum power speed at the density altitude specified in the ODM.
14. In the selection of height and speed at which to fly the wind velocity should be carefully considered in case it should be advantageous to fly at a height and speed which is at variance with that recommended in the relevant ODM.

## CHAPTER 11 - HELICOPTERS - WEIGHT AND BALANCE

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## Introduction

1. The captain of a helicopter will often be faced with the responsibility for loading his aircraft in the field and the importance of keeping the all-up weight (AUW) and centre of gravity (CG) within the permitted limits cannot be over-stressed. An incorrectly loaded aircraft may be capable of being flown under favourable conditions but its operational efficiency will be impaired and it may not be able to safely complete a flight. The method of calculating the AUW and the CG is considered in this chapter.

## Definition of Terms

2. The following terms are applied to helicopter weight and balance:
a. Basic Weight. This is the weight of the aircraft including all basic equipment and unusable fuel and oil, to which it is necessary to add only the weights of variable, expendable and payload items in the various roles to arrive at the AUW. The basic weight and moment can be found in the MOD Form 701 - Leading Particulars, at the front of the aircraft's Form 700.
b. Normal Maximum AUW. The AUW is the basic weight plus the disposable load - the crew, fuel and oil, passengers and cargo. The normal maximum AUW is found in the limitations section of the aircrew manual.

Note: A table of removable equipment, included in the basic weight is given in an aircraft's Form 700. The table is not an exhaustive list of all removable items of equipment included in the basic weight, since, in this context, the term 'removable' refers to:
(1) Those readily removable items of basic equipment without which the aircraft could still be flown safely although possibly without some particular facility.
(2) Those items of equipment about which some reasonable doubt could exist in user units as to whether or not they are included in the basic weight.
c. Basic Equipment. This is the non-expendable equipment which is common to all roles for which the aircraft is designed and includes inconsumable fluids, coolant, hydraulic and pneumatic systems.
d. Variable Load. The variable load consists of those items which may vary between sorties and which are not expendable in flight, such as crew and equipment and role equipment.
e. Expendable Load. This includes fuel, oil, armament and cargo/stores which may be airdropped, including parachutists.
f. Payload. The payload is the total load of cargo or passengers carried.
g. Operating Weight. The operating weight is the sum of the basic weight and variable load. When operating weight is subtracted from maximum AUW the result is the lifting capacity of the helicopter.

## Factors Affecting Take-off Condition

3. AUW. The maximum permitted AUW is a design figure which allows a laid down minimum rate of climb outside ground effect. Since rate of climb is affected by atmospheric conditions and wind strength, these factors must be considered when calculating the aircraft's take-off AUW. The operating data manual for the aircraft will contain the graphs which enable the pilot to make these calculations.
4. Longitudinal Balance. Not only must the helicopter be within the calculated take-off AUW but the load must be positioned to ensure that the CG remains within the fore and aft limits. In still air conditions with the rotor disc level the fuselage CG will be in line with rotor thrust. However, the fuselage attitude when in the hover will vary with the CG position and it may be necessary to use cyclic stick to keep the disc level, Fig 1. Provided that the CG remains within the permissible limits, the cyclic range available will be adequate for the permitted flight envelope.

## 12-11 Fig 1 Longitudinal Change in CG

Fig 1a CG on Datum


Fig 1b CG Aft of Datum

5. Lateral Balance. The lateral position of the CG normally changes very little with internal or underslung loads, but a weight on the winch can have an effect. Lateral displacement of the CG requires a compensating cyclic movement if the disc is to remain level. To avoid running out of cyclic stick control, particularly if there is an adverse side wind, it is important not to exceed the maximum permitted weight on the winch (see Fig 2).

## 12-11 Fig 2 Lateral Displacement of CG

Fig 2a Weight Internal/Underslung


Fig 2b Weight Internal Plus Winch


## Calculating the Position of the Centre of Gravity

6. The CG position is determined by finding the turning moment of individual items of equipment about a given datum, adding together all the moments and dividing the total moment by the total weight (see Volume 2, Chapter 22, para 14). The turning moment is found by multiplying the weight of the object by its distance from the datum. If the turning moment is clockwise it is considered to be POSITIVE and if anti-clockwise, NEGATIVE. A simple example of calculating the CG is shown in Fig 3.


The CG position is therefore 10 inches to the right, or on the positive side of the datum.
12-11 Fig 3 Calculating the CG - Positive Moments


Zero Moment
7. Provided that all the moments are taken about the same datum it is immaterial where the datum lies, as is shown by the following example (see Fig 4) where, using the same figures as in Fig 3, the datum has been taken as being 7 inches to the right of the left-hand 10 lb weight.

i.e. the CG position is 3 inches to the right, or on the positive side of the datum.

12-11 Fig 4 Calculating the CG - Positive and Negative Moments

> + = CG Position
> - $=$ CG Datum Point

8. The CG of a loaded helicopter can be calculated in the same way. For example, assume that the Form 700 of a helicopter records a basic weight of $5,000 \mathrm{lb}$ and a moment of 0 lb in . A flight is planned with a crew of two in the front seats, three passengers, 500 lb of baggage and a full fuel tank. Reference is made to the Aircraft Maintenance Manual and the following information extracted and added to the basic weight and moment:

| Basic Weight | $5,000 \mathrm{lb}$ | Basic Moment | 0 lb in |
| :--- | ---: | ---: | ---: |
| Pilot and Crew | 400 lb | Moment | $-16,200 \mathrm{lb} \mathrm{in}$ |
| Passengers | 600 lb | Moment | $-19,800 \mathrm{lb}$ in |
| Baggage | 500 lb | Moment | $-3,000 \mathrm{lb}$ in |
| Fuel | $\underline{1,500 \mathrm{lb}}$ Moment | $+31,000 \mathrm{lb}$ in |  |
| Totals | $\underline{8,000} \mathrm{lb}$ | $-\overline{8,000} \mathrm{lb} \mathrm{in}$ |  |
|  |  |  |  |
|  | CG position is $\frac{-8,000 \mathrm{lb} \text { in }}{8,000 \mathrm{lb}}=-1 \mathrm{in}$ |  |  |

Reference should now be made to the Aircrew Manual to see if this CG position lies within the permissible limits. The datum used for calculating moments will be found in the Aircraft Maintenance Manual.

Note: The datum is usually the rotor axis of rotation and moments are calculated with the helicopter facing to the left. Thus, minus CG values will give a nose-down attitude and plus CG values will give a tail-down attitude.

## Constructing a Graph for Plotting CG Position

9. To obviate the need for making mathematical calculations for every flight, the movement of the CG resulting from using fuel in flight, or by varying the load as the flight develops, can be presented graphically. The graph is constructed by drawing horizontal lines, equally spaced and at any convenient scale, to represent the varying weight of the helicopter (see Fig 5). A vertical line drawn
on the graph represents the datum. Where the permissible CG movement can be forward or aft of the datum this vertical line is drawn in the centre of the graph. Where the CG movement is all positive, the left-hand edge of the graph represents the datum.

## 12-11 Fig 5 Graph for Plotting CG Position


10. To indicate the CG position, sloping lines are drawn up from the base of the graph and numbered consecutively from the datum, positive values to the right, negative values to the left. The datum has a value of zero. The lines (representing the CG) slope because the CG position of a helicopter of, say, $4,000 \mathrm{lb}$ weight with a moment of $\pm 4,000 \mathrm{lb}$ in will be $\pm 1 \mathrm{in}$, but a helicopter of $8,000 \mathrm{lb}$, having the same moment, has a CG position of only $\pm 0.5 \mathrm{in}$. The $8,000 \mathrm{lb}$ helicopter therefore requires twice the moment for a CG of $\pm 1$ in.
11. To arrive at the correct degree of slope, first mark the base line of the graph at some suitable scale to indicate one-inch changes of CG position, say, one inch measured distance equals one inch change in CG position. Then on the horizontal line which has a value equal to twice the base line weight, marks are made to indicate the CG position, but at double the scale used for the base line. Sloping lines are then drawn to connect corresponding marks (see Fig 5). Similarly, the fore-and-aft limits of the CG position may be plotted.

## Using the CG Position Graph

12. The completed graph is used as follows. Using the figures given in para 8, as an example, the basic CG is calculated and plotted:

| Basic Weight | Basic Moment | CG |
| :---: | :---: | :---: |
| $5,000 \mathrm{lb}$ | 0 lb in | 0 in |

The weight and moment of the pilot and crew are then added to the basic weight and moment, and a new CG calculated. This process is continued for all items being added; the plotted positions being joined consecutively (see solid line in Fig 5 and the figures in Table 1).

Table 1 Example CG Data

| Item | Item Weight <br> $\mathbf{l b}$ | Cumulative <br> $\mathbf{W e i g h t ~} \mathbf{~ b}$ | Item Moment <br> $\mathbf{l b} \mathbf{~ i n}$ | Total Moment <br> $\mathbf{l b} \mathbf{~ i n}$ | CG Position <br> $\mathbf{i n}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Helicopter | 5,000 | 5,000 | 0 | 0 | 0 |
| Pilot and Crew | 400 | 5,400 | $-16,200$ | $-16,200$ | -3 |
| Passengers | 600 | 6,000 | $-19,800$ | $-36,000$ | -6 |
| Baggage | 500 | 6,500 | $-3,000$ | $-39,000$ | -6 |
| Fuel | 1,500 | 8,000 | $+31,000$ | $-8,000$ | -1 |

13. To use the graph for helicopters of the same type but having different basic weights and moments and carrying different loads, first calculate the basic CG; this will be the starting point on the graph. As items are loaded the CG will move in the direction of the appropriate plotted line, the distance along the line varying according to the weight being added. An example is shown by the broken line in Fig 5, where the basic weight and CG position is considered to be $5,000 \mathrm{lb}$ and +3 in , with the helicopter being loaded with a pilot ( 200 lb ), baggage ( 800 lb ), fuel ( $1,000 \mathrm{lb}$ ) and passengers ( 500 lb ). The CG of the loaded helicopterhas now become +0.8 in .
14. The CG will change as fuel is used, and in order to find its new position after 500 lb have been used, draw a line parallel to the fuel line starting from the CG position of +0.8 in and stopping when it cuts the helicopter weight line for $7,000 \mathrm{lb}$. This gives a new CG position of -0.6 in . It is important to note that, in some helicopters, there is a large change in the CG position as a result of using fuel and, although the CG may be within limits for take-off, it can go outside the limits during the flight.

## CHAPTER 12 - HELICOPTER FLYING TECHNIQUES

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## Introduction

1. Control of the helicopter usually presents some difficulty to the experienced fixed-wing pilot during the early stages of instruction because of the new sensations associated with hovering, sideways and backwards flight, vertical climb and descent and the ability to remain airborne at zero airspeed. However, this initial difficulty is soon overcome and unless restraint is exercised until experience is gained, overconfidence in the capabilities of both the aircraft and the pilot may be bred; the helicopter pilot should always remember that although his aircraft is capable of a wide variety of tasks and can operate from places that are inaccessible to any other type of vehicle, he may be let down, figuratively and literally, with little warning and with surprising speed, if he mishandles the aircraft. If, on the other hand, the pilot is careful and uses the recommended flying techniques, the helicopter can be confidently and safely flown to its limits.

## BASIC TECHNIQUES

## Ground Taxiing

2. Helicopters are required to ground taxi on a regular basis particularly when manoeuvring in crowded dispersals, moving to suitable take-off areas and entering confined spaces. In conditions of very high or gusty wind conditions, it may be necessary to start the aircraft in a hangar and ground taxi out, or to ground taxi into a hangar before shutting down.
3. To ground taxi a helicopter, the rotor rpm (Rrpm) for take-off must be selected. Depending on aircraft type, a combination of cyclic and/or collective will be used to move the aircraft forwards and to control its speed once moving. In addition, wheel brakes may also be used to control forward speed which should never exceed a fast walking pace. The controls used to adjust the aircraft's forward speed will also be used to stop it. Under normal circumstances, the rotor disc will not be tilted back beyond the horizontal. The method of achieving directional control depends on aircraft type. Some aircraft are turned by normal use of the yaw pedals while others use nosewheel steering. Additionally, to aid stability in the turn, the cyclic is either moved into the turn or left in the laterally neutral position, precise techniques varying with aircraft type.
4. Great care is necessary when taxiing over rough, or soft ground, or up a gradient because the power required to move forward may cause the aircraft to begin to lift off and if this occurs, the attempt must be abandoned. The helicopter must also be brought to rest if severe lateral oscillation or fore-and-aft pitching develops. The latter is particularly dangerous and no attempt must be made to correct it with the cyclic stick, as this would involve tilting the rotor in the opposite sense to that of the fuselage, with a consequent danger of the rotor striking the tail cone.
5. When ground taxiing, particular care should be taken when the danger of ground resonance exists (see paras 42 to 44). A helicopter which is unserviceable to fly is also unfit to ground taxi.

## Control in Hovering Flight

6. In forward flight the effects of the controls of the helicopter are very similar to those of fixed-wing aircraft, although their use may differ slightly because of the addition of an extra control, the collective lever, which is used specifically to control height. However, at the hover the use of some of the controls changes slightly in order to compensate for this new mode of flying. The helicopter can be
said to be hovering when the following three conditions of flight are fulfilled: constant position over the ground, constant height and constant heading.
a. Position over the Ground. The position over the ground is controlled by the cyclic stick. Assuming a perfect hover, if the stick is moved the rotor disc tilts, followed closely by the fuselage, both tilting in the same direction as the stick has been displaced. After a perceptible lag, the aircraft moves bodily over the ground in the same direction as the stick is moved. This lag between change of attitude and movement over the ground is caused by aircraft inertia and if the pilot corrects the attitude during the lag, the aircraft's position will not alter. Aircraft attitude, controlled by the stick, is of prime importance when hovering.
b. Aircraft Height. The height is controlled by the collective lever, working in the natural sense: raising the lever will increase the height and vice versa.
c. Aircraft Heading. The yaw pedals vary the magnitude of the tail rotor force which is required to counteract the torque reaction of the main rotor. The pedals act in the natural sense: applying right pedal results in a yaw to the right and vice versa. In tandem rotor helicopters there is no need for a tail rotor as the two main rotors can be tilted in such a way as to produce the yaw required. In forward flight the yaw pedals provide balanced flight as in a fixed-wing aircraft. However, as there is no slipstream effect in the hover, the stick cannot be used to turn the aircraft: its use would only result in bodily movement over the ground.

## Effect of Wind on Control

7. When hovering in strong wind conditions the rotor disc will tend to flap-back from the wind and unless corrective action is taken the helicopter will drift down-wind. To hover in a wind, therefore, the aircraft has, in effect, to fly into the wind at the wind speed. Thus, to maintain the hover, the pilot must tilt the disc into the wind, the amount of stick displacement from the central position varying with the wind strength. Cyclic and yaw control limits determine the maximum wind speed in which the helicopter can hover crosswind or down-wind.

## Effect of Wind on Power Required to Hover

8. As the disc must be tilted to maintain a hover in a wind, so the resultant airflow down through the disc is modified in such a way that the mass flow is altered, thus enabling the hover to be sustained using less power. This effect is known as translational lift and is a very important factor in helicopter operation, especially at the lower end of the speed range. However, translational lift is not present when hovering in still-air conditions, but another factor, ground effect, becomes important.

## Ground Effect

9. Hovering the helicopter near the ground in still-air conditions will require less power than is required at 50 or 100 ft . This phenomenon is known as ground effect. It is only present in still air, or very light winds, and its greatest effect will be when the helicopter is at its lowest hover. However, ground effect is apparent up to heights equal to approximately two thirds of the rotor diameter. For example, if a helicopter has a rotor diameter of 30 ft , ground effect is felt up to about 20 ft . The nature of the ground will affect the amount of benefit gained by the ground cushion. A smooth, level surface produces most ground effect while a rough, sloping surface tends to minimize the effect.

## Normal Take-off and Landing

10. It is normal practice to hover the helicopter immediately prior to landing and immediately after take-off. This enables the pilot to correct for any lateral motion before touching down and also allows him to check that the helicopter has been correctly loaded before committing the aircraft to forward flight. The hover height chosen will be a compromise between exploiting the maximum ground effect, where less power is needed to hover, and the need to maintain a safe clearance between the aircraft and the ground for possible manoeuvring.
11. Take-off. A take-off into the hover is accomplished by raising the collective lever and thus increasing the pitch on all the rotor blades. When the resulting increase in rotor thrust more than offsets the weight of the helicopter, the aircraft leaves the ground and climbs vertically, the lever then being adjusted to maintain the desired hover height. During the take-off, the correct hovering attitude is selected with the cyclic stick and any tendency to yaw, as torque is increased, is corrected by use of pedal. The stage when the landing gear is in only light contact with the ground should not be prolonged - the aim being for a smooth unstick - as any lateral movement at this stage could induce ground resonance.
12. Landing. Although a landing is basically a reversal of the take-off technique, the variations in helicopter design lead to slight differences. In general, the helicopter is first settled in a hover and then height is gently reduced by use of the lever. The aim is for a firm but smooth contact with the ground, with no movement except in the vertical plane. As soon as the landing gear is firmly in contact with the ground the whole weight of the helicopter is transferred to the ground with a smooth but firm downward movement of the lever, continuing the movement until the lever is fully down. Throughout the landing the hover attitude is maintained to prevent the helicopter from drifting; any tendency to yaw is checked by use of pedal.

## Take-off and Landing out of Wind

13. Ideally, the take-off and landing should be made into wind, but there will be times when this is not possible. The basic landing and take-off techniques apply equally in crosswind conditions, but in strong winds certain control limitations exist which must be anticipated and allowed for by the pilot.
14. During any out-of-wind take-off the tendency for the rotor disc to 'flap-back' in relation to the wind must be checked by use of the cyclic control, otherwise the aircraft will drift sideways down-wind. On landing, this drift will be corrected by maintaining a steady hover prior to touchdown, but on take-off, the pilot must be prepared to incline the rotor disc slightly into wind by use of the cyclic control as the aircraft leaves the ground. In some helicopters the amount of rearwards cyclic control available is less than the amount of forward control. Loss of control can, therefore, occur whilst attempting to obtain a steady hover following a down-wind take-off, or when approaching the hover prior to a down-wind landing in a strong wind. In addition, during a down-wind take-off or landing, the weathercock effect tends to make the aircraft directionally unstable.
15. Added to the drift problems associated with an out-of-wind take-off, landing or hover is the impairment of directional control. This becomes increasingly critical where a crosswind tends to weathercock the aircraft in the same direction as the main rotor torque because, in the extreme case, the combined weathercock and torque effect will exceed the counteracting force which can be applied by the appropriate yaw pedal. In such a condition, directional control could not be maintained. Aircrew Manuals should be consulted for limitations.
16. When hovering cross-wind the attitude of the fuselage will be affected and on touching down the landing gear on one side of the fuselage will make contact with the ground before the other, resulting in a rolling moment of the fuselage as the landing is completed. The tendency for the disc to follow the fuselage must be prevented with the stick, the stick not being centralized until the collective lever is in the fully down position. Tail rotor roll will also affect fuselage attitude. When the tail rotor is below the level of the main rotor, the tail rotor drift corrective force being produced by the main rotor will create a rolling couple with the tail rotor thrust, causing the helicopter to hover one wheel, or skid, low.
17. Approach and landing down-wind should only be made when there is no alternative. Such a necessity implies an obstructed landing area requiring a steep angle of approach at a low forward speed; in a strong tailwind this may mean that the helicopter has an effective backward airspeed which is potentially dangerous because of impaired directional control and reduced aft cyclic control. Whenever there is a tailwind component, translational lift will be lost completely before the helicopter comes to the hover and during this period the rate of descent must be kept very low (less than 500 fpm in most helicopters - consult Aircrew Manual for each type) to avoid encountering a vortex ring state (see paras 39 to 41). Before taking-off down-wind, the ground in front of the aircraft should be examined to see that it is suitable for a run-on landing, which will be necessary if the rearward limits of the cyclic stick are reached whilst attempting to hover.
18. The limiting wind speeds for take-off and landing out of wind vary between types of helicopters. In some tandem rotor configurations, the take-off and landing is more easily accomplished in crosswind conditions as this eliminates the rotor interference which occurs when the aircraft is headed into wind.

## Landing on Sloping Ground

19. The degree of slope on which a complete landing, ie when the whole aircraft's weight is transferred to the undercarriage, may be safely made is not very great. Since the angle and direction of the gradient may be difficult to detect in a confined area, all landings on unfamiliar ground must be approached with caution. The technique is basically the same as that used for normal landing but great care must be taken to maintain a horizontal disc attitude and constant fuselage heading while transferring the aircraft's weight from the rotor to the undercarriage.
20. As shown by Fig 1a, when landing across the slope, first contact with the ground is made by the up-slope landing gear. Transfer of the weight to the down-slope landing gear must be made by continuing the downwards movement of the collective pitch lever, at the same time preventing the rotor from following the fuselage movement by maintaining the rotor disc as near to the horizontal as flying controls will allow (see Fig 1b) with the cyclic stick 'held into the slope'. This stage of the landing must be carried out carefully; if the cyclic control reaches its limiting stop before the whole of the landing gear is on the ground, the attempt must be abandoned, as beyond this point the aircraft will try to slide down the slope. If the undercarriage has a castering nose/tail wheel, the aircraft may tend to yaw down the slope during the landing. To assist in maintaining direction the wheel brakes and locks should be applied before attempting to land.

## 12-12 Fig 1 Landing on Sloping Ground

Fig 1a


Fig 1b

21. When the complete landing gear is on the ground the collective pitch lever must be lowered carefully but stopped if the helicopter begins to slide down the slope. In this condition the helicopter is prone to ground resonance and must be lifted clear of the ground immediately if this develops. Provided the lever can be fully lowered and cyclic limits have not been reached, the cyclic can be relaxed to the central position but great care must be taken to ensure that this movement does not initiate a slide or yaw, in which case an immediate take-off may have to be made. For semi-rigid and rigid rotor heads the technique will vary slightly because of restrictions in the lateral movement of the cyclic at minimum pitch on the ground. If an immediate take-off is required, then the cyclic must be moving towards the central position before the collective is raised in order to reduce the risk of dynamic rollover (for a detailed explanation of dynamic rollover, see Volume 12, Chapter 5). It may be found impossible to reduce collective pitch completely but, with care, passengers or freight can be transferred to or from the aircraft. If, during the take-off, the helicopter is allowed to pivot too quickly about its up-slope skid or wheel there is a very real risk of dynamic rollover. If this condition seems possible, the pilot should swiftly but gently reduce collective pitch; rapid lowering of the lever may lead to the helicopter bouncing off the down-slope skid or wheel and rolling the other way.

## Sideways and Backwards Flight

22. For the purpose of manoeuvring in confined spaces, the helicopter can be flown sideways or backwards by simply moving the cyclic stick in the required direction. Because the airspeed in sideways and backwards flight is limited, the amount of translational lift obtained is also low, therefore relatively high power is required and these manoeuvres should normally be done at ground cushion height.
23. In sideways flight the airflow acting on the tail cone causes a tendency to weathercock in the direction of flight and this must be corrected by use of yaw pedal. However, excessive speed in sideways flight may result in loss of directional control because the amount of yaw pedal required may be insufficient to counteract the weathercock effect.
24. In backwards flight directional control is difficult to maintain because of the tendency to weathercock. Additionally, as backwards airspeed increases, the disc will flap-back relative to the airflow, but forward relative to the fuselage, and a further rearward movement of the stick will be required to maintain the original disc attitude. If the backwards airspeed is allowed to increase to the point where it is necessary to have the stick fully back, any further flapping forward of the disc cannot be corrected and the aircraft is likely to pitch forward out of the control. Care must be taken when stopping backwards flight, even at low speed. A small forward cyclic stick movement will act with the disc, which is flapping-back relative to the rearwards movement, and can cause a large forward rotation of the fuselage. This may cause the rotor to strike the tail boom.

## Turning on the Spot

25. A turn on the spot is a manoeuvre where the helicopter is yawed through $360^{\circ}$ whilst hovering over a point on the ground and where a constant rate of yaw, constant Rrpm and height are maintained throughout.
26. In executing a spot turn, the rate of turn is controlled by the yaw pedals, position is maintained with the cyclic stick and height with the lever. In calm conditions, it should not be necessary for the cyclic stick to be moved from the normal hover position and there should be very little displacement of the yaw pedals. In windy conditions, the cyclic stick will have to be moved throughout the turn to prevent any tendency to drift down-wind and the yaw pedals used to prevent any changes in the rate of turn due to the varying weathercock effect. In the case where a turn is required in an aircraft's own length rather than about the main rotor axis, a certain cyclic stick displacement in the direction of the turn will be required. The length of the tail cone should always be remembered and a good look-out maintained in the opposite direction to the turn to ensure that no obstructions endanger the tail or main rotor. The centre of gravity and windspeed limitations of the aircraft should be checked before carrying out a spot turn to avoid the danger of reaching aft cyclic limits when hovering down-wind.

## Transitions

27. The change from hovering to horizontal flight, or vice versa, is called a transition. To move from the hover into forward flight the rotor disc is tilted forwards by a forward movement of the cyclic stick. As the speed starts to increase, the aircraft moves away from the ground cushion, the height being maintained with the lever and, as forward speed further increases, translational lift is gained and the aircraft starts to climb. During this acceleration forwards it will be necessary to move the stick forward to prevent the disc from flapping back.
28. Transition to forward flight down-wind should be avoided if possible as more time and distance are needed due to the late onset of translational lift. Furthermore, the initial forward movement of cyclic must be very gentle as a harsh movement produces a large forward tilt to the disc which, with the wind behind the disc, results in severe nose-down pitching and the possibility of reaching aft cyclic limits.
29. The transition from forward flight to the hover is initiated by a rearwards movement of the cyclic stick, adjusting the lever to maintain height by a progressive increase in power. When forward flight ceases the aircraft must be levelled with cyclic to the hover attitude to prevent the aircraft moving backwards. Further adjustment with the lever will be necessary as the ground cushion is reestablished. During transitions, the torque will vary as the power is changed and any tendency for the aircraft to deviate from its heading must be corrected with the yaw pedals.

## Circuit Patterns

30. The flying characteristics of the helicopter may make the standard, fixed-wing circuit procedures unsuitable. Moreover, it is undesirable for the helicopter to conform to these procedures since they seriously reduce its natural flexibility of operation and potential usefulness. Unless a special procedure is used, the helicopter, due to its low speed and small turning radius, is likely to constitute a hazard and a distraction to fixed-wing pilots. It is, therefore, essential to have a circuit pattern which allows the maximum flexibility of operation and which, coincidentally, offers the minimum interference with fixed-wing aircraft.
31. When helicopters are operating from permanent bases which are also used by fixed-wing aircraft, aircrews should be thoroughly conversant with circuit procedures as circuit patterns may vary in height or direction. If the runway in use must be crossed, then it should be crossed at right angles over the centre of the runway.
32. When approaching an unfamiliar airfield the most convenient and accepted procedure is to remain outside the circuit area at a height of not more than 500 ft until called in by the air traffic controller. If the runway in use has to be crossed, this should be done at right angles at the centre of the runway and the helicopter flown to the indicated landing position.
33. Where only helicopters are operating from the airfield and it is desired to fly a circuit as a precision exercise, the circuit pattern should be based on that shown at Fig 2.

## 12-12 Fig 2 Basic Helicopter Circuit



## MISHANDLING

## Overpitching

34. Overpitching is the condition arising from the use of insufficient engine power to maintain Rrpm which is falling due to the high pitch angle and drag coefficient of the blades. If it is impossible to regain the Rrpm, then an overpitched state has been reached and at this stage the only method of recovery is to reduce pitch. However this is not always feasible because a reduction in pitch means a reduction in height and, when hovering, loss of height may not be acceptable.

## Overtorqueing

35. The large increase in power available from turbine engine helicopters may make it possible to overstrain or 'overtorque' the transmission. Since torque = power/rpm, any increase in shaft power (sp), or decrease in Rrpm for the same sp will increase the torque loading.
36. Overtorqueing can be avoided by monitoring the torque gauge fitted to most helicopters.
37. The manufacturer can guard against overtorqueing by restricting fuel flow and, therefore, power, but only at sea level conditions. Because of the increasing efficiency of jet engines with height, power available increases with altitude and the need to control accurately power and temperature within the laid down limits, to prevent overtorqueing, cannot be over-emphasized.
38. The inherent danger in overtorqueing is the possibility, in some turbine engine helicopters, of exceeding the fatigue life of a transmission component before its final overhaul life is complete. This can result, even if the aircraft is flown within its transient power, temperature and Rrpm limitations, and especially if the pilot does not observe the need to reduce the maximum torque with altitude. The torque limitation with height is designed to give a constant shp up to the aircraft's ceiling and any excursion past that limit will increase the torque and, therefore, the transmission loading, beyond its limits.

## Vortex Ring State

39. The vortex ring state occurs most commonly during a powered descent with a very low airspeed, although the rate of descent at which the effects become apparent will vary with aircraft type. The symptoms are normally pronounced juddering throughout the airframe, a tendency for the aircraft to yaw, a slight variation in Rrpm, a rapidly increasing rate of descent which, if allowed to continue, can produce pitching and/or rolling, or, in perfect conditions, a smooth vertical descent at a very high rate.
40. The probability of vortex ring developing quickly with little warning is at its highest during the final stages of an approach to land; particularly if the approach has been made with a tailwind, giving an acceptable groundspeed but a low, or zero, airspeed. It is, therefore, of vital importance that the pilot should check the local wind conditions before making an approach to land, and restrict the rate of descent when the airspeed is low. Vortex ring may also be induced by applying power to recover from a zero airspeed autorotation without first regaining forward speed or by allowing the aircraft to lose height in a steep nose-up attitude when executing a quick stop.
41. As the vortex ring state develops only when the aircraft is descending in the direction of its own downwash, the corrective action must be to move the aircraft forward, by use of the cyclic stick, away from this flight condition. As soon as positive and increasing airspeed has been achieved, power should be
applied to check the rate of descent, but the application of power when the airspeed is very low will only aggravate the situation and prolong the subsequent recovery. It must be appreciated that, probably, there will be a time lag after the stick has been moved forward and before the aircraft gains forward airspeed and that during this period height will continue to be lost. It follows, therefore, that to allow the vortex ring state to develop when flying close to the ground would result in a condition from which it could be impossible to recover. When carrying out a vertical descent, or steep approach at zero or low airspeed, the rate of descent should not be allowed to exceed 500 fpm. (See Volume 12, Chapter 5, para 14).

## Ground Resonance

42. Ground resonance is the condition wherein there exists a severe sympathetic oscillation between the rotor system and the undercarriage of a helicopter. Any out-of-balance force set up in the rotors (by faulty blade damping, sideways motion on landing or wheel 'bouncing') may give rise to ground resonance. During take-off an excessive time spent sharing support of the helicopter's weight between the rotor and the undercarriage must be avoided and the aircraft must be lifted positively and cleanly off the ground as soon as it begins to feel 'light'; for the same reason the collective pitch must be reduced smoothly and fully on touchdown. The helicopter is most prone to ground resonance during a running take-off or landing, whilst taxiing or when landing on sloping ground.
43. The corrective action to be taken if ground resonance occurs varies slightly according to the prevailing conditions but, basically, as the phenomenon results from contact with the ground, the aircraft should be lifted clear immediately. In some conditions, where the power setting is too low to lift the helicopter clear of the ground quickly enough, the collective lever should be lowered fully as quickly and smoothly as possible, the engine disengaged or stopped and the rotor brake applied - the intention being to change the Rrpm by the quickest possible means available.
44. Ground resonance is a most dangerous condition. The likelihood of ground resonance occurring is eliminated as far as possible in the design of the aircraft, but the conditions which can cause it should also be avoided (see Volume 12, Chapter 5 for a detailed explanation of ground resonance).

## EMERGENCIES

## Engine Failure

45. A free-wheel unit is normally fitted in the rotor drive system to allow the rotors to turn independently of the engine(s). If a total loss of power occurs during flight, the Rrpm will decay rapidly if significant collective pitch is maintained and the aircraft will yaw in the direction of the main rotor rotation. The collective pitch must be reduced immediately to the autorotative range to maintain Rrpm and corrections made to counter pitch and yaw.
46. In autorotation the aircraft descends at a steep angle but good control and manoeuvrability are retained. The aircraft can be autorotated to a suitable landing area within range, speed reduced prior to touchdown and the landing cushioned by use of lever, involving a reduction in Rrpm.
47. The best airspeed for autorotation, ie minimum rate of descent, usually approximates to the recommended climbing speed but, within certain limits, the angle of descent may be reduced and range increased by increasing the airspeed. Range may also be increased still further by raising the collective lever and reducing the Rrpm to a specified minimum. Down to a certain limit, this results in increased blade efficiency and, therefore, reduced rate of descent, but it is important to regain Rrpm before landing.
48. In light helicopters, range may be reduced and angle of descent increased by reducing the airspeed, to zero if necessary, to give a near vertical descent, depending on wind speed. At a high rate of descent, positive airspeed should be maintained, but in moderate wind conditions the aircraft can be allowed to drift backwards over the ground while still maintaining positive airspeed. Prior to landing, the rate of descent can be reduced and Rrpm increased by increasing the airspeed to normal. The change in attitude can be quite marked and because the height loss in regaining normal airspeed can be considerable, a low speed autorotation should not be continued below approximately 1000 ft AGL. It is essential to ensure that the aircraft does not land with negative groundspeed.
49. Approach and Flare-out. On approaching ground level following a normal or range autorotation, the forward speed must be reduced sufficiently to permit a safe touchdown. This is achieved by flaring (a positive rearward inclination of the rotor and fuselage), which also has the effect of increasing Rrpm and, reducing the rate of descent. In the late stages of the flare, the collective lever is then raised slightly to reduce the rate of descent and the aircraft is then returned to a level attitude at a low or zero groundspeed and the collective lever raised to check the descent completely just before touchdown.
50. Touchdown Technique. On touchdown the Rrpm will be low and the coning angle high and, therefore, the lever should be lowered smoothly so as to avoid the blades flexing and flapping down excessively. Rapid lowering of the lever must be avoided; this applies particularly to helicopters with a skid-type undercarriage since lowering the lever violently whilst still moving forward over the ground will cause the aircraft to stop abruptly, possibly causing strain to the rotor mast bearing.
51. Speed Control. Under true forced landing conditions the aim should be to touch down with zero forward airspeed. However, on a good surface, a touchdown speed of up to 15 kt may be accepted with safety, provided the aircraft is kept level and landed without drift. Because of the high rate of descent in vertical autorotation and the difficulty in judging the final hold-off, forward speed should be reduced at as low a height as safely possible. It is also important that the flared attitude should be restored to a level attitude in good time before touchdown because once the lever is raised, the Rrpm reduces and this causes a progressive loss of stick control and an increased tendency to yaw as tail rotor rpm fall. Incorrect attitudes or headings cannot easily be rectified at this stage and landing with drift may cause the aircraft to roll over.
52. Safety Height Margins. During the transition period from powered flight to autorotation a rapid loss of height may occur, the height loss varying inversely with the airspeed at the time of engine failure. If the engine fails at normal cruising speed the height loss during the transition may be greatly reduced by flaring. This increases the Rrpm and rotor thrust and also aids the establishment of autorotation by inducing the upwards inflow more quickly. If, however, the airspeed is zero, then 400 ft or more will be lost before full autorotation is established. Unless operationally necessary, therefore, flight at low airspeed at low level should be avoided in a single-engine helicopter.
53. Handling at Very Low Levels. Because of the loss of height, and the reduction of speed by flaring is only gradual, engine failure at very low heights may have serious consequences if the airspeed is high. If, however, engine failure does occur at low level and at speed, then the aircraft should be flared immediately, for maximum speed reduction, and the lever lowered. This will greatly assist in regaining lost Rrpm and, depending upon the airspeed at the time, height can also be gained in the flare. As speed is lost and the aircraft is about to descend, the aircraft must be levelled and the touchdown cushioned with the lever, running-on at the minimum residual speed. A typical airspeed/altitude graph for safe autorotative landing is shown at Fig 3. It should be noted that this graph does not have general application.

12-12 Fig 3 Airspeed/Altitude Graph for Autorotative Landing

54. Wind. It is desirable that the final part of the approach for an autorotative landing should be carried out into wind, bearing in mind that a considerable loss of height will occur during an autorotative descent if turns through more than $180^{\circ}$ are carried out. Loading in the turn may make it necessary to control Rrpm by raising the lever, the lever being lowered on completing the turn in order to maintain Rrpm. Airspeed must be maintained. When practicable, the minimum height on a crosscountry flight should be such as to allow for turns into wind. Because of the steep angle of descent in autorotation, flying over towns, heavily wooded areas and large stretches of water should be avoided in a single-engine helicopter.
55. Practice. The engine-off capabilities of the helicopter provide a degree of safety not found in other aircraft. Regular practices of engine-off landings and autorotation to flare recovery will promote personal confidence in the aircraft and improve pilot judgement under varying conditions.

## APPLIED AND OPERATIONAL TECHNIQUES

## Operating at Maximum All-up Weight

56. When operating at maximum all-up weight (auw) the following considerations must be borne in mind:
a. An increase in auw requires more power to hover and thus reduces the excess power available for the climb.
b. Performance varies considerably between types of helicopter but full power may be required in some types to hover at maximum auw outside the ground cushion even at sea level and moderate temperatures.
c. Whilst cruising flight (with translational lift) presents no problem, flight with little or no forward speed should only be attempted at ground cushion height.
d. Large changes in pitch attitude should be avoided, particularly when moving from the hover into forward flight and vice versa, because a substantial power increase is required to maintain height due to the loss of lift caused by the reduction in ground effect and the inclination of the total rotor thrust.
e. In forward flight, the higher the auw the lower will be the airspeed at which the symptoms of retreating blade stall will occur.
f. It is important to remember that the maximum auw limitation is imposed for structural as well as performance reasons.

## Centre of Gravity Considerations

57. In single rotor helicopters the safe range of movement of the centre of gravity (CG) is very small, often being as little as four or five cm fore and aft of the CG datum, which is usually, but not necessarily, directly below the rotor shaft. The natural hang of the fuselage when hovering in still air conditions changes with CG position, becoming nose-down as the CG moves forward and tail-down as the CG moves back, in relation to the datum.
58. The position of the cyclic stick to maintain the hover will also be affected by the CG position; the stick being closer to its forward stop when the CG is aft of the datum, and vice versa. A condition could be reached where the CG is so far aft that the cyclic stick will be on its forward stop, purely to maintain the hover, thus making forward flight impossible. If the aircraft is loaded beyond the maximum aft CG position whilst on the ground, the pilot will find that, on take-off, the aircraft will move backwards and he will have no forward cyclic control left to stop this movement. The reverse effect will occur if the position of the CG is beyond the forward limit.
59. Since operational use of the helicopter involves the carriage of widely differing loads, it is essential that pilots should take care to assess the weight to be carried and load the aircraft to keep the CG within safe operating limits. On some helicopters the CG will change as the result of using fuel. The method of calculating these factors is considered in detail in Volume 12, Chapter 11.
60. In the tandem rotor configuration the range of CG movement is much greater than in the single rotor helicopter since the pitching moments of the fuselage can be corrected by differential collective pitch of the rotors.

## Limited Power Operations

61. Many helicopter operations have to be carried out in ambient conditions which limit the power available, or in conditions when maximum power is available but inadequate. When operating in tropical conditions, knowing the density altitude becomes of paramount importance, eg with a pressure altitude of 500 ft and an ambient temperature of $35^{\circ} \mathrm{C}$, the density altitude may be as high as $3,000 \mathrm{ft}$. It is, therefore, important to know the power limitations of the aircraft so as to be able to assess accurately what may be achieved with the power margin available after take-off and before landing.

Depending on the power margin available, different take-off and landing techniques are required for safe operations; the exact amounts of power required for each type of manoeuvre vary with the type of helicopter being flown.
62. Ideally, the aircraft's performance should be calculated before take-off as part of the pre-flight planning, so that the pilot should be in no doubt as to his power requirements or which technique to use. However, the information required for pre-flight planning may not be readily available and in such cases the pilot will have to rely upon 'rule of thumb' methods to determine the aircraft's capability.
63. Take-off. The method of assessing the power in hand for take-off is:
a. Hover at normal hover height in the ground cushion and note the power required.
b. Check the maximum power available under the prevailing conditions.

The difference between sub-para a and sub-para b represents the power margin available and indicates the type of take-off and transition possible.
64. The different types of take-off and transition are:
a. Running Take-off. When the power is limited to such an extent that the aircraft cannot be brought to the hover or only to a very low hover, a running take-off is advisable provided that the take-off run is over smooth flat ground, that no obstacles exist in the take-off path and that the aircraft has a suitable undercarriage for this type of take-off. The method of making a running take-off is to taxi forward into wind and then allow the speed to increase and fly the aircraft off, counteracting any nosedown tendency at unstick with cyclic control; accelerating gently while allowing the aircraft to climb until the chosen speed is reached. The initial acceleration will be slow and a considerable distance flown before climbing speed is reached. Depending upon obstacles, it may be necessary to climb at the speed that will give the best angle although not the best rate of climb. Gentle movements of the cyclic stick are essential or the aircraft will lose height and could strike the ground. Where fitted, nosewheel locks should be in at the beginning of the take-off run.
b. Cushion Creep Take-off. From the hover, slightly below normal hover height, the aircraft should be gently eased into forward flight. The aircraft will gradually accelerate and, as the effect of the ground cushion is left behind, translational lift will be gained and the aircraft will continue to gain speed. With full power applied and when the speed to give the best climbing angle or the correct climbing speed is reached full climb may be started. It is essential that a clear flat take-off path is available and that all control movements are made gently.
c. Vertical Inside Ground Effect (VIGE) Take-off. Where a take-off has to be made from a confined area, with obstacles no more than two thirds of the rotor diameter high, the VIGE transition may be appropriate but the power margin must be sufficient to ensure some vertical climb out of the ground cushion. From a low hover, maximum power is applied and the aircraft climbed vertically. Shortly before the vertical climb stops and when clear of forward obstacles, the aircraft is eased into forward flight, converting rate of climb into forward speed and gaining translational lift. The climb should be gauged in relation to the obstacles to be cleared and the aircraft flown to pass over the lowest of the obstructions and, when clear, accelerated to normal climbing speed.
d. Vertical Outside Ground Effect (VOGE) Take-off. Above a certain power margin it will be possible to climb vertically out of the ground cushion, clear all obstructions and then make a transition into forward flight. Unless there is no other way of safely leaving the area, a vertical climb is not recommended because once the climb has started, ground reference is easily lost. Assuming the use of a fixed power setting, the rate of climb will deteriorate with increasing height and eventually become zero. If the pilot attempts to continue the vertical climb beyond the limit imposed by his power setting there is a danger of overpitching and overtorqueing.
65. Landing. The method of assessing the power in hand before landing is based on similar principles to that used for the take-off, except that it is done in forward flight. Whilst maintaining forward flight the appropriate performance graphs are consulted to determine the power required for the selected landing point.
66. The different types of approach and landing are:
a. Zero Speed or Running Landing. This type of landing may be carried out where the power margin is small and the indications are that the aircraft is unlikely to be able to come to even a low hover. The speed of run-on, from zero to the maximum permitted for the type of aircraft, will vary according to the power margin. The landing area for a running landing should meet the following requirements:
(1) Flat and reasonably smooth.
(2) A good escape route should exist for overshooting in the case of a missed approach.
(3) The approach path should not be steep.

A thorough inspection of the landing area should be made and a height selected below which it would be dangerous to overshoot (committal height). A low circuit should be flown, the prelanding checks done and a constant angle approach started. Speed must be gradually reduced, but not allowed to fall below the translational lift speed of $15-20 \mathrm{kt}$. If it is necessary to use all the power before committal height is reached, an overshoot should be considered or the aircraft flown at a speed that allows less-than maximum available power to be maintained. Once the committal height has been reached, airspeed and rate of descent should be reduced together. Ideally, the touchdown point should be reached with the wheels just above the ground and the speed at zero, with a small amount of power still available. The lever is then gently adjusted to place the wheels firmly on the ground. If full power has been applied before the speed falls to zero, the aircraft should be flown on at this speed and no attempt made to reduce the speed further, otherwise the rate of descent will increase rapidly. The landing should be controlled throughout and any tendency to overpitch or overtorque should be avoided.
b. Bare Wheel Clearance. With slightly more power available than that required for a zero speed landing, the helicopter may be brought to a low hover in the ground cushion. The landing area should again be examined for a suitable flat approach, escape routes and a surface suitable for the establishment of a ground cushion; committal height should also be determined. Power, speed and height should be closely co-ordinated so that, as translational lift is lost, a strong cushion is established. It is essential that some speed be maintained until the aircraft is within the landing area and at a height where the ground cushion is to be expected. From the low hover a normal landing may be made.
c. High Hover. Where the landing area is unsuitable for the establishment of a ground cushion, or because of obstacles, it will be necessary to establish a high hover and a considerably greater power margin will be required. Careful co-ordination of power, speed and height is necessary and the final part of the approach should be made about 3 m above the ground or obstacles.
d. Emergency Run-on Landing. In the event of partial power failure, it may be necessary to land with a power margin less than those tabled. A suitable speed in relation to rate of descent should be maintained and the aircraft flown on at that speed. The lever should be used to lower the aircraft gently on to the ground and, with the lever fully down, the wheel brakes should be applied.
e. Overshooting. The decision to overshoot should be taken as early as possible. Height, speed and escape routes are valuable when power is limited and height and speed should never be lost unnecessarily as they can be converted into translational lift.

## Operating from Confined Areas

67. Operating helicopters in the field will frequently involve landing and taking-off from small areas, often surrounded by high trees, buildings etc. Special care must be taken to ensure a safe entry into and exit from the area, and to meet this requirement the following special technique is employed.
68. At some convenient place prior to reaching the landing site a power check as detailed for the aircraft type should be made to determine whether power available is adequate to enter and leave the site. The local wind velocity should also be determined.
69. A thorough reconnaissance of the landing site and the surrounding area should be made on arrival, special note being made of:
a. The size, shape, surrounds, surface and slope of the landing site.
b. The best approach and exit paths, with special reference to escape routes and committal height, the cleared area and the touchdown point, the altitude of the landing site and any turbulence on the approach and exit paths.

The information obtained from the power check and reconnaissance is used to plan a detailed circuit, approach, landing, take-off and exit from the site. An initial proving circuit is flown, usually at 200' above obstacles, and, if satisfactory, the final circuit is started. Once the aircraft is within the confines of the site it is essential to ensure, by means of a reconnaissance, that the tail rotor will not foul obstacles on touchdown. The surface of the landing point must be free from erosion and sufficiently firm to support a laden helicopter. It must also be free from potholes, tree stumps and any debris that could be blown up into the rotor blades; dusty or sandy areas should be avoided where possible. The ground should be relatively level, the slope not exceeding the limit for the aircraft type.
70. If a change of load has taken place whilst on the ground, the CG should be checked in the hover and a power check carried out to ensure that the power margin is sufficient for the type of take-off required. The take-off and transition should follow that decided by the reconnaissance.
71. Landing Points. The size and the approach/exit angles of the landing point will depend on the type of helicopter for which it is planned.
72. Suitable Areas for Landing. Practice landing points can be constructed to meet training requirements but, operationally, they will have to be constructed to meet the needs of the ground forces. The choice of a landing point should first be judged in relation to its entry and exit path and the following are suitable places to build a landing point:
a. On top of a piece of ground higher than the immediate surrounding area.
b. On a 'pimple' in a valley where an up-valley approach and down-valley exit is possible, taking account of any prevailing wind.
c. On a curve of a river which is wide enough for the helicopter to be flown over the water on the approach and exit.
d. In the centre of a saddle where the approach may be made across it and the exit carried on in a straight line.
e. On a ridge in the side of a hill where the approach and exit can be made parallel to the hillside

## Mountain Flying

73. Mountain flying poses several special problems and aggravates many others. An appreciation of mountain wind effects, the ability to assess aircraft performance accurately and an understanding of the physiological problems involved are necessary if the pilot is to fly the aircraft safely and confidently. Although a general pattern may be laid down for the approach and landing on to specified features, because of the changing wind effects, no two approaches are likely to be the same. Smooth, accurate flying is particularly important because on many occasions it will be necessary to fly to the limits of the aircraft's performance and the pilot's ability. This subject is discussed in some detail in Volume 12, Chapter 16.

## Low Flying

74. The nature of helicopter operations is such that much flying is done at low level and pilots must have a clear understanding of the problems involved. Because of the low speed of helicopters there will be a large variation in groundspeed between the into-wind and down-wind case in strong wind conditions, and the effects of turning cross-wind will also be very marked. Any inclination to reduce the airspeed when flying down-wind, in an attempt to maintain a constant groundspeed, must be done with care. When carrying out a low-level creeping line ahead search, a start should be made from the downwind end of the area and all turns made into-wind. Where turns down-wind are unavoidable, sufficient airspeed should be maintained to ensure a forward airspeed when the turn has been completed.
75. The maintenance of a good look-out and, where necessary, taking prompt avoiding action, is of paramount importance. The following are the most satisfactory methods of avoiding obstacles that cannot be cleared laterally:
a. When flying approximately into wind, make a quick stop by flaring to reduce speed rapidly, at the same time lowering the lever to avoid gaining height.
b. When flying down-wind, turn through $180^{\circ}$ and flare.

A quick stop is not normally attempted if flying down-wind. To clear a high obstacle or rising ground, collective pitch and power are increased and the attitude, and therefore airspeed, maintained with the cyclic control. The natural tendency to want to make a 'cyclic climb' when nearing an obstacle should be avoided unless power limitations are reached.

## Flying at High Altitude

76. As height is gained, control response decreases because of the reduction in air density and added care must be taken to maintain control of attitude. Density altitude will be the same as pressure altitude when the ambient temperature conforms to ISA conditions, but when the temperature for any given height is not the ISA temperature, density altitude should be calculated using a density altitude graph, to ensure that the flight will be within the flight envelope. For example, at a pressure altitude of $6,000 \mathrm{ft}$ with air temperature of $+15^{\circ} \mathrm{C}$, the density altitude would be $7,400 \mathrm{ft}$.
77. For the best rate of climb, IAS must be reduced as height is gained so that a TAS is maintained at which maximum excess climbing power is available. Maximum indicated cruising speed must also be reduced with height because the higher blade angle of attack required to obtain the necessary rotor thrust in the less dense air results in the retreating blade reaching its stalling angle at a lower forward IAS than at sea level. Control response of the main and tail rotors is reduced and violent manoeuvres and steep turns at altitude should be avoided since the sudden onset of blade stall will produce a nose-up rolling attitude from which it may be difficult to recover.

## Instrument Flying

78. Aircrew Manuals for different types of helicopter will specify the limitations placed on the aircraft for the purposes of instrument flying and these normally include maximum and minimum airspeeds, maximum altitude and maximum angle of bank. Unless a flight control system or hover meter is fitted, it is impossible to hover a helicopter by sole reference to instruments.
79. Control of Attitude and Airspeed. A change of attitude in the pitching plane is synonymous with a change of airspeed and height. The instruments used to determine a change of attitude are the attitude indicator (AI), airspeed indicator and vertical speed indicator. In the rolling plane, bank attitude is shown on the Al (with turn shown on the rate of turn indicator, where fitted).
80. Control of Height. Change of height is effected by use of the collective lever. At a constant attitude/airspeed in level flight, tendencies to climb or descend are detected by reference to the vertical speed indicator and corrected by small adjustments of the collective lever. A change of attitude/airspeed will result in a height change, but any attempt to recover to the original conditions must be treated as two separate control movements, firstly attitude change, to restore the original airspeed and, secondly, a collective lever movement to restore the original height.
81. Control in the Yawing Plane. A conventional slip indicator is fitted to assist the pilot to maintain balanced flight and a gyrocompass provides the necessary heading information. Any movement of the collective lever will require a corresponding adjustment of the yaw pedals to counteract the alteration in main rotor torque.
82. Approach Aids and General Instrument Flying. Within the helicopter's speed range all normal types of airfield and runway approach procedures can be flown, although initial difficulty may be experienced by the ground controller because the slow speed of the helicopter often necessitates relatively large corrections to compensate for drift.
83. Icing. Individual aircraft icing limitations must be adhered to. Apart from engine icing, airframe icing is a serious hazard because:
a. Blade loading is high and a small amount of ice accretion on the blades is likely to cause a large deterioration in rotor performance.
b. An increase in blade weight due to ice accretion causes a significant increase in the centrifugal reaction which may impose unacceptable loads on the rotor hub.
c. Even small inequalities in the amount of ice accretion on individual blades will cause blade imbalance and since blade balance is very critical, severe vibration may result.

See also Volume 12, Chapter 15 and Volume 8, Chapter 2.

## Formation Flying

84. Leadership. The duties of a leader of a formation remain essentially the same as for fixed-wing formation flying: all matters relating to the safety, positioning and tactics of the formation being his responsibility. In tactical and battle formation, the spacing of individual aircraft is much greater and, therefore, the safety of each aircraft becomes the responsibility of individual pilots, but the leader still retains overall control of navigation and tactics.
85. There are three categories of formation flying:
a. Close Formation. Close formation is used mainly for demonstration and display purposes.
b. Tactical Formation. There are 2 types of tactical formation, they are as follows:
(1) Tactical formation is used for all mutual operations and particularly when a large number of helicopters are involved in the dropping of troops into a forward area, or when a large supply drop is required.
(2) Battle Formation. Battle formation enables individual aircraft to provide mutual support and can involve any number of aircraft.
86. Close Formation - Basic Positions. The following types of formation (described further in Volume 8, Chapter 21) can be flown:
a. Vic - three or more aircraft.
b. Box - four aircraft.
c. Finger Four - four aircraft.
d. Echelon - two or more aircraft.
e. Line Astern - two or more aircraft.
f. Line Abreast - two or more aircraft.

All horizontal spacing is related to rotor diameter - usually one rotor diameter between tips of main rotor blades of adjacent aircraft. In line astern, aircraft are displaced vertically or 'stepped-up' above aircraft ahead. Since there is only a small margin between the formation's cruising and maximum speeds, angles of bank are kept low in all types of formation, except line astern, to assist pilots in keeping the correct position and spacing.
87. Trail Formation. Formations consist of two or more aircraft and horizontal spacing is at least two rotor diameters between aircraft. Vertical spacing is not so critical as in close formation because of the greater distances between aircraft. This formation may be flown at very low level.
88. Battle Formation. Formations consist of two or more aircraft or formation elements. Lateral spacing is from 1 to 4 km in transit, closing to two rotor spans for landing. This formation can be flown at medium or low level and is particularly useful when there is a threat from fixed wing or other rotary aircraft.
89. Pre-planning. Operational tactical formations of this type will require extensive pre-planning of routes to achieve maximum protection and to avoid known obstacles and enemy positions.

## Night Flying

90. Support and SAR helicopters are fitted with a full range of flight instruments and night flying presents no particular problems. Internal lighting may be configured to support operations with night vision devices (NVD), since these are now in frequent use (their use is discussed in para 92). External illumination is provided by white (and in most instances infra-red) landing lights, standard navigation lights, downward identification lights and NVD compatible formation and anti-collision lights. A Nightsun Searchlight or its equivalent may be fitted as role equipment. Similarly, hand held lights and illuminating flares may be carried for special purposes and their use will be pre-briefed. Brightstar floodlights provide useful background illumination.
91. Many landing sites will be illuminated by natural light only. Others may have landing aids at night in the form of lights set up in various configurations to give the pilot azimuth and elevation indications. These lights may vary from hand held electric torches, through crossed headlights provided by stationary vehicles, to a NATO illuminated ' $T$ '. The NATO ' $T$ ' lighting may or may not be configured for NVG operation but some of the alternative lighting mentioned may be too bright for NVG.

## Operating with Night Vision Devices (NVD)

92. The design and construction of NVD are described in Volume 7, Chapter 17. The following paragraphs give general details which may be applicable to a sortie with NVD. Some hazards and limitations are also discussed.
93. In total darkness NVD would be useless. They are light intensifiers and require some light to function. Light is quantified in lux or millilux where 1 lux $=1,000$ millilux $=1$ lumen per square metre. At night, the main sources of natural light are the moon, stars and residual solar light. In addition, some illumination comes from sunlight reflected from particles in the upper air or debris in space and this is termed background illumination. Reference to the day's Astronomical Data Service (ADS) sheet will provide the times of sunset, sunrise, moonset and moonrise and the times of nautical twilight. It also gives the expected light levels for each hour of the night for varying cloud conditions. Normally, NVD are effective down to about 1 millilux but operation below this is quite feasible in areas of high environmental or cultural lighting (i.e. from street lamps, towns etc). Cultural lighting may provide a bonus in particularly cloudy conditions. However, the ADS does not provide details of these conditions.
94. Minimum Heights for NVD Flying. Air Staff Orders prescribe the minimum heights to be flown when wearing NVG and the following terms may be used:
a. Obstacle Plane Value (OPV). The OPV is the height over a specified area above which any obstacle may be clearly ascertained by reference to maps of the area. For example, if the OPV in a particular area is 200 ft , all obstacles with a vertical extent above 200 ft will be marked on the appropriate in-date low flying maps.
b. Minimum Operating Height ( MOH ). MOH is the minimum height for NVG operations. $\mathrm{MOH}=$ $\mathrm{OPV}+50 \mathrm{ft}+$ distance below aircraft of any underslung load rounded up to next 50 ft (if not already a multiple of 50 ). MOH is not increased further to account for obstacles above the OPV.
c. Minimum Safe Height (MSH). The MSH is calculated to allow for obstacles en route higher than the OPV. MSH = height of highest obstacle within 1.5 nm of track +100 ft rounded to the next 50 ft .

Pilots are generally permitted to operate on NVD down to MOH. However, if a marked obstruction, which penetrates the OPV, is not sighted when the aircraft is estimated to be 1.5 nm from being abeam its stated position, a climb is to be initiated to reach MSH at least 0.5 nm from the abeam position. A descent back to MOH may only be made when a positive fix is obtained showing the aircraft to be clear of the obstruction. If, at MOH , the obstruction is sighted it must be avoided by the minima set out in the appropriate Air Staff Order.
95. NVD Sortie Planning. NVD sortie planning must follow published standard operating procedures for the aircraft type, role and theatre of operation. However, the following general points should be taken into consideration:
a. Action in the event of NVD failure must feature as part of the sortie briefing.
b. Standard map marking conventions should be employed but colours and line boldness should be optimized for viewing with NVD.
c. Some geographical features and other structures which are prominent by day often look different on NVD. Accordingly, when planning a route, selection of check features needs careful attention.
d. Most airfields will be well lit and departures should be planned to be flown without the use of NVD. Crews should be briefed as to when to expect transition to NVD.
96. Hazards and Limitations. Flying with NVD brings some great advantages. However, some problems are discussed below.
a. Tunnel Vision. When flying on NVD all peripheral vision is lost and the field of view is about $40^{\circ}$ compared with $160^{\circ}$ without goggles. The tunnel vision produced makes it difficult to assess closing rates and speeds and it is essential to scan to the side to assess speed over the ground.
b. Spatial Disorientation. When flying over large expanses of featureless terrain or the sea, spatial disorientation can ensue. The surface appears as a flat area of monochromatic shading with
nothing to aid depth perception. This may lead to disorientation which can be avoided by searching for other objects such as trees, hills and villages and features on the horizon using a slow scan.
c. Directional Disorientation. Because of the restricted field of view, loss of sense of direction can occur when orbiting resulting in difficulty in relocating objects previously seen. This is best avoided by focusing on other familiar objects, if possible, and frequently bringing the compass into the scan.
d. Poor Acuity. With perfect eyesight, normal day vision is assessed as 20/20. However, NVD acuity under the same circumstances is only 20/80. In other words, an object which would normally be seen at 80 metres in daylight would just come into vision at 20 metres under NVD. Although, at night, this is still better than the unaided eye, power cables, pylons, masts and other less visible obstructions may not be seen at all until it is too late. Careful planning and use of the map should overcome this problem.
e. Poor Height Judgement. Partly because of the poor acuity, judgement of height and slope is markedly more difficult than by day. Frequent cross checking of the instruments and the use of the infra-red landing lamp will help to alleviate this problem.
f. Unusual Reflective Effects. On NVD some materials are less prominent than in daylight and vice versa. This is a function of how well materials reflect infra-red light as opposed to visible light. Cold metal is a poor infra-red reflector whereas trees and wood in general are good. However, with a grassy background, leafy trees can be almost invisible to a NVD wearer.
g. Monochromatic Images. When wearing NVD, everything outside the cockpit appears in monochrome. Besides inducing fatigue, this can have flight safety implications. Although infrared lights will show up well, normal red obstruction lights, for example, will not appear red and will tend to merge with background lights. To reduce this danger, it is necessary to look out under the NVD from time to time when there is a possibility of being in the vicinity of such an obstruction.
h. Fatigue. The extra weight of NVD, in addition to that of the helmet, may cause neckache and fatigue, especially in view of the head movement when scanning.
i. Cockpit Obstructions. It is easy, in some aircraft, to knock the NVD against parts of the cockpit when making head movements. Care should be taken when reaching out or leaning forward to adjust switches until the environment becomes familiar.
j. Red Eye. Removal of the NVD, after use for some hours, may produce unnaturally red vision for a short while. This is a harmless phenomenon caused by the brain trying to readjust to the colour world.
k. Cloud, Snow and Rain. NVD do not allow the wearer to see through cloud. However, the light that does get through is amplified and may give the impression of seeing through cloud. Similarly, it is possible to fly unwittingly into a snowstorm. There is no indication on the windscreen and, if flying without lights, the snow is undetectable until the lights on the ground suddenly disappear. Snow is best detected by the use of landing lights but red strobes or navigation lights may also enable it to be seen more easily. Rain on the windscreen severely degrades NVD performance, especially if the screen is dirty and/or greasy. If any of these conditions is expected, early warning of problems can be given by looking out under the goggles frequently.

## CHAPTER 13 - EXTERNAL LOAD CARRYING

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## Introduction

1. The helicopter can, more conveniently, carry a greater variety of loads externally than internally, particularly in the case of bulky or unusually-shaped items. Such loads may be carried on external load fitments such as panniers, or by attachment to underslung load-carrying gear. These facilities also permit loading and unloading in areas where the aircraft cannot land.

## Crew Training

2. It is essential that all personnel concerned with flying, marshalling and loading be thoroughly briefed and fully understand the role of other team members and the problems associated with their tasks.

## Aircrew Pre-flight Checks

3. Before commencing external load carrying all the necessary equipment should be checked for serviceability with particular reference to release and emergency release mechanisms.

## External Loads

4. Loading Teams. When hand-marshalling, the loading team consists of a minimum of two men: one marshaller and one hooker When voice-marshalling, the team may be reduced to one hooker, although it is preferable to have a marshaller as well, in case of difficulty.
5. Hand-marshalling. In the event of a hand-marshalling operation, the marshaller is positioned in front of the helicopter and by using the hand signals depicted in the table overleaf, he directs the helicopter to a position over the load. The movement of the helicopter is monitored by the hooker and when it is suitably positioned he hooks up the load and signals the marshaller when the load is secure and ready for pick-up.
6. Voice-marshalling. During voice-marshalling, the crewman directs the helicopter over the load and informs the pilot when the load is ready for lifting.

## Off-loading

7. Marshalled. The marshaller stands in a position some 20 to 30 m up-wind of the point where the load is to be dropped. The helicopter comes to a high hover over the pre-selected point and is thereafter directed by the marshaller. The pilot is then directed to descend slowly until the load is on
the ground. Further marshalling signals are given to release the load and to indicate that the load is clear and the aircraft free to depart.
8. Unmarshalled. Loads may be off-loaded without the aid of a marshaller. The pilot positions the helicopter, under the directions of a crewman, over the selected point of off-loading and slowly descends; as the weight is taken on the ground the load can be released either automatically or manually by the crewman or pilot.

Table 1 Hand marshalling Signals
This dispersal or marshaller

## Pilot Technique

9. During Loading. The pilot must identify the particular marshaller as soon as possible on the approach and should follow the marshalling instructions throughout the operation.
10. Hook-up. Once hook-up is complete, power is applied gently to take up the slack and at the same time small corrections may be made to ensure that the helicopter is vertically above the load. This will prevent dragging of the load as the weight is taken up and also minimizes any swinging of the load as it is lifted clear of the ground.
11. Lifting Technique. After take-off, a power margin check is made to ensure that sufficient power is available to climb away. A towering technique is normally employed to ensure clearance of immediate obstacles.
12. During Flight. It is essential that smooth control movements are made to obviate any possibility of causing unnecessary load-swinging or exceeding the aircraft's airframe limitations.
13. Load Oscillation. If swinging of the load does develop, it is felt as an aircraft oscillation and any attempt to dampen it by use of the cyclic control normally leads to over-controlling and so worsens the situation. The normal procedure is to reduce forward speed gently but if this does not have the desired effect, application of bank and/or power may provide a centrifugal force to dampen the oscillation. An uncontrollable load should be jettisoned.
14. Area Safety. Built-up areas should be avoided and, during low flying, a safe load clearance above obstacles must be maintained. Load switch procedures, designed to obviate load jettison, should be adhered to.

## CHAPTER 14 - TROPICAL AND COLD WEATHER OPERATION

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## Introduction

1. The tropical regions of the world cannot be classified under any one set of characteristics; similarly, the term 'cold weather operation' can properly be applied to operations inside the Arctic or Antarctic Circles or, in a hard winter, to day-to-day operations in the European area. Although aircrew will be given specific information on the problems involved during training, they should also seek local knowledge whenever they are operating in a particular area.

## TROPICAL OPERATION

## High Temperature and Humidity

2. High ambient temperatures reduce air density and adversely affect performance in much the same way as an increase in altitude; furthermore, this reduction in performance is aggravated by high humidity. Both engine and rotor performance are adversely affected. Under certain conditions, the maximum all-up weight may have to be restricted to something below the maximum quoted in the Aircrew Manual to ensure adequate performance.
3. Particular care is required if operating from marginal landing sites well above sea level in conditions of high temperature and humidity, where all the circumstances combine to reduce performance margins.

## Sand and Dust

4. Most helicopters have more operating parts exposed to the eroding action of sand and dust than comparable fixed-wing aircraft and special precautions are necessary. Prolonged hovering over sandy or dusty ground should be avoided otherwise main and tail rotor blades may be seriously damaged and rapid wear will occur in any bearing penetrated. Bearings should be purged regularly with grease and all moving parts and mechanisms inspected frequently.
5. Engine intake air filters should be fitted; these normally consist of a fine-mesh sand filter or a centrifugal-action separator. However, care should be taken when operating in rain with the fine-mesh filter because the combination of water and dust can result in a serious restriction of the airflow to the engine.
6. Because of the downwash of the main rotors, helicopters can produce their own "dust storm" and thus create difficulties for themselves and others. It is particularly important that the inside of all helicopters should be kept clean at all times otherwise dust and sand stirred up by the rotor blades will enter the eyes of crew and passengers and foul equipment. Air and ground crews should wear goggles or eye-shields, and aircraft doors, hatches and windows should be kept shut. Ground marshallers should position themselves as well clear of the landing area as circumstances permit. For the same reasons, formation take-offs and landings should be restricted and the distances between aircraft may have to be increased.

## Weather

7. Rapid changes in weather are likely in some tropical areas and violent thunderstorms and sandstorms can form with little warning and extend quickly over a large area. The turbulence arising from such storms can be very severe and the associated electrical disturbances are likely to cause serious deterioration in radio communications. Unless it is of vital operational necessity, helicopter pilots should avoid flying in or near such storms and should acquaint themselves with the weather conditions likely to be encountered in their area of operations generally, and before every flight, although this may not always be possible when operating away from fixed bases.

## Navigation

8. The tropical areas of the world are vast and radio communications are not usually as readily available as in, say, North America or Europe; moreover, tropical storms can cause serious deterioration in the facilities that do exist. An additional problem is that some maps of these areas are not entirely reliable. The vagaries of the weather, poor radio facilities, inadequate maps and the fact that helicopters operate at comparatively low altitudes and speeds, all combine to make navigation difficult.
9. Routes should be planned to make maximum use of clearly defined line features and, where it is necessary to fly over inhospitable country, frequent position reports should be made and any deviations from flight plan passed to the controlling authority. In jungle country, in particular, where the jungle canopy can close over a crashed or force-landed aircraft, flights should, if possible, be planned to follow well defined lines of communication, eg roads, railways, rivers etc.

## COLD WEATHER OPERATION

## Icing

Note: See also Volume 12, Chapter 15 (Helicopter Icing) and Chapter 16 (Mountain Flying and Winter Operations).
10. Icing presents a particular problem when considered in relation to helicopter flying. In the really cold areas of the world, severe icing is not usually encountered because it is often too cold and because the cumulus and cumulo-nimbus-type cloud is not often found in these areas. The winter in the more temperate regions is potentially more hazardous.
11. In turbine-engine helicopters, engine icing is a problem and although an anti-icing system is fitted, it can deal with only comparatively light icing.
12. Ice accretion on main and tail rotor blades is a serious problem and the matter of providing an effective de-icing system is under constant review and development. The rotor blades are finely balanced and any uneven build-up of ice on them creates severe vibration and handling difficulties, quite apart from the fact that the aerodynamic qualities of the blades are modified and diminished.

Severe icing would quickly result in insufficient lift being generated to support the aircraft's weight, particularly if general airframe icing had increased the weight significantly.
13. Icing may or may not form on the windscreen and is not a reliable indicator of icing conditions. However, any increase in collective lever position (power) to maintain straight and level flight could be indicative and this and any unusual vibration or 'nibbling' sensation on the controls must be regarded as warning that icing conditions exist.
14. The hydraulic jacks or manually-operated pitch operating arms are more or less exposed and are therefore subject to ice formation. In some cases, ice or pack snow melts on the warm transmission and, having run down on to the jacks or control runs, refreezes. Prolonged flight under these conditions may cause the collective lever to freeze solidly and the cyclic may be restricted within the small diameter of its normal travel for maintaining cruising flight. To keep them free, periodic exaggerated movement of both collective and cyclic controls is recommended.
15. Windscreen icing can 'blind' the pilot completely and although some helicopters have an effective de-icing system, in aircraft with the bubble-type canopy there is little that can be done except to quit the icing conditions, or land as soon as possible.
16. Starting up or closing down with ice on the rotor blades can cause ground resonance. To prevent this, never start up with ice on any part of the blade.

## Handling Techniques

17. Ice under the skids or wheel may cause the helicopter to spin during rotor engagement or when the engine is throttled back quickly during engine and transmission checks. Care is necessary, to ensure that the cyclic is held in the central position during these checks.
18. If the aircraft has been landed in a dispersal which is covered with slush or wet snow, the supercooled skids may cause the helicopter to freeze to the ground and thus present an unwary pilot with a problem on the next take-off. In this case, the lever should be used to reduce the weight on the skids and the aircraft should then be yawed carefully to ensure that it is free for take-off.
19. Where there has been a new fall of snow, a prolonged run-up should be employed to blow away the fresh snow. However, because of the reduced visibility caused by the resulting "snow cloud", it will be necessary to use a reference point within the periphery of the rotor blades. Movement at a low hovering height should be avoided and a vertical climb-out technique employed. Similarly, returning to the hover over areas of fresh snow can be hazardous and an approach should be made to a specific object, such as a bush or tree stump, which should be used as a hovering reference whilst the snow is being blown away by the rotor downwash. This reference should be held inside the rotor periphery to prevent it being lost from sight in the disturbed snow. In cases where no such reference is available, several low flypasts may blow away most of the loose snow; they should be made at sufficient speed to ensure that there is always an area of clear vision in front of the aircraft.
20. Depth perception is difficult over large areas of unbroken snow, particularly for the uninitiated, and such areas should be avoided for practising quick-stops or autorotations with a powered recovery. Tree lines, fences, clear roads, tracks, etc will provide references to assist in judging height.
21. A landing on fresh snow, particularly at an unfamiliar site, should be tackled with extreme care since there will be doubt about the depth of snow and the condition and nature of the underlying ground. The weight of the aircraft should be transmitted to the landing gear carefully, gradually, and vertically so that an assessment can be made of the ability of the site to take all the aircraft's weight
and permit shutdown. Throughout this procedure the pilot must be ready to take-off immediately should circumstances warrant it.
22. Snow that has a strong crust must be treated with extreme caution; the crust may give way during landing, causing a violent roll. If the crust allows the skids to penetrate to the underlying soft snow, care must be taken not to allow any yaw during the final settling, since a skid which has slid underneath the hard crust may give an unexpected off-balance lateral force on the next take-off. If, during landing, the undercarriage penetrates below the top surface of the snow, the tail rotor will be much nearer to the surface of the snow.

## Navigation

23. Navigation across wide expanses of unbroken snow is always difficult and maximum use must be made of any line features such as power cables, trees, ridges, etc. During blizzard conditions, when the pilot has both a navigation and orientation problem, it may be advantageous to follow line features on the down-wind side, so that any gusting will tend to yaw the helicopter towards the line feature, ie in a direction which permits the pilot some ground reference, rather than where he will be confronted with an expanse of snow. Similarly, following a line feature on the downwind side means that only a small turn will be required to force-land into wind should the need arise and a visual reference will be available throughout.

## Snow Clearance in Dispersals

24. The dispersal area is likely to become congested during snow clearance operations and movements into, through and out of dispersal should be made with extreme caution. This is particularly important when various agencies are involved in snow clearance, e.g. helicopters, snow ploughs, blowers etc where each may be creating individual and separate 'snow storms'. Ridges of hard snow are often formed during snow clearance operations and these are sometimes hard to see; any hover taxiing should be done at a sufficient height to avoid these ridges.

## CHAPTER 15 - HELICOPTER ICING

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## Introduction

1. Through practical experience, a wealth of knowledge has been accumulated operating fixed-wing aircraft in icing conditions; there are some other considerations, however, with rotary-wing aircraft. See also, Volume 8, Chapter 2 (Aircraft Icing), Volume 10 Chapter 11 (Icing) and Chapter 22 (Helicopter Meteorology) and Table 1 at the end of this chapter.
2. Conditions for Ice Formation. The conditions in which ice formation is possible are given below:
a. Icing may occur in conditions of high humidity when the ambient air temperature is at or below $0^{\circ} \mathrm{C}$.
b. Due to local reduction in pressure, icing may occur in conditions of high humidity when the ambient air temperature is above $0^{\circ} \mathrm{C}$.

High humidity occurs in all forms of precipitation, cloud and fog, or in air close to these conditions.
3. Categories. For convenience, helicopter icing is considered under three headings, in the following order of priority:
a. Rotor system icing.
b. Engine icing.
c. Airframe icing.

## ROTOR SYSTEM ICING

## Icing Effects on Main Rotor System

4. The primary effect of ice on the rotor system is drag; the secondary effect is loss of lift due to the change in aerodynamic efficiency of the blade. The way in which ice forms on the blade is affected by five main factors.
a. Temperature.
b. Liquid content and droplet size.
c. Kinetic energy.
d. Blade section.
e. Mechanical flexion and vibration.
5. Some blade forms produce more kinetic heating than others and this can be related to the design of the blade and its speed of rotation.
6. Continuous operation in rain ice/freezing rain is impossible; this is because the water content is so high that ice will form all over the blade surface giving maximum drag and change of aerodynamic shape at the same time. Ice shedding (see paras 16 to 25 ) will tend to worsen this condition.

## Blade Icing Characteristics

7. Each time a blade rotates in continuous icing conditions, a thin layer of ice is deposited on $20 \%$ of the leading edge, spanwise from the tip. If a section of this ice, which has been formed in temperatures below $-10^{\circ} \mathrm{C}$, is examined, it will be seen to have bands of slightly differing colour tone which can be seen by the naked eye. These bands are, in fact, growth bands and the greater the number of rotations, the greater the growth of ice.

## Ice Formation on Different Blade Types

8. High Performance Blade. On a blade with a characteristically high performance profile and a high rotational speed, ice forms readily on the leading edge because the radius is small and the boundary layer shallow (see Fig 1); super cooled droplets can easily penetrate this layer allowing the formation of ice.

## 12-15 Fig 1 High Performance Blade


9. High Lift Blade. A blade having typical high lift characteristics, is deep in section, has a large tip radius and a slow rotational speed. Because the tip radius is greater than that of the high performance blade, the boundary layer which surrounds it is deeper and most of the super-cooled droplets that penetrate this layer are centrifuged
off again and only a small proportion form ice on the leading edge (see Fig 2). This is a better blade configuration in icing conditions than the high performance blade.

## 12-15 Fig 2 High Lift Blade


10. Tail Rotor Blades. So few problems have been encountered with icing of the tail rotor blades that it is unnecessary to go into great detail; ice is picked up on only $20 \%$ of the blade from the root end towards the tip. Although ice does build on the pitch change mechanism, this can be kept clear by regularly cycling the controls.

## Ice Formation at Different Temperatures

11. Ice Formation at, or Just Below, Freezing Point. Between $0^{\circ} \mathrm{C}$ and $-3^{\circ} \mathrm{C}$ ice will form in natural icing conditions on the leading edge of the blades from the blade root towards the tip covering about 70\% of the span and $20 \%$ of the chord from the tip of the leading edge, the remaining $30 \%$ of the span at the tip being free of ice due to kinetic heating. If the blade ice is allowed to build up, the maximum accretion point will be the mid-point of this area, with another area of high accretion around the blade root caused by turbulence (see Fig 3). The ice formed on the leading edge at these relatively high temperatures will have the classical mushroom shape. At the blade root there may also be a degree of run-back which, in itself, is not important as little lift is produced in this area.

## 12-15 Fig 3 Blade Ice Coverage at Temperatures Just Below Freezing Point


12. Ice Formation at Temperatures Between $-3^{\circ} \mathrm{C}$ and $-15^{\circ} \mathrm{C}$. It has been shown that at $-3^{\circ} \mathrm{C}$ about $70 \%$ of the leading edge span will be covered by ice. As the temperature decreases, ice is deposited further along the blade until $100 \%$ coverage from root to tip takes place (see Fig 4) the lower temperature having overcome the kinetic heating. With $100 \%$ coverage of the leading edge, drag becomes very high and, if this ice cannot be shed, the drag will increase to a point where power is limited.

## 12-15 Fig 4 Blade Ice Coverage at Temperatures Between $-3^{\circ} \mathrm{C}$ and $-15^{\circ} \mathrm{C}$


13. Leading Edge Ice Formation at Temperatures Above $-\mathbf{1 0}^{\circ} \mathrm{C}$. Fig 5 shows the ice formation on the leading edge at a temperature above $-10^{\circ} \mathrm{C}$ with a definite depression at the stagnation point (point A ). The ice build-up at point $B$ is heavier than at $A$ because only the freezing fraction, which is the smallest part of the supercooled droplet, freezes on impact, the remainder runs back towards point $B$ and freezes between $B$ and $C$. The drag factor produced by this type of ice accretion is high.

## 12-15 Fig 5 Leading Edge Ice Formation at Temperatures Above $-10^{\circ} \mathrm{C}$


14. Leading Edge Ice Formation at Temperatures Below $-10^{\circ} \mathrm{C}$. At temperatures below $-10^{\circ} \mathrm{C}$, ice forms on the leading edge in a different way; there is no longer a concave depression at the stagnation point and the formation is more symmetrical (see Fig 6). This is because the freezing fraction of the supercooled droplet is much larger with very little run-back; consequently, the drag factor is not so high but the problem of asymmetric shedding is now posed. The rate of accretion is much slower because the air is drier.

## 12-15 Fig 6 Leading Edge Ice Formation at Temperatures Below - $\mathbf{1 0}^{\circ} \mathrm{C}$



## Icing Effects on Rotor Head Control Rods

15. Although icing of the rotor head control rods will occur in flight, the control rod ends are always in a condition of movement and this keeps the vital area clear and does not normally restrict control movement. However, it is highly desirable to keep these areas as clear as possible from ice accretion and this is done by fitting an airflow deflector plate forward of the control rod area; a secondary reason for keeping the control rods free of ice is that in some designs they are adjacent to the engine intake and any shedding can result in engine ice ingestion (see also para 27).

## Natural Ice Shedding

16. All main rotor blades have some degree of self-shedding and this always starts at a point $30 \%$ outboard from the blade root and continues to the tip. The reason for this is that, at this point, the blade is subject to mechanical forces and flexion and vibration are at their maximum here. The characteristics of the high lift blade are much better for natural shedding than those of the stiffer, high performance blade with its weak boundary layer.
17. Before any shedding can take place in the natural shedding range, sufficient ice must have been built up; this varies with different types of helicopters and blade design.
18. Flight in continuous icing conditions is not dangerous provided that the helicopter is not flown in temperatures at which natural shedding cannot be guaranteed; this temperature limit is known as the critical shedding temperature.
19. Determination of Critical Shedding Temperature. The critical shedding temperature is determined by test flying, at the hover, in an icing rig over a wide range of temperatures, water content and droplet size. The temperatures at which shedding is no longer reliable are carefully bracketed, but have to be exceeded under carefully controlled test conditions. These temperature limits are clear cut and the icing rig test flying is followed by free flight over a wide time and condition range in icing cloud, freezing fog and wet and dry snow. There is a need to repeat many of these conditions in free flight with varying quantities of ice on the blades. This is because, whilst it may appear that conditions are satisfactory in the hover and low speed manoeuvres where the ice has been retained, in forward flight (eg climbing, descending, steep turns and autorotation), asymmetrical shedding may take place.
20. Asymmetric Shedding. Below critical shedding temperature, ice may be retained on all blades for some time; however, one or more blades can suddenly shed its ice, giving an asymmetric condition. If asymmetric shedding occurs in flight it can cause violent vibration, possibly leading to destruction. In such conditions, the only course is to land immediately and shut down as soon as possible, even if this means using the rotor brake harshly.
21. Damage to the Tail Rotor by Shed Ice. The incident rate of damage to the tail rotor from ice shed from the main rotors is very low and may amount only to slight denting of the leading edge, not sufficient in itself to cause vibration or balance problems.

## Blade Anti-icing

22. The equipment for blade anti-icing consists of an electrical matrix which covers $20 \%$ of the leading edge chordwise from the tip along the length of the blade. Heat is phased into this matrix in different sectors, timed to coincide with the natural shedding cycle, ie when sufficient ice has built up.
23. This works well until the heat application and the natural shedding cycle get out of phase; heat may then be applied at the wrong time. This causes run-back, the ice reforming further back along the chord line, causing the blade CG to move backwards which, in turn, causes imbalance and flutter; it can also cause a residual build-up of ice. The extreme case is the failure of heating to one blade causing asymmetric problems.
24. The power supply for the matrix equipment is a drain on the electrical resources and, since the only satisfactory solution would be to heat the whole blade, a generator large enough to do this would impose weight installation problems.
25. Much research is going into solving this problem, but no clear solution is imminent. The only free, untapped source of heat that exists is from the engine efflux, but, until this can be harnessed to provide an efficient de-icing system, natural shedding and its restrictions must be accepted.

## ENGINE ICING

## Turbine Engine Icing

26. The only ice produced on a turbine engine is at the throat near the first compressor stage. This is not an insurmountable problem as there is sufficient heat available from hot air bleeds and hot oil, to heat this area, and the inlet guide vanes (where fitted).
27. Because of their delicate construction however, there is a problem of ice ingestion by high performance turbines. A sudden slug of slush, even as low as 350 cc water equivalent, can put out the engine flame. Momentum separators are effective in preventing the ingestion of ice and slush and the multi-purpose air intake system, when in the anti-icing mode, separates out any ice particles which may be present and deposits them in an evacuation compartment.

## AIRFRAME ICING

## Problem Areas

28. The main airframe icing problems are:
a. Intakes. It has been found that some intakes, although heated, allow ice to form. Generally, engine intakes must be very clean in design, avoiding any projections; even rivet heads will cause sufficient turbulence to form an accretion point. If the intakes are hinged to give engine access, the sealing at the hinge point must not offer any leakage.
b. Windscreen Anti-lcing. Electrically-heated windscreens are completely satisfactory and also reliable, even in the most severe conditions.
c. Outside Air Temperature (OAT) Gauge. Once in the icing range, temperatures are critical and an OAT gauge that is accurate to one degree is essential.
d. Pitot/Static Systems. Most pitot heads are heated and operate satisfactorily in icing conditions. The combined pitot/static probe is excellent because both its sources are combined and the whole heated.
e. Grilles. Most helicopters are fitted with a grille which may cover a fire-fighting access point or serve to ventilate a small gear-box. These grilles are usually made of expanded metal or wire mesh and are natural catchment areas and ice traps.

## Appearance of Airframe Ice

29. At temperatures between $-5^{\circ} \mathrm{C}$ and $-10^{\circ} \mathrm{C}$, ice usually appears clear; between $0^{\circ} \mathrm{C}$ and $-5^{\circ} \mathrm{C}$ it may appear granulated because it will have been formed from fairly large droplets. At lower temperatures, ie at $-15^{\circ} \mathrm{C}$ and below, ice appears whitish and opaque. At the higher temperatures $\left(0^{\circ} \mathrm{C}\right.$ to $\left.+3^{\circ} \mathrm{C}\right)$ the ice, because of its appearance, may appear much more dangerous than it is; it is certain that at these temperatures the weight of fuel being burnt will be greater than the weight of ice deposited but this is not the case with rain ice/frozen rain which will deposit clear ice faster than fuel is being used and will not shed naturally at temperatures normally safe to fly in.

## OPERATING CONSIDERATIONS

## Indications of Main Rotor Blade Icing and Natural Shedding by Instrument Interpretation

30. Before a pilot contemplates flying in cloud in natural icing conditions it is essential that he can interpret these conditions by reference to his instruments; it is equally important that he is aware of the aircraft temperature limits in these conditions and at no time is it wise that he should attempt to exceed them - except in an emergency and then he must be aware of the consequences.
31. Depending on the temperature and water liquid content of the cloud, ice will start to form on the main rotor blades. This ice will produce increased drag which, in turn, will demand more power from the engine to maintain the rotor rpm. When this extra power is demanded, it is shown by an increase in torque for a set collective angle, i.e. the torque will be seen to increase although no alteration has been made to the position of the collective lever.

Furthermore, a stage in the deterioration in the aerodynamic section may be reached such that maintaining Rrpm in autorotation is not possible; this being at a time when the engine(s) are susceptible to damage from ice ingestion.
32. As the ice builds up on the leading edge of the blades, the torque will show a steady rise up to $20 \%$ of its original value and at the same time a slight increase in the general vibration level will be apparent. At the point where sufficient ice has been built up to shed, natural shedding takes place and the engine torque returns to its original value, as will the vibration level. A steady cycling of this nature will continue as long as the helicopter remains in icing conditions.

## Aircraft Limitations

33. Limitations on flying in icing conditions are defined in the relevant Aircrew Manual and are mandatory; flight in icing conditions is only permitted if the aircraft is suitably equipped or is modified to the necessary standard (e.g. intake door configuration, OAT gauge, lighting etc).
34. The Aircrew Manual or Release to Service for the particular helicopter may also need to state the following:
a. The accuracy of the OAT gauge and, therefore, the maximum indicated temperature at which $0^{\circ} \mathrm{C}$ ambient air temperature can be expected.
b. The maximum temperature at which engine icing could be expected.
c. The minimum gas generator rpm, with time limits, for effective engine anti-icing.
d. The areas where icing may be expected at temperatures above $0^{\circ} \mathrm{C}$.

Table 1 Types of Icing and their Properties
Reference: The Handbook of Aviation Meteorology - HMSO - 1994

| Type | Occurs | Appearance | Effect | Action |
| :---: | :---: | :---: | :---: | :---: |
| Hoarfrost | Occurs in clear air on a surface whose temperature is reduced below the frost-point (1) of the air in contact with it. <br> Occurs on clear nights when there is a fall in temperature to a value below $0{ }^{\circ} \mathrm{C}$. <br> May occur in flight when moving rapidly from air well below $0^{\circ} \mathrm{C}$ to warmer and more humid air. Should soon disappear as the aircraft warms up. May affect radio reception, and may cause frost on the windscreen and instruments. | White crystalline coating, normally of a feathery nature. | Weight of the deposit is unlikely to be serious. It can interfere with the airflow over the wing and thus the attainment of flying speed during take-off. <br> Can also affect vision through the windscreen, the free working of control surfaces and radio reception. | Should be removed before take-off. |
| Rime Ice | Occurs when small supercooled water drops freeze on contact with a surface at a temperature below $0^{\circ} \mathrm{C}$. At ground level it forms in freezing fog. <br> In flight it may form in clouds of low water content composed of small droplets, comparable with those of freezing fog. <br> Most liable to occur at low temperatures where small, unfrozen cloud droplets freeze almost instantaneously. | Tiny ice particles between which air is entrapped to give a rough crystalline deposit. <br> Forms and accumulates on leading edges with no spreading back. <br> Trapped air gives a white opaque appearance. | Usually breaks away quite easily. <br> Usually little weight. <br> Alters the aerodynamic characteristics of the wings and may block air intakes. | If present, it should be removed before take-off. |
| Clear Ice (Glaze Ice) | Occurs in dense cloud of convective or orographic type. <br> Forms when large water drops, not far below 0 ${ }^{\circ} \mathrm{C}$, are encountered in flight. <br> Results from water flowing over a cold airframe before freezing. <br> Drop unite while liquid and little air is trapped. May also occur when an airframe, below $0{ }^{\circ} \mathrm{C}$, descends rapidly through large raindrops. <br> May also occur where there is an inversion where rain falls from a level above $0{ }^{\circ} \mathrm{C}$ to a layer where it is below $0^{\circ} \mathrm{C}$. Typically associated with warm fronts where the icing layer occupies a narrow range of altitude below the frontal surface. | Transparent or Translucent coating with a glassy surface. <br> Ice surface is smooth but may have bumps and undulations. | Tough and sticks closely to the surface of the aircraft and cannot be broken away easily. If it breaks away, it sheds in large pieces which may be dangerous. <br> Will affect the aerodynamics and increase weight. May cause unequal loading of the wings, struts and propeller/rotor blades. | Avoid if possible. Use aircraft anti-icing/de-icing systems. <br> Try to avoid the danger area associated with warm fronts Cross the front at right angles if possible. |
| Cloudy (Mixed Ice) | Rime and Clear ice are the extreme forms of ice accretion experienced by aircraft in flight through cloud and rain. <br> As a large range of drop sizes may be encountered at any temperature between $0^{\circ} \mathrm{C}$ and $-40^{\circ} \mathrm{C}$, a wide range of icing exists between the two extremes. These varieties are usually described as Cloudy or Mixed ice. | The smaller the drops and the lower the temperature, the rougher and more cloudy will be the build up on the leading edges. <br> A smoother and more glassy ice formation, spreading back over the airframe will occur with large drops and a temperature closer to $0^{\circ} \mathrm{C}$. | Effects as above depending on droplet size and temperature. <br> Where ice crystals are present in a cloud, these may stick to a wet airframe and freeze, along with the cloud drops, to give a formation of rough cloudy ice. <br> If snowflakes are present they are trapped in the ice as it forms, producing an opaque deposit with the appearance of tightly packed snow. | Avoid if possible. Use aircraft anti-icing/de-icing systems. |

(1) Frost-point is the temperature to which moist air must be cooled in order to just reach the condition of saturation with respect to a plane ice surface. Further cooling induces deposition of ice in the form of hoar-frost on solid surfaces, including other ice surfaces.

## CHAPTER 16 - MOUNTAIN FLYING AND WINTER OPERATIONS

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## Introduction

1. An ability to transport personnel and equipment efficiently in mountainous terrain is fundamental to Support helicopter operations. An awareness of the effects of altitude on helicopter performance and a sound knowledge of the techniques which may be used to cope with unusual and often extreme meteorological conditions are essential for safe mountain flying.

## Density Altitude and Performance

2. Helicopters are affected by variations in air density, caused by a change in altitude or temperature or a combination of both. Before operating in mountainous regions pilots need to be aware of the prevailing density altitude and its effect on helicopter performance.
3. Density Altitude. Density altitude is defined as the height in the standard atmosphere (above or below mean sea level) to which the actual density at any particular point corresponds. Density altitude may be determined from graphs found in helicopter Operating Data Manuals (ODM), and Volume 2, Chapter 8. Alternatively the following formula may be used:

$$
\text { Density Altitude }=\text { Pressure Altitude } \pm(120 t)
$$

(where $t$ is the difference between local air temperature at pressure altitude and the standard temperature for the same pressure altitude. Note: if the air temperature is higher than the standard, (120t) is added to the pressure altitude; if it is lower it is subtracted).

Pilots should be aware that large local variations in temperature within confined areas (e.g. bowls and valleys) will have a significant effect on density altitude.
4. Effects of Decreased Air Density. A lower than usual air density affects helicopter performance in several important ways. Details of particular aircraft performance may be obtained from the Operating Data Manual (ODM) but the following considerations apply to most types of helicopter in service:
a. Power Available. Generally, the power available to a gas turbine powered helicopter is limited by transmission considerations. As density altitude increases gas turbine engines will accelerate to maintain the power required until the limits of engine speed or temperature have been reached. If density altitude is further increased then the power available will reduce.
b. Power Required. Rotor profile power requirements decrease as height is gained because of the reduction in air density. At the same time, however, the rotor will have to be operated at higher pitch settings and angles of attack, giving rise to an increase in the induced power required (see Volume 12, Chapter 13). If initially the blades are moving towards an optimum combination of RRPM and pitch, the reduction in rotor profile power may be greater than the increase in induced power. However, once the optimum setting has been reached, any further reduction in air density will result in the induced power requirement increasing faster than the reduction in rotor profile power, and the overall power required will therefore increase. At higher density altitudes the induced power demand becomes progressively more predominant, leaving only a reduced power margin.
c. Handling. The response from the flying controls will reduce as density altitude increases because for a given control input a change in pitch on the blades will give a smaller control force in less dense air. With increasing altitude the effectiveness of yaw control from the tail rotor will also decrease and the limits of tail rotor control authority may be reached.
d. Stability. There is an overall reduction in a helicopter's dynamic stability at increased altitude because of the reduction in rotor damping in the less dense air.
e. IAS/TAS Relationship. For a constant IAS, TAS will increase with density altitude and this will have several effects on the operation of a helicopter:
(1) Groundspeed. If an approach is flown with reference to IAS the corresponding groundspeed will be higher.
(2) Turning Circles. The radii of turning circles will be increased.
(3) Inertia. The inertia of a helicopter is a function of TAS; from a higher TAS the helicopter will take longer to decelerate. More power will also be required to bring it to the hover from a descending approach.
f. Hazardous Flight Configurations. A greater degree of anticipation will be required to maintain safe flight configurations when manoeuvring at high density altitudes. Because helicopters will have to be flown with higher pitch settings and angles of attack and with reduced control response, pilots should be aware that they will be flying closer than normal to the potential dangers of retreating blade stall, vortex ring, and the limits of tail rotor control.

## Physiological Effects

5. All crew members must be alert to the physiological and psychological effects of flying in mountainous terrain. Knowledge and training is required for crews to have the confidence to operate successfully in the mountains.
6. Lack of Oxygen. Below $10,000 \mathrm{ft}$ pressure altitude, atmospheric pressure provides a normally healthy person with sufficient oxygen to undertake both physical and mental tasks in daytime without significant degradation of performance - there is, however, a significant reduction in night vision above $5,000 \mathrm{ft}$. Crews should be aware of the dangers of over-confidence and a reduction in judgement and ability with reduced oxygen levels.
7. Lack of External Horizon. The external horizon will often be obscured by surrounding terrain or weather. This can have two significant effects:
a. Levelling to a False Horizon. In the absence of a true horizon there will be a tendency to level the aircraft laterally to false horizontal cues such as rock strata, sloping ridge or cloud lines, and longitudinally to sloping valley floors.
b. Disorientation. Hovering against steeply sloping terrain or flying at low level across a ridge line with a deep valley on the other side can cause disorientation.

These effects are best countered by reference to the instruments, in particular to the artificial horizon. A check of instruments should be made before entering mountainous areas.
8. Apprehension. Apprehension, leading to indecision and tenseness on the controls, is a normal pilot reaction when first undertaking mountain flying operations and confidence must be gained through knowledge, practice and familiarity. Experience in a wide variety of situations will bring the confidence required for safe operations when extreme conditions are encountered. Nevertheless, a healthy respect for the hazards to be encountered in the mountains must be maintained regardless of experience.

## Mountain Winds

9. By far the most important weather factor in mountain operations is the wind. Over open country the assessment of wind strength and direction and its effect on flying presents few problems. In the mountains the wind flow may be modified markedly, with significant upward and downward movement as well as horizontal variations, depending on the nature of the air mass and the incidence of topographical friction.
10. Wind Flow Over Isolated Hills or Pinnacles. When a wind flow is interrupted by an obstacle such as an isolated hill or pinnacle, it will divide and accelerate over and to each side, causing updraughts to windward, and turbulent down-draughts with eddies in the converging air on the lee side. The intensity of these disturbances will depend on the speed of the wind and the cragginess of the obstacle, varying from mild up-and down-draughting over gentle slopes to more volatile vertical and horizontal mixing when strong winds encounter rough, irregular features. This is illustrated in Figs 1 to 4, showing an increasing risk of severe up and down draughts, with localized reverse flows which, in extreme cases, may exceed the normal maximum rates of climb and descent of a helicopter. The demarcation line between up-draughting and down-draughting air will, typically, become steeper and move towards the windward edge of the feature as wind speed increases. The demarcation line should not be considered to be a discrete line between the up-draughting and down-draughting air but more accurately as an area through which the wind characteristics change. This also implies that the line is not necessarily a straight feature. Figs 1 to 4 illustrate where the demarcation is likely to be present.

## 12-16 Fig 1 Light Wind Flow



12-16 Fig 2 Moderate Wind Flow


## 12-16 Fig 3 Strong Wind Flow



12-16 Fig 4 Strong Wind Flow Across Craggy Obstacle

11. Wind Flow Over a Ridge or Line of Hills. In the case of a ridge or a continuous line of hills, the effects will be further complicated as the wind flow is divided and channelled through valleys or forced to ascend and react with a more stable layer of air above. Turbulence can be particularly severe below the lee side of a ridge when strong winds flow across it at right angles. In certain weather conditions, as well as the disturbances close to the ridge, helicopter crews may experience standing waves several miles downwind, or rotor streaming turbulence at levels well above the height of the peaks:
a. Standing waves develop when a deep current of air in which direction is constant and speed increases sharply with height, is forced to rise over a ridge line stretching at or near to the perpendicular across its path. Gravity, and reaction with a layer of more stable air above, causes oscillations in the stream which may cause turbulence several miles downwind. This turbulence may be severe in well developed waves, with reverse winds in the rotor zones beneath the first crests, where roll clouds may also form if there is sufficient moisture present (see Fig 5).
b. Rotor streaming requires a strong, shallow, current of air in which speed decreases sharply with height to a stable, slow moving, layer above. When forced to rise by a ridge or line of hills the air flow decelerates quickly as it ascends, mixes vigorously and tumbles down to cause severe turbulence (see Fig 6). There is no lee wave activity but turbulence may occur in the layer level with, and possibly up to twice the height of, the ridge line.

12-16 Fig 5 Standing Waves


12-16 Fig 6 Rotor Streaming Turbulence

12. Wind Flow Over and Through Valleys. The pattern of wind flow across a valley will depend largely on the strength of the wind and the depth of the valley. If the wind is light and the valley shallow, the wind stream will follow the smoothed outline of the depression, giving rise to gentle up and down draughts (see Fig 7). When the wind is stronger and the valley deeper, the wind may flow across the top of the feature and curl into it, giving rise to down-draughting on the down-wind side and up-draughting on the up-wind side of the valley (Fig 8).

## 12-16 Fig 7 Wind Flow Over Shallow Valley



## 12-16 Fig 8 Strong Wind Flow Across Deep Valley



When obstructed by a line of hills, wind will tend to funnel along the valleys and localized effects will occur depending on the valley shape and size. In a winding valley, areas of local up-draughting may occur as shown in Fig 9, and there are likely to be passages of increased wind speeds in the narrower straighter sections. In the absence of a prevailing wind stream, diurnal effects may need to be considered, the katabatic wind blowing down the valley sides by night, and the anabatic wind blowing up the valley sides by day.

12-16 Fig 9 Plan View of Valley Wind Flow


## Transit Flying

13. Wide variations in terrain type, weather conditions (particularly wind and temperature), tactical requirements and aircraft performance considerations preclude the possibility of laying down procedures that can apply in all circumstances. Nevertheless, there are several basic 'rules' that may be followed to achieve safe mountain operations.
14. Flight Planning. Above all, an appreciation of the nature of the wind in the mountains will be needed for sound planning. Pilots should plan to avoid areas of down-draughting and turbulence. Areas of updraughting may be used to reduce power settings. Pre-flight planning should include a study of:
a. The terrain and features en route including obstacles and power cables. This is particularly significant where power cables may be strung across deep valleys.
b. The meteorological conditions, including wind, turbulence, weather, cloud, temperature levels and the risk of icing.
c. Performance planning, including the power margin required, flight envelope considerations, single engine performance and safety altitudes.

After these factors have been studied a route can be selected. This may be a direct 'over the top' line or a valley route. Tactical considerations may of course be over riding, dictating both the route and the height to fly.
15. Escape Routes. Mountain flying operations will frequently require the helicopter to be flown into locations and environments where a change in weather conditions, or a failure of an aircraft service could put the aircraft into a hazardous situation. The maintenance of an escape route along which the helicopter may be flown safely away from obstacles if it is not possible to continue along the intended flight path is fundamental to safe mountain flying operations, and should be both a planning consideration and a matter for constant awareness during flight. The nature of the escape route will vary depending on the weather, terrain and type of aircraft flown. A combination of adequate height above the ground and/or sufficient airspeed and power margin will be required for the aircraft to be flown away from hazards. Consideration should be given to selecting a flight path where a powerreducing, pedal-assisted turn will facilitate manoeuvre away from obstructions. Passengers and crew must remain secure during those stages of flight where it is not possible to maintain an escape route, typically during, the latter stages of the approach, hover and initial stages of the departure.
16. Action When Caught in Severe Down-draughting. In the event that the aircraft is caught in severe down-draughting where it is not possible to maintain height using full power, the pilot should turn the helicopter to take his escape route. Maximum power should be applied and the helicopter flown at the best rate of climb/angle of climb speed to clear the area. If it is not possible to fly clear of down-draughting, the helicopter should be flown towards a flat area where the effect may be less severe. The rate of descent may be reduced by flaring the aircraft and applying any remaining power available.
17. Retreating Blade Stall. At high speeds and weights at altitude, retreating blade angles of attack will be high. A sharp gust or manoeuvre could induce retreating blade stall. Pilots should continuously monitor the aircraft's performance and reduce pitch angles and airspeed if turbulence is encountered.
18. Engine Emergencies. Pilots of single engine helicopters should, where possible, fly at a safe height within autorotative range of a reasonable landing area. Pilots of twin engine helicopters should be aware of the maximum altitude that can be maintained in the event of a single engine failure and plan their operations accordingly.
19. Wind Assessment En Route. During the transit the wind may be assessed continually by reference to and comparison of on-board navigational systems (GPS, TANS), smoke, wind lanes and patterns on water, blowing vegetation and cloud formations.
20. Ridge Crossing. The safest technique for crossing ridges will vary depending on the wind strength and whether the crossing is to be from lee to windward or vice versa. The basic rule is to approach the ridge diagonally to provide the best escape route should the helicopter be unable to complete the crossing safely. In strong winds crossing from lee to windward should be carried out with ample clearance above the top of the ridge to avoid down-draughting and turbulence, see para 10. Crossing from windward to lee poses less obvious hazards; the clearance will be assisted by up-draughts but, if a low cloud base exists, the aircraft may be carried up into cloud even with minimum power applied. If the crossing is carried out with insufficient clearance the aircraft may encounter turbulence to the lee of the ridge.
21. Valley Flying. Valley flying constitutes a major part of mountain flying, especially under operational conditions. The following points are relevant to operating in valleys and bowls below the tops of major features:
a. When flying along a valley the aircraft should normally be flown on the up-draughting side to conserve the power margin; the aircraft should be flown close enough to the valley side to allow for a $180^{\circ}$ turn escape route, although in strong winds this requirement must be balanced against the need to avoid localized terrain turbulence.
b. Weather and tactical considerations may dictate heights at which to fly. Where possible, the aircraft should be flown at sufficient height to allow for free manoeuvre towards an escape route in the event that the selected height or course cannot be maintained. In strong winds the aircraft will have to be flown below the turbulent layer which may extend down into the valley.
c. In poor weather conditions, with a low cloud base, severe down-draughting or turbulence, it may be safer to fly at low level close to the valley floor. Pilots will then need to monitor their flight instruments closely to avoid the dangers of flying to false horizons and sloping valley floors. However, flying at low level in a valley should be avoided if an escape route cannot be maintained.
d. Crews must maintain a continuous watch for pylons on the tops and sides of valleys as a guide to the presence of power cables.

## Operating Site Procedures

22. Power Check. Before committing the aircraft to an approach, the pilot should carry out a check to confirm that the power margin available is sufficient for the intended operation.
23. High Reconnaissance. A high reconnaissance should be made at a safe height above the site, flown at minimum power speed and normally in a race-track pattern. The following points need to be established:
a. The nature of the terrain surrounding the operating site and the obstacles which will affect possible flight paths.
b. The general wind affecting the site, noting any local features which may cause turbulence, and marked up- or down-draughting.
c. The approximate height of the site.
d. The size, shape and surrounds of possible landing or hover points.
e. Provisional approach and departure paths.
f. Escape routes, which should ideally lie within $45^{\circ}$ of the approach and departure paths and will normally be down slope.
g. The circuit pattern and a low level reconnaissance plan.
24. Low Reconnaissance. The low reconnaissance is flown at the minimum power speed to pass at low level close to, or over, the intended operating point. If the wind is light, or has already been accurately established, the low reconnaissance may be flown as a dummy approach. The proposed approach path should be followed to an overshoot, to confirm the optimum approach, overshoot, departure and escape routes in both elevation and azimuth and the slope and surface of the landing or hover point including any alternatives. A sudden or marked increase in the power required to
maintain the approach is an obvious indication of turbulence. If this occurs in the latter stages of an approach an alternative plan may have to be considered.
25. Localized Wind Finding. Wind has been described as the most important weather factor affecting helicopter mountain operations. In particular an accurate assessment of the wind affecting the operating point is required to effect a safe approach and departure. The reconnaissance must therefore have been preceded by, or should include, this assessment, which may be achieved by one of the following methods:
a. Aircraft Navigational Equipment. If the aircraft is fitted with GPS or Doppler equipment a direct read out of the calculated wind or Doppler along-and-across velocities may be used.
b. Cloverleaf Drift Pattern. The cloverleaf drift pattern may be used to assess the wind affecting the aircraft. On completion of the procedure the aircraft should be pointing into the local wind over a feature (see Fig 10):
(1) Fly (usually on a cardinal heading) at the minimum power speed and at a safe height ( 200 ft ) above the selected point. Note the direction of drift; this will place the wind within a $180^{\circ}$ segment.
(2) Turn the aircraft in the direction of drift through $270^{\circ}$ and fly across the point at right angles to the previous run. Note the direction of the drift; this will place the wind within a $90^{\circ}$ quadrant.
(3) Turn the aircraft in the direction of drift to fly across the point, bisecting the $90^{\circ}$ quadrant. The wind will now be within a $45^{\circ}$ arc. Further similar runs may be made; however, usually on the third run the track of the aircraft can be adjusted to fly over the point into wind.

The full procedure is particularly lengthy and is normally used only in the early stages of training. Operationally, if the approximate wind direction is already known, a run across the operating point, adjusting the heading to eliminate drift, will be sufficient to determine the surface wind direction.

## 12-16 Fig 10 Cloverleaf Drift Pattern


c. Orbit. In light wind conditions an orbit may be flown to assess the wind. The aircraft is flown directly over the operating site at a safe height. A constant speed turn is flown maintaining the same angle of bank through 360 degrees. On return to the start heading the aircraft will be directly down-wind of the site.
d. Groundspeed/Airspeed/Power Comparison. This method may be used in most situations but is particularly suitable for use in restricted areas such as valleys and bowls. The aircraft is flown into and out of the area at constant airspeed. A comparison between heading, track and groundspeed into and out of the area will confirm the wind direction and speed. If the power required to maintain height is noted, an assessment of the location and strength of areas of upand down-draughting can be made.
e. Smoke. If available, smoke is probably the best site wind evaluator. It should be used both on the site and on features close to the site which could have an effect on the wind during any part of the approach or departure. Care must be taken in the use of pyrotechnics both in their handling inside the aircraft and for their effect on the ground during and after the operation.
f. Cloud. Cloud above an area can provide information on wind direction, strength, and up- or down-draughting affecting the site.
g. Vegetation. Close examination of the way the wind affects long grass and other vegetation can often give a good indication of the wind at a site.
h. Water Features. Wind lanes on water surfaces near the site can provide valuable information on wind direction and strength, and ruffled water may be an indication of downdraughting air.
26. The Final Approach Path. The direction of the final approach path will be towards the operating point as near to into wind as practicable whilst providing a good escape route.
27. The Circuit. Typically the circuit will be orientated to the final approach path. It will normally be flown at the minimum power speed at approximately 200 ft above the operating point. The aim of the circuit is to place the helicopter on the final approach path at a safe speed to start the approach. Prelanding checks should be completed.
28. The Approach. Fig 11 illustrates a demarcation plane resulting from a strong wind blowing perpendicular to a line feature. A constant angle approach should be flown, above the demarcation plane, but otherwise as close to normal as possible. If the onset of turbulence would thereby cause the approach to be too steep, the approach may be offset laterally, further out of wind to give a longer, more shallow descent to the operating site. The approach should not, however, be so flat that the margin for an escape route is reduced during the latter stages, and the angle should, whenever possible, be monitored using the backdrop technique. At the same time, speed and rate of descent must be watched carefully on the flight instruments.

## 12-16 Fig 11 Demarcation Plane and Angle of Approach


29. Overshoot. An overshoot must be initiated if:
a. The helicopter is forced to deviate far from the chosen approach path in azimuth or elevation.
b. A safe power margin cannot be maintained.
c. High rates of descent are encountered at low IAS.

The selected escape route should be flown if a normal overshoot cannot be achieved.
30. Hover and Landing. The helicopter should normally be brought to a slightly higher hover to maintain the aircraft clear of obstacles until overhead the selected landing point. If the landing point is in turbulent air the helicopter should be brought to the hover over a clear area in smooth or up-draughting air and then manoeuvred carefully to the landing point. Once the final detailed positioning has been completed the helicopter may be landed using sloping ground techniques. Great care needs to be taken to ensure that an adequate clearance is maintained between the rotors and rising ground. Passengers should be made aware of the dangers of reduced rotor clearance with sloping ground.
31. Take-off. The helicopter should be lifted to the normal hover height and its power margin and takeoff path confirmed. A vertical take-off should be initiated to clear the helicopter from near obstacles. When the take-off path is clear a transition should be commenced, preferably over a downslope, and climbing speed achieved before manoeuvring the helicopter further. Flight instruments should be checked to confirm that a positive rate of climb is maintained until clear of all obstacles.

## Standard Features

32. There are five standard features that are used in basic mountain flying training. The techniques applied to making an approach and landing at these features provide a sound basis for advanced mountain flying operations. The basic techniques applied to these features are discussed below.
33. Pinnacles. A standard circuit may be flown to place the helicopter on the final approach path. The final approach path should be offset from the wind to keep the helicopter outside the area of downdraughting and turbulence to the lee of the summit. An out-of-wind approach will also provide a good escape route away from the feature towards smoother air to the side of the pinnacle (see Fig 12). On the final stages of the approach the helicopter is turned into wind and established in the hover over the landing point.

## 12-16 Fig 12 Pinnacle Approach


34. Ridges or Saddles. A standard circuit may be flown to place the helicopter on the final approach path. The final approach path should be orientated at an angle to the ridge line to provide an escape route away from the feature (see Fig 13). The escape route will necessarily be towards the lower ground below the top of the ridge in down-draughting air; sufficient height will be needed to fly the helicopter away from obstacles. On the final stages of the approach the helicopter is turned into wind and established in the hover over the landing point. A saddle may be approached in the same manner as for a ridge; account will have to be taken of the effect of the wind from the higher sides of the saddle.

## 12-16 Fig 13 Ridge Approach


35. Spurs and Ledges. Spurs and ledges on valley sides often present significant landing problems, as they may be subject to abrupt and considerable variations in the prevailing main feature wind due to localized topographical effects. The circuit will need to be orientated to use the space available within the valley and the approach path may need to be curved. The approach should be flown to provide greater ground clearance in the latter stages; pilots should be ready to take their escape route at any time, usually away from the main feature towards the valley floor (Fig 14).

12-16 Fig 14 Spur Approach

36. Valleys. The recommended approach to a landing point on a sloping valley floor depends mainly on whether the wind is blowing into the valley (up-slope), out of the valley (down-slope), or across it.
a. Up-slope Wind. If the wind is blowing into the valley the circuit will be orientated along its length, with the final approach path into wind, descending along the downward sloping valley floor to the landing point. The downwind leg should be flown at a constant altitude with reducing ground clearance as the helicopter is flown towards rising ground. The turn onto the final approach path will be determined by the slope of the valley floor to the landing point. If the slope is shallow a normal, level turn may be made but if the slope is steep the aircraft will have to be descended towards the valley floor in the turn in order to fly a normal approach angle. In light winds the rate of descent must be monitored closely to avoid the possibility of vortex ring. The escape route (and subsequent takeoff and transition) will be directly into wind over descending ground (see Fig 15).

## 12-16 Fig 15 Valley Approach - Up-slope Wind


b. Down-slope Wind. The approach to a valley landing site into wind towards rising ground poses, in itself, few problems. However, there may be neither a practical escape route nor a subsequent safe take-off path if the slope is severe and the wind strong. In this case an alternative landing point should be considered unless operational considerations are paramount.
c. Across-slope Wind. An approach to a valley landing site with the wind blowing across the valley will normally be made in the same way as for the wind blowing into the valley, in order to maintain a good escape route. Great care will need to be exercised to keep the rate of descent within limits to avoid vortex ring. If this is not practicable the approach may be made up-slope. On the final stages of the approach the helicopter is turned into wind and established in the hover.
37. Bowls. Generally, the considerations applicable to a landing point on the floor of a bowl are similar to those for a valley landing site. However, the close confines of a bowl impose the need for particular attention to the following points when the wind is blowing into it:
a. Reconnaissance. A detailed assessment of the wind effects within the bowl must be made. The helicopter should be flown at the minimum power speed around the bowl as high and as close to the sides of the bowl as practicable to assess areas of turbulence and up- and downdraughting. The direction of flight will be a compromise between the requirements for the pilot to be close to the obstacles to assess safe clearances, and for a power-reducing, pedal-assisted turn away from obstructions should it be necessary to take the escape route into the centre of the bowl. Reversing the direction of the reconnaissance will often give a greater overall view of the feature. Further lower orbits around the bowl should be flown at a suitable height until it has been established that a safe approach may be made to the landing point. Speed may be reduced once it has been assessed that it is safe to do so.
b. Approach. The final approach path will be commenced from a suitable height above the landing point as a descending curve starting from the mouth of the bowl, flying the helicopter forward and down around the sides of the bowl until a normal sight picture approach can be completed into wind (see Fig 16).

## 12-16 Fig 16 Bowl Approach


c. References. There will be no natural horizontal references in the bowl and the floor of the bowl may be sloping. The tendency will be for the pilot to reduce airspeed in an attempt to maintain a reasonable groundspeed whilst flying downwind and to climb as he approaches rising ground towards the back of the bowl. It is essential that the flight instruments are monitored carefully to assess the helicopter's attitude and to maintain a minimum safe airspeed for manoeuvring until the helicopter is established in smooth or up-draughting air. The forward and down-curving approach may be achieved by selecting markers along the proposed path and flying the helicopter towards them until the landing point is visible.
d. Escape Route. The escape route will be to turn the aircraft away from obstructions towards the centre and out of the bowl. Because the helicopter will be flown close to the sides of the bowl and with minimal vertical clearance, it is essential that sufficient airspeed be maintained to manoeuvre the helicopter away from obstructions before it can be allowed to descend. The minimum speed will vary, depending on the type of helicopter, between 20 kt and 40 kt .

## Advanced Techniques

38. Although the standard and simplest helicopter approach is made into wind, there may be occasions when tactical considerations dictate the use of modified, more advanced techniques. The deployment of troops or equipment may require the helicopter to be hovered close to a cliff face below the tops of the surrounding features, or it may be important to avoid being seen above the sky line. In the latter case, if a final approach from above the sky line is inevitable, the reconnaissance should be carried out whilst keeping the helicopter concealed, and the aircraft only climbed to intercept a normal approach path to avoid down-draughting air in the final stages. The departure too would need to be modified to achieve a lower profile on take-off until it became safe to descend again below the sky line. Two advanced approach techniques can be used:
a. Up-draughting Approach. The up-draughting approach is particularly applicable for an approach to a landing point on a ridge line or saddle; indeed, with experience, it may be the preferred approach under normal circumstances. The helicopter is flown at low level in the up-
draughting air on the windward side of the ridge. As the landing point is approached the groundspeed is reduced and the aircraft turned into wind and established in the hover. This technique can be adapted for other features (such as a pinnacle). The helicopter is established flying into wind to the side, either level with or just below the top, of the feature. The helicopter is climbed in the up-draughting air until abeam the landing point; it is then manoeuvred across the top of the feature towards the landing point.
b. Level Approach. A level approach will normally be made when the helicopter is required to be brought to the hover below the top of the major feature. The helicopter is flown close to the cliff face, level with the operating point. As the operating point is approached the groundspeed is reduced gradually and the hover established. This technique has several advantages:
(1) The approach is easy to fly with good references close to the helicopter.
(2) The approach requires no more power than is required for the hover because no additional power is required to stop a rate of descent. This is particularly applicable if the helicopter is to be hovered in down-draughting air.
(3) A good escape route is maintained throughout the operation.

## WINTER OPERATIONS

## Winter Operations - Day

39. There are several additional hazards that may be encountered whilst flying in mountainous terrain during winter. These are associated with the cold environment, the volatility of the weather, and in particular the effect that snow can have on visual references. In all cases the requirement to maintain a good escape route is paramount. When flying in such conditions the need for good crew cooperation cannot be over emphasised. For example, it is very likely that when flying in mountainous terrain, that the escape route will be the reciprocal of the aircraft heading. As the weather conditions can change rapidly, it is vital that, in this situation, a crew member keeps the pilot informed of the conditions behind the aircraft. The most significant hazards associated with flying in such conditions are noted below:
a. Weather. Bad weather can arrive suddenly in the mountains. In winter, severe conditions of low cloud, hail, snow, and poor visibility may be encountered with little warning.
b. Icing. The dangers of engine, airframe, and control icing are significant.
c. White-out. White-out conditions are particularly hazardous as all reliable external references may be lost suddenly, and without warning.
(1) Frequent reference to flight instruments is essential.
(2) The radar altimeter should be used as the primary height reference.
(3) The low warning bug should be selected at just below the minimum safe transit height to provide a warning of rising ground.
(4) The decision to follow an escape route must be taken early to clear white-out conditions, and then an alternative route selected.
d. Glare. On bright days reflected sunlight may cause a blinding glare. Crews must ensure that their visors are clean and use their tinted visors to prevent snow blindness.
e. Physiological effects. The problems of disorientation and vertigo when mountain flying is well documented. In arctic conditions these problems are dramatically accentuated. Whiteout conditions are the prime cause of disorientation. When the sky is overcast and light levels are low, all contrast is lost and mountains blend in with the cloud. All height and slope perception disappears. Bends in valleys and spurs can be impossible to see, which can make safe navigation extremely difficult. To overcome the physiological effects, it is vital to maintain visual references ahead of the aircraft, which should be updated continuously. The Attitude Indicator (AI) and Radar Altimeter should be referred to regularly to ensure that the aircraft is in a level attitude and at a safe height. Although flying in extreme conditions, the crew should still try to be relaxed but not complacent. Once the feeling of relaxation begins to wane, it may be an indication that the crew are beginning to reach the limits of their abilities in the current conditions, and that to 'press on' would be a poor decision.
40. Snow Landing. The downwash of the helicopter in the hover will blow loose snow causing it to recirculate and envelop the helicopter. This may create white-out conditions, making a safe landing impossible unless crews are prepared by ensuring that adequate hover references are available before committing themselves to a landing. The following snow landing techniques are recommended:
a. Hover Reference Marker. The landing site must be reconnoitred carefully to ensure that it is free from obstructions. An object (such as a small tree) is selected as a hover reference marker (HRM) which will remain visible to the crew when white-out occurs. However, in snow conditions it is not recommended to approach to a low hover and the term HRM, can be considered to be a misnomer in this context. Instead, a continuous approach is made to a zero speed landing, as described in this Volume 12, Chapter 12, Para 66a, at a point where the HRM can remain in sight in the pilot's 2 o'clock position close to the helicopter. This gives the best chance that the snow cloud will not obscure the pilot's view of the touch-down point. However, the pilot must remain prepared for the snow cloud to envelop the helicopter and it may be necessary to use the technique in b . below. In white-out conditions, there is a danger of target fixation leading to disorientation. Pilots must overcome the tendency to drift towards the HRM with the possibility of striking it. The aim, in these circumstances, is to execute a soft and careful (rather than a firm) landing, allowing the helicopter to sink into the snow and settle, but avoiding obstructions beneath the snow that might cause damage. Once the helicopter has landed, gentle cyclic movement may be used to allow it to settle fully. Throughout the landing the pilot must be prepared to overshoot on instruments as soon as a mislanding is evident.
b. High Hover. If an HRM is not available, the top layers of snow will have to be removed to reveal sufficient references for a landing to be made. The helicopter is brought to a high hover, clear of blowing snow, over the landing point. Forward and lateral reference markers are selected to hold position, and the helicopter's downwash is used to blow the snow clear. Once the initial blowing snow has been cleared the hover height is reduced by 15 to 20 ft and again the snow cleared. This procedure is continued until an object that can be used as a HRM is visible. The landing is then completed as above.
c. Running Landing. It is not advisable to attempt a running landing on snow covered surfaces.
d. Overshooting. An overshoot should be called by any member of the crew who has any doubt over the safety of the aircraft. If an overshoot is called when in recirculating snow, the pilot should transfer references immediately to the instruments and initiate a collective-led vertical climb until clear of the snow cloud and any obstructions, before transitioning into forward flight. There is a constant risk of disorientation when operating in snow. Due to the often sudden onset of white-out it cannot be over emphasised to overshoot sooner rather than later.
41. Snow Take-off. During the initial stages of the take-off it is important that the helicopter rises evenly from the ground. This can be achieved by small applications of yaw and cyclic control to break the grip of frozen snow. As the helicopter breaks from the ground the hover attitude is established and a vertical instrument take-off continued without delay to clear the helicopter from blowing snow. A retractable undercarriage should be cycled to clear any wet snow to prevent freezing of the undercarriage and brakes. A running take-off on a snow covered surface is not advisable.
42. Navigation. Visual navigation across wide expanses of unbroken snow is always difficult and maximum use must be made of any line features such as power cables, trees, ridges, etc. During blizzard conditions, when the pilot has both a navigation and orientation problem, it may be advantageous to follow line features on the down-wind side, so that any gusting will tend to yaw the helicopter towards the line feature, i.e. in a direction which permits the pilot some ground reference, rather than where he will be confronted with an expanse of snow. Similarly, following a line feature on the downwind side means that only a small turn will be required to force-land into wind should the need arise and a visual reference will be available throughout. If a forced landing is necessary, a decision will have to be made as to whether to keep the rotors running or to shut down. If a shut down is necessary, the aircraft should be protected wherever possible by fitting covers and blanks. A thorough check of the aircraft will be needed before a re-start.

## Winter Operations - Night

43. The flying techniques required for night operations in snow (with or without night vision devices (NVD)) are an extension of the basic day flying techniques. Even so, the difficulties and hazards associated with operations in snow are exacerbated at night. When operating at night in snow consider the following:
a. Overshoot. The correct overshoot technique, as described in the Aircrew Manual, is as important as that for landing. If in doubt, overshoot early. Note also the comments regarding blowing snow in d. below.
b. Navigation. Ambient weather conditions, cloudbase, visibility and precipitation are more difficult to assess at night, with or without NVD assistance. Even with NVD assistance, navigation across wide expanses of unbroken snow is more complicated than in daylight and should be avoided; routes should be selected to remain below the tree line where possible. Assessing the nature of the terrain below the aircraft, or below the snow, is also harder than in daylight.
(1) When flying below the tree line (BTL), there are generally sufficient visual reference features. However, wires and unlit masts are very difficult to see so accurate navigation is vital.
(2) Flight above the tree line (ATL) is a demanding and potentially dangerous area even when the light conditions provide sufficient contrast. The pilot has to devote as much time to the instruments as he/she does to looking outside, and the workload inside the cockpit increases dramatically.
c. Use of NVD. Depth perception is reduced when using NVD. When operating with NVD in snow a constant and careful scan is required of the radar altimeter and the flight instruments to
check the aircraft attitude and height. Good crew cooperation is paramount. White snowy terrain, when coupled with bright lights, tends to increase the 'blooming' effect on NVD.
d. Hazards of Blowing Snow. Recirculation of blowing snow will present even greater problems in the hover than in daylight, therefore the selection of good hover references with strong contrast is essential. Suitable references may be supplied by tree stumps, bushes, dark rocks or suitable light sources. As the HRM may be inside the rotor disc on landing the pilot must be certain of it's height. If a NATO ' $T$ ' is available then consider using the ' $T$ ' base light as a lateral marker. An NVD compatible NATO 'T' allows the pilot to remain operating with NVD assistance; a non-NVG compatible 'T', dependent on ambient light conditions, may not. Time in the hover should be kept to a minimum, and take-off and approach made as if by day. Applying a significant amount of pitch to the rotor disc for a period before lifting will often clear away the loose snow and reduce recirculation in the hover. Disorientation is an even greater hazard during an overshoot at night. The non-handling pilot should monitor all approaches closely and be prepared to call for an overshoot, or take control. If references are lost at any stage of an approach or landing, overshoot early and be prepared for an immediate transfer to instruments with a vertical climb at maximum power until clear of the snow cloud and able to establish visual flight with suitable external references. A sudden entry into a cloud of blowing snow, with complete loss of external references, is extremely disorientating.
e. Lighting. Snow intensifies and reflects light, whether NVD-compatible IR lighting or white light. This intensification of light hinders the use of aircraft external lighting in a snow cloud. Each external light source fitted to a helicopter has its own merits but there is a limit to the assistance that it can give. Experimentation with the external lighting system, and aids available, will determine the most appropriate mix to use for the conditions being experienced. If using NVD then careful use of available lighting on the ground can be beneficial. Day flares are normally far too bright for NVD operations whilst Arctic Smoke Grenades tend to sink into the snow and leave a stain, which is visible on NVD but is easily lost in the snow cloud in the final stages of an approach. 'Cylume' chemical light sticks can offer limited hover references if they are thrown from the aircraft overhead a suitable landing point, however they must be attached to a suitable stick or pole to stop them sinking into the snow. The best approach and hover reference is provided by troops huddled together with a suitable white light source. Such troops must be suitably briefed and be aware of the danger to themselves, and to the aircraft, of sloping ground. Any white light source at a landing point must not be too bright and must be directed away from the aircraft or it will have the detrimental effect of reducing the gain of the NVD thus reducing the effective visibility of the pilot. If the aircraft lights are to be used, whether IR or white light, it is recommended that they be turned off prior to the aircraft being engulfed by the snow cloud. Should aircraft external lights, especially high intensity landing lamps, be left illuminated in a snow cloud, the snow crystals will reflect so much light that the NVD gain is reduced, effectively limiting visibility to the snow cloud only. The lighting systems of individual helicopters allow many combinations and permutations. Accordingly, captains should consider and brief the responsibilities for light switching before any approach.

## External Loads

44. The basic techniques for operating with underslung loads in snow conditions are the same day or night. However, the difficulties and hazards associated with operations in snow are exacerbated at night. Underslung loads normally require accurate position adjustments within the ground cushion; however, hovering in the ground cushion can rapidly lead to white-out. White-out can sometimes be dispersed by a short period in the high hover.
a. Pick-up. The aircraft is lifted to the hover using a prominent hover reference, the hover height may have to be between 30 ft to 80 ft agl dependent on snow conditions. After the white-out has
been allowed to dissipate the aircraft is manoeuvred to overhead the load, with the pilot maintaining good visual references. Clearly defined lateral and forward hover references, combined with those from the radar altimeter are essential to maintain an accurate hover. Once sufficient visibility has been achieved the aircraft may descend, in steps if required, and the load attached. Consideration should be given to using a length of strop or strops which will allow a higher hover and expedite the process. When the load clears the ground the radar altimeter height should be noted. At this stage there is a significant chance that the increased downwash will induce further recirculating snow; as at any stage in the pick up, if references are lost the pilot must immediately transfer his scan to the flight instruments and climb vertically clear of the snow cloud.
b. Drop off. A normal approach should be flown to a higher than normal hover some distance short of the drop point, aiming to keep the snow cloud below the aircraft whilst keeping the landing point and marshallers in sight. As the snow cloud begins to clear, the aircraft should be marshalled forward and down to place the load on the ground. No attempt should be made to place the load on the ground using a zero speed technique in recirculating snow.
c. Protection. Snow or ice will increase the weight of the load. The helicopter crewman is especially subject to the chill effect of the slipstream and should wear face and eye protection. The hook-up team needs substantial protective clothing, and static build-up is far greater at low temperatures. If possible the load should be attached from inside the aircraft using the 'shepherds crook' load attachment pole.

## Formation Flying

45. Day or night tactical formation flying in snow conditions can be conducted using the normal procedures laid down in aircraft SOPs. The following additional points should be considered when operating in a winter environment:
a. Configuration. Formation size and composition will require modification to cater for the conditions. A formation landing in recirculating snow conditions can be extremely difficult and potentially hazardous.
b. Camouflage. Arctic camouflage is very effective in the snow, even above the tree line; it may be necessary to reduce inter-formation distances.
c. Obstacle Clearance. Each aircraft captain is responsible for his own terrain clearance and care must be taken not to fly into hidden features, such as snow covered ridges, as a result of fixation on the other formation aircraft.
d. Escape Route Brief. During transit flying in mountainous terrain the formation leader should consider briefing his formation escape route to allow other aircraft to position accordingly.

## Tandem Rotor Operations

46. In general, the principles of mountain flying for tandem rotor helicopters are the same as for single main rotor aircraft. The enhanced ability to operate out of wind may be a significant advantage because the pilot will have a greater choice of approach and departure paths to avoid turbulent air.
47. Differential Lift. Pilots should be aware of the dangers of flying tandem rotor helicopters in environments where the two rotors may be experiencing significantly different air flows causing differential lift and potential control problems. Hovering into wind close to the edge of a ridge or a pinnacle, where the front rotor may be in up-draughting air and the rear rotor in level or (in extreme cases) down-draughting air, should be avoided.

## CHAPTER 17 - EMBARKED OPERATIONS FROM ROYAL NAVY SHIPS

1. The procedures to be followed when operating aircraft from ships vary with aircraft and ship type. Details of the procedures can be found in the Royal Navy publication BR766, Embarked Aviation Operating Handbook. Before operating from ships, aircrew should be familiar with the relevant sections of BR766 and have received the appropriate pre-embarkation training.


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## CHAPTER 1 - FRACTIONS AND DECIMALS

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## Fractions

1. Description. Vulgar fractions are numerical quantities which are not whole numbers, expressed in terms of a numerator divided by a denominator. There are two types: proper fractions which are less than 1 and improper fractions which are greater than 1. Mixed numbers include whole numbers and vulgar fractions.
2. Comparing Two or More Fractions. To compare two or more fractions they must first be given the same denominator-

Example:
Arrange $\frac{5}{7}, \frac{11}{14}$, and $\frac{3}{4}$ in order of size.
The lowest common multiple of 7,14 and 4 is 28 .

$$
\frac{5}{7} \times \frac{4}{4}=\frac{20}{28} \quad \frac{11}{14} \times \frac{2}{2}=\frac{22}{28} \quad \frac{3}{4} \times \frac{7}{7}=\frac{21}{28}
$$

In order of size $\frac{5}{7}<\frac{3}{4}<\frac{11}{14}$
3. Reducing a Fraction to the Lowest Terms. Reducing a fraction to the lowest terms, or simplest form, means finding the equivalent fraction with the smallest possible numerator and denominator.

Examples:

$$
\begin{aligned}
& \frac{12}{18}=\frac{2}{3} \text { after dividing numerator and denominator by } 6 \\
& \frac{20}{35}=\frac{4}{7} \text { after dividing numerator and denominator by } 5 .
\end{aligned}
$$

4. Addition and Subtraction of Fractions. If the denominators of the fractions are the same the numerators may simply be added or subtracted.

Example:

$$
\frac{3}{5}+\frac{7}{5}+\frac{1}{5}=\frac{3+7+1}{5}=\frac{11}{5}=2 \frac{1}{5}
$$

If the denominators are different it is necessary to find the lowest common multiple so that the fractions may be rewritten with the same denominator.

Example:
$\frac{2}{3}+\frac{4}{5}+\frac{4}{6}$ The lowest common multiple is 30.
The fractions may be expressed as $\quad \frac{20}{30}+\frac{24}{30}+\frac{20}{30}=\frac{64}{30}=2 \frac{2}{15}$
5. Multiplication of Fractions. Fractions may be multiplied by first multiplying the numerators together and then multiplying the denominators.

Example:

$$
\frac{10}{15} \times \frac{9}{7}=\frac{10 \times 9}{15 \times 7}=\frac{90}{105}=\frac{6}{7}
$$

6. Division of Fractions. To divide one fraction by another, the divisor should be inverted and the fractions then multiplied.

Example:

$$
\frac{3}{4} \div \frac{5}{7}=\frac{3}{4} \times \frac{7}{5}=\frac{21}{20}=1 \frac{1}{20}
$$

## Decimals

7. Description. Decimals are fractions in which the denominators are powers of 10. Decimals are written using a decimal point, instead of in the fraction form.
8. Changing Fractions to Decimals. A fraction may be converted to a decimal by dividing the numerator by the denominator.

Example:

$$
\frac{7}{8}=7 \div 8=0.875
$$

It is also possible to convert a fraction to a decimal by expressing the denominator as a power of 10 .
Example:
By multiplying numerator and denominator by 4

$$
\frac{13}{25}=\frac{52}{100}
$$

and so

$$
\frac{52}{100}=0.52
$$

9. Changing Decimals to Fractions in their Lowest Terms. To change a decimal to a fraction, the decimal should be written as a numerator with a denominator of a suitable power of 10.

Example:
Express 0.68 as a fraction.

$$
0.68=\frac{68}{100}=\frac{17}{25}
$$

10. Addition and Subtraction of Decimals. When adding or subtracting decimals it is essential to ensure that the decimal points are in line.

Example:
$212.2+14.9+6.3+0.36$
212.2
+14.9
$+6.3$
$+0.36$
233.76
11. Multiplying Decimals By Powers of 10. When multiplying by powers of 10, the decimal point is moved one place to the right for each power of 10.

Example:

$$
\begin{aligned}
0.7 \times 10 & =7.0 \\
0.7 \times 100 & =70.0 \\
0.7 \times 10,000 & =7000.0
\end{aligned}
$$

12. General Multiplication of Decimals. To multiply decimals the numbers should be multiplied together and then the number of decimal places should be counted and the point set accordingly.

Examples:

$$
\begin{aligned}
& 0.9 \times 0.007=0.0063 \\
& (\text { decimal places } 1+3=4) \\
& 2.652 \times 0.04=0.10608 \\
& (\text { decimal places } 3+2=5) \\
& 410 \times 0.12=49.20 \\
& (\text { decimal places } 0+2=2)
\end{aligned}
$$

13. Division of Decimals by Powers of 10. When dividing decimals by powers of 10 the point should be moved one place to the left for each multiple of 10.

Example:

$$
\begin{aligned}
0.7 \div 10 & =0.07 \\
0.7 \div 100 & =0.007 \\
0.7 \div 10,000 & =0.00007
\end{aligned}
$$

14. General Division of Decimals. To divide decimals, both numbers should be multiplied by whatever power of 10 is required to convert the denominator into a whole number, then division may be carried out in the normal fashion.

Examples:
$4.55 \div 0.5$, may be written as
$\frac{4.55}{0.5} \times \frac{10}{10}=\frac{45.5}{5}=9.1$
$42.6 \div 0.03$, may be written as
$\frac{42.6}{0.03} \times \frac{100}{100}=\frac{4260}{3}=1420$
$0.0272 \div 0.4$, may be written as

$$
\frac{0.0272}{0.4} \times \frac{10}{10}=\frac{0.272}{4}=0.068
$$

15. Significant Figures. If a number is given as an approximation it may be rounded to a multiple of 10. Thus, 76,282 may be given as 76,000 which is accurate to the nearest thousand or to 2 significant figures, the 7 and the 6 . To 3 significant figures it would be 76,300 because 76,282 is nearer to 76,300 than to 76,200 . The general rule is to consider the next digit to the right of the one to which 'significant figure' accuracy is required. If it is greater than 5 , then the previous figure should be increased by one, and the appropriate number of noughts appended. If it is less than 5 , then the previous figure should stand, again with the appropriate number of noughts added.
16. Decimal Places. Numbers are often rounded off or given correct to a certain number of decimal places, depending on the degree of accuracy required. A calculator may give pi as 3.141592654 which, for most purposes, will be given to 3 decimal places and written as 3.142.
17. For some fractions, the division never ends, but numbers (or a series of numbers) are repeated:

$$
\begin{aligned}
& \frac{1}{3}=0.3333 \ldots \\
& \frac{4}{7}=0.571428571428 \ldots \text { etc. }
\end{aligned}
$$

Such decimals are called recurring decimals. The repeating pattern can be shown by placing a dot over the first and last digits in the recurring group:

$$
\begin{aligned}
& \frac{1}{3}=0 . \dot{3} \\
& \frac{4}{7}=0 . \dot{5} 7142 \dot{8}
\end{aligned}
$$

## CHAPTER 2 - PERCENTAGES AND PROPORTIONS

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## Definition of Percentage

1. Percent means 'per hundred'. A percentage is a fraction with a denominator of 100. Thus 13 percent means 13 divided by 100 or $\frac{13}{100}$, and is written as $13 \%$ or 13 pc.

## Percentages as Fractions or Decimals

2. To convert a percentage into a fraction or a decimal, it should be divided by 100.

Examples:

$$
42 \% \text { expressed as a fraction } \quad=\frac{42}{100}=\frac{21}{50} \text { or as a decimal } 42 \%=0.42
$$

$261 / 3 \%$ expressed as a fraction $=\frac{26 \frac{1}{3}}{100}=\frac{79}{300}$ or as a decimal $=0.263$
$9.8 \%$ expressed as a fraction $\quad=\frac{9.8}{100}=\frac{98}{1000}=\frac{49}{500}$ or as a decimal 0.098.

## Fractions or Decimals as Percentages

3. To convert a fraction or decimal to a percentage it should be multiplied by 100.

Examples:
$\frac{3}{5}$ as a percentage $=\frac{3}{5} \times 100 \%=60 \%$
$\frac{7}{8}$ as a percentage $=\frac{7}{8} \times 100 \%=871 / 2 \%$
0.62 as a percentage $=0.62 \times 100 \%=62 \%$

## Finding a Percentage

4. To find a percentage of a given quantity the quantity should first be multiplied by the required percentage and then divided by 100.

Example:

$$
\begin{aligned}
& \text { Find } 36 \% \text { of } 180 \\
& 180 \times \frac{36}{100}=64.8
\end{aligned}
$$

## Expressing One Quantity as a Percentage of Another

5. To express one quantity as a percentage of another, first express one as a fraction of the other and then multiply by 100.

Example:
Express 49 miles as a percentage of 392 miles.
$\frac{49}{392} \times 100=12.5 \%$

## Percentage Increase or Decrease

6. To increase or decrease an amount by a given percentage the amount should be multiplied by the new percentage.

Examples:
Increase 650 by 6\%
$650 \times \frac{106}{100}=689($ or $650 \times 1.06=689)$
Decrease 650 by 6\%
$650 \times \frac{94}{100}=611($ or $650 \times 0.94=611)$
7. To find an original quantity, given a quantity which has been increased or decreased by a percentage, it is necessary to first divide the quantity by the new percentage, and then multiply by 100.

Example:
After an increase of $8 \%$ a quantity is 178 , what was the original quantity?
The increased quantity is $108 \%$ of the original.
$108 \%$ of original quantity is 178
$1 \%$ of original is $\frac{178}{108}$
So $100 \%$ of original is $\frac{178}{108} \times 100=164.81$

## Ratios

8. A ratio enables the comparison of two or more quantities of the same kind, and is calculated by dividing one quantity by the other.

Example:
The ratio of 375 to $500=\frac{375}{500}=\frac{3}{4}$ and is written as 3:4.
To find the ratio of 5 km to 700 m : Ratio $=\frac{5000}{700}=50: 7$
9. Division in a Given Ratio. To divide a quantity according to a ratio $3: 4: 5$, the quantity is first divided by $3+4+5$, then 3 parts, 4 parts and 5 parts are allocated.

Example:
Divide 2400 in the ratio $3: 4: 5$
$\frac{2400}{3+4+5}=\frac{2400}{12}=200$
Sums are 600, 800 and 1000
10. Increasing and Decreasing in a Given Ratio.

Example:
If fuel consumption of 60 kg per minute is increased in a ratio of $5: 4$ what is the new consumption?
Consumption $=\frac{5}{4} \times 60=75 \mathrm{~kg}$ per min.

## Scales

11. If a map has a scale of $1: 50,000$ it means that I cm on the map represents $50,000 \mathrm{~cm}$ on the ground. In the same, way I km on the ground is represented by

$$
\frac{100,000}{50,000} \mathrm{~cm} \text { or } 2 \mathrm{~cm} .
$$

The scale 1:50,000 could also be given as ' 2 cm to I km'.

## Proportion

12. If two quantities are in direct proportion then an increase in one quantity causes a predictable increase in the other. An inversely proportional relationship means that an increase in one quantity causes a predictable decrease in the other.

Examples:

## a. Direct Proportion.

If 400 cards cost $£ 28$ find the cost of 650 cards.
Cost of 650 cards $=\frac{650}{400} \times 28=£ 45.50$
b. Inverse Proportion.

If it takes 6 men 12 days to paint a hangar, how long will it take 9 men?
Time for 9 men $=\frac{6}{9} \times 12=8$ days

## The 1 in 60 Rule

13. The 1 in 60 rule is used as a method of assessing track error and closing angle, and has long been favoured as a mental deduced reckoning (DR) navigation technique because of its flexibility, ease of use and relative accuracy (up to about $40^{\circ}$ ). The 1 in 60 rule postulates that an arc of one unit at a radius of 60 units subtends an angle of one degree (see Fig I).

## 13-2 Fig 1 The 1 in 60 Rule

$$
1^{\circ}
$$

60 Units
1 Unit
60 Units
14. In practical use, this 1 in 60 rule may be applied equally well to a right-angled triangle. It may be accepted that, in a right-angled triangle, if the length of the hypotenuse is 60 units, the number of units of length of the small side opposite the small angle will be approximately the same as the number of degrees in the small angle (see Fig 2).

## 13-2 Fig 2 Application to a Right-angled Triangle

## $1{ }^{\circ}$

## 60 Units

## 1 Unit

This approximation can be compared with the exact computation below:

| Short Side | Sine of Angle | Angle |
| :---: | :---: | :---: |
| 1 unit | $1 / 60=.0167$ | $0^{\circ} 57^{\prime}$ |
| 10 units | $10 / 60=.1667$ | $9^{\circ} 36^{\prime}$ |
| 20 units | $20 / 60=.3333$ | $19^{\circ} 28^{\prime}$ |
| 30 units | $30 / 60=.5000$ | $30^{\circ}$ |
| 35 units | $35 / 60=.5833$ | $35^{\circ} 41^{\prime}$ |
| 40 units | $40 / 60=.6667$ | $41^{\circ} 49^{\prime}$ |

15. Furthermore, since the navigator is likely to have distances on the required track marked on his map, the approximation is just as good if the distance gone is measured along the required track (see Fig 3). In either case, the distance gone is compared with the distance off track and the ratio of one to the other is reduced to an angle.

$$
\text { Track error (degrees) }=\frac{\text { Distance off Track } \times 60}{\text { Distance along Track }}
$$

## 13-2 Fig 3 Calculation of Track Error



Thus, an aircraft passing over a feature 2 miles port of the required track, after flying 30 miles has a track error of:

$$
\frac{2}{30}
$$

## CHAPTER 3 - AVERAGES

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## Introduction

1. Averages are discussed in some detail in Volume 13, Chapter 16, where it can be seen that an 'average' might mean any one of three quite different values. Of these the most useful and most commonly used is more accurately described as the arithmetic mean.

## Arithmetic Mean

2. The arithmetic mean of a set of values is defined as:

The sum of all the values
The number of values

Examples:
a. A rugby scrum has players of weights $92 \mathrm{~kg}, 89 \mathrm{~kg}, 86 \mathrm{~kg}, 94 \mathrm{~kg}, 97.5 \mathrm{~kg}, 97 \mathrm{~kg}, 96 \mathrm{~kg}$, and 95.5 kg . The arithmetic mean (or average) weight of the players may be calculated as follows:

$$
\begin{aligned}
\text { Average weight } & =\frac{92+89+86+94+97.5+97+96+95.5}{8} \\
& =93.375 \mathrm{~kg}
\end{aligned}
$$

b. The times taken to travel to work from Monday to Friday are $1 \mathrm{hr} 12 \mathrm{~min}, 1 \mathrm{hr} 18 \mathrm{~min}$, $1 \mathrm{hr} 14 \mathrm{~min}, 1 \mathrm{hr} 21 \mathrm{~min}$, and 1 hr 22 min . The average time taken in travelling to work can be calculated as follows:

$$
\text { Average time }=\frac{72+78+74+81+82}{5} \mathrm{mins}
$$

$$
=1 \mathrm{hr} 17.4 \mathrm{mins}
$$

3. The arithmetic mean is useful for presenting large amounts of data in a simplified form, and is most accurate when used in calculations involving data which do not include extreme values. This form of average may also yield data which are capable of further statistical analysis or mathematical treatment. It uses every value in a distribution, and is the most readily understood and commonly accepted representation of the term 'average'.

## Limitations in the Use of Arithmetic Mean

4. The arithmetic mean may produce distortions because of extreme values in a distribution.

Example:

The values of stamps to be auctioned are estimated at $£ 15, £ 17, £ 23, £ 24, £ 20$, and $£ 500$. The average (arithmetic mean) of their values is given by:

$$
\frac{15+17+23+24+20+500}{6}=£ 99.83
$$

However, it would clearly be misleading to describe the stamps as being of average value of approximately $£ 100$. A more accurate and fair description would be that with one exception the average value of the stamps is approximately $£ 20$.
5. The arithmetic mean can also produce impossible quantities where data is necessarily in discrete values, (e.g. 1.825 children in an average family).

## Weighted Averages

6. When calculating an average from more than one set of data, the figures cannot be combined without giving due regard to the relative sizes of the samples.

Example:
A class of 40 students score an average of 60 marks and a class of 20 students score an average 68 marks. The average might be calculated as:

$$
\frac{60+68}{2}=64 \text { but this is clearly incorrect. }
$$

The marks should be weighted according to the number of students in each group thus:

$$
\frac{60(40)+68(20)}{40+20}=\frac{2400+1360}{60}=62.66
$$

This is termed the weighted average, and it gives a more accurate measure in this type of situation. Weighted averages may also be used when it is desired to give certain quantities greater importance than others within a distribution.

## CHAPTER 4 - BASIC VECTOR PROCESSES

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## Introduction

1. Many physical quantities like mass, volume, density, temperature, work and heat, are completely specified by their magnitudes. Such quantities are known as scalar quantities or scalars. Other physical quantities possess directional properties as well as magnitudes, so that each magnitude must be associated with a definite direction in space before the physical quantity can be completely described. It is found that some, though not all, of these directed quantities possess a further common property in that they obey the same triangle (or parallelogram) law of addition. Directed quantities which obey the triangular law of addition are known as vectors.
2. Definition of a Vector. Any quantity which possesses both magnitude and direction, and which obeys the triangle law of addition is a vector.

## Graphical Representation of Vectors

3. A Vector quantity may be represented by a line having:
a. Direction.
b. Magnitude.
c. Sense.

Fig 1 shows a vector having a direction defined by the angle $\theta$, a magnitude equal to the length OP and a sense indicated by the arrow. The vector OP may be represented by a symbol which may be either in bold type or underlined, eg OP may be represented by a. When a vector is shown graphically, a scale should be given.

## 13-4 Fig 1 Graphical Representation of a Vector



## The Resultant Vector

4. The resultant of a system of vectors is that single vector which would have the same effect as the system of vectors.

## The Resolution of Vectors into Components

5. It may be convenient to resolve a vector into two components acting at right angles to each other. In Fig 2, vector OP is at an angle of $30^{\circ}$ to Ox . The vector may be divided into two components OA and OB at right angles to each other. Resolved graphically, the component OA may be measured as 6.9 units and OB as 4 units. To calculate the magnitudes of OA and OB mathematically, use is made of the trigonometrical ratio of OA to OP , thus:

$$
\begin{aligned}
& \frac{O A}{O P}=\cos 30^{\circ} \\
\therefore & O A=O P \cos 30^{\circ}=6.93 \text { units }
\end{aligned}
$$

and $\frac{\mathrm{OB}}{\mathrm{OP}}=\frac{\mathrm{AP}}{\mathrm{OP}}=\sin 30^{\circ}$
$\therefore \mathrm{OB}=\mathrm{OP} \sin 30^{\circ}=4$ units

13-4 Fig 2 Resolution of a Vector into Components


## Addition of Vectors

6. Co-linear Vectors. The simplest case of the addition of vectors occurs when the vectors are parallel. There are two cases to consider:
a. Parallel Vectors Acting in the Same Direction. Consider two forces acting on a body, one of magnitude 4 units and the other of magnitude 3 units. The two forces act in the same direction. Fig 3 shows the two vectors, $\underline{a}$ of four units and $\underline{b}$ of three units. The sum of the vectors is $\underline{a}+\underline{b}=4+3=7$ units.

13-4 Fig 3 Addition of Parallel Vectors Acting in the Same Direction

b. Parallel Vectors Acting in Opposite Directions. When two forces acting on a body are parallel and in opposite directions the vector representation is as Fig 4. The forces are of magnitude seven units and three units. The sum of the vectors is $\underline{a}-\underline{b}=7-3=4$ units.

## 13-4 Fig 4 Addition of Parallel Vectors Acting in Opposite Directions


7. Non Co-linear Vectors. Any two vectors may be added together. Fig 5 shows a triangle of vectors. A displacement from $O$ to $A$ is represented by vector $\underline{a}$, and a further displacement from $A$ to $B$ is represented by vector $\underline{b}$. The sum of the displacements is equivalent to a displacement from $O$ to $B$, or $\underline{a}+\underline{b}$.
8. Vector Difference. The difference of two vectors may be represented as $\underline{a}+(-\underline{b})$, the vector $-\underline{b}$ being $\underline{b}$ rotated through $180^{\circ}$ as shown in Fig 6.

## 13-4 Fig 5 Triangle of Vectors



## 13-4 Fig 6 Vector Difference



## The Polygon of Vectors

9. When more than two vectors are to be resolved they can be added or subtracted two at a time as shown in Fig 7. To resolve $\underline{a}+\underline{b}+\underline{c}+\underline{d}, P Q$ is drawn to represent $\underline{a}$, then from the terminal point of $P Q, Q R$ is drawn to represent $\underline{b}$, and so on until all the vectors are represented. $P R=\underline{a}+\underline{b}$ and, in the triangle $P R S, P S=P R+R S=\underline{a}+\underline{b}+\underline{c}$. In the triangle $P S T, P T=P S+S T=\underline{a}+\underline{b}+\underline{c}+\underline{d}$. Any number of vectors can be summed in this way.

## 13-4 Fig 7 The Polygon of Vectors



## CHAPTER 5 - INDICES AND LOGARITHMS

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## INDICES

## Introduction

1. When a number is successively multiplied by itself, it is said to be raised to a power. Thus:
$4 \times 4$ is four raised to the power of 2 (or 4 squared)
$5 \times 5 \times 5$ is five raised to the power of 3 (or 5 cubed)
$6 \times 6 \times 6 \times 6 \times 6$ is six raised to the power of 5
This is usually written as the number that is to be multiplied, known as the base, together with the number of times it is to be multiplied as a superscript, known as the index. Thus:


$$
\text { and } 6 \times 6 \times 6 \times 6 \times 6=6^{5} .
$$

The notation is not confined to actual numbers; algebraic symbols and expressions may be similarly expressed. Thus:

$$
\begin{aligned}
& m \times m \times m \times m=m^{4} \\
& \text { and, }(a+2) \times(a+2) \times(a+2)=(a+2)^{3} .
\end{aligned}
$$

## Multiplication and Division Rules

2. Multiplication. Suppose it is necessary to multiply $2^{5}$ by $2^{3}$. Now, $2^{5}=2 \times 2 \times 2 \times 2 \times 2$ and $2^{3}=2 \times 2 \times$ 2.

$$
\therefore 2^{5} \times 2^{3}=(2 \times 2 \times 2 \times 2 \times 2) \times(2 \times 2 \times 2)=2^{8}
$$

i.e. the result is obtained by adding the indices, e.g. $2^{16} \times 2^{8}=2^{24}$
3. Division. If it is necessary to divide, say, $2^{5}$ by $2^{3}$ then this may be written as:

$$
\frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2}
$$

Cancelling the terms yields the result $2^{2}$ i.e. the result is obtained by subtracting the indices, e.g. $m^{10}-m^{6}=m^{4}$. Consider $m^{4} \div m^{4}$. Clearly, any number or expression divided by itself $=1$. By the subtraction rule $m^{4} \div m^{4}=m^{0}$. Therefore, $m^{0}=1$. Indeed, by the same reasoning, any number or expression raised to the power zero $=1$.

## Negative Indices

4. Consider $2^{5} \div 2^{6}$. This is equivalent to:

$$
\frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2 \times 2}=\frac{1}{2}
$$

By the division rule $2^{5} \div 2^{6}=2^{-1}$. So $2^{-1}=\frac{1}{2}$, i.e. the negative index indicates a reciprocal. Similarly, for example, $\mathrm{m}^{-3}=\frac{1}{\mathrm{~m}^{3}}$

## Fractional Indices

## 5. Consider the problem:

"What number, when multiplied by itself $=2$ ?" Expressing this in index form, and using the multiplication rule:

$$
\begin{aligned}
& 2^{a} \times 2^{a}=2^{1} \\
& \therefore a+a=1, \text { i.e. } 2 a=1 \\
& \therefore a=\frac{1}{2}
\end{aligned}
$$

Thus, the fractional power $\frac{1}{2}$ has the meaning of square root. Similarly, $\frac{1}{3}=$ cube root, $\frac{1}{4}=$ fourth root and so on.

## Power of a Power

6. Consider the expression $\left(2^{2}\right)^{3}$. This is equivalent to:

$$
(2 \times 2) \times(2 \times 2) \times(2 \times 2)=2^{6}
$$

The result is obtained by multiplying the indices. In general terms: $\left(a^{m}\right)^{n}=a^{m n}$

## Standard and Engineering Forms

7. In science and engineering a very wide range of numerical values are frequently encountered. For example, the velocity of light is approximately $300,000,000$ metres per second whilst the wavelengths of light are in the approximate range of 0.0000000008 metres to 0.0000000004 metres. Such very large and small numbers are clearly cumbersome in use and often difficult to comprehend quickly. In order to overcome this difficulty, it is common practice to express numbers in a standard form or in an 'engineering' form, making use of index notation.
8. The Standard Form. The standard form of a number consists of only one digit in front of the decimal point which is then multiplied by the appropriate power of 10 , i.e. in the form:

$$
A \times 10^{n}
$$

where A is between 1.0000 and 9.9999 , and the index, n , is the required power of 10 . Thus, for example:

$$
\begin{aligned}
& 67.9 \text { in standard form }=6.79 \times 10^{1} \\
& 679 \text { in standard form }=6.79 \times 10^{2} \\
& 300,000,000 \text { in standard form }=3 \times 10^{8} \\
& 0.00679 \text { in standard form }=6.79 \times 10^{-3} \\
& 0.0000000008 \text { in standard form }=8.0 \times 10^{-10}
\end{aligned}
$$

9. Engineering Form. Engineering notation is also commonly available on calculators. It differs from the standard form in that the power of 10 is always a multiple of 3 . For example:

$$
\begin{aligned}
& 300,000,000 \text { in engineering form }=300 \times 10^{6} \\
& 0.679 \text { in engineering form }=679 \times 10^{-3}
\end{aligned}
$$

## Summary

10. In summary the rules for the handling of numbers or algebraic expressions in index form are as follows:

$$
\begin{gathered}
a^{0}=1 \\
a^{m} \times a^{n}=a^{m+n} \\
a^{m} \div a^{n}=a^{m-n} \\
\left(a^{m}\right)^{n}=a^{m n} \\
a^{-m}=\frac{1}{a^{m}} \\
a^{\frac{1}{m}}=\sqrt[m]{a}
\end{gathered}
$$

## LOGARITHMS

## Introduction

11. The concept of logarithms is closely associated with the notion of indices. If a positive number, $y$, is expressed in index form with a base a, i.e.

$$
y=a^{x}
$$

then the index, $x$, is known as the logarithm of $y$ to the base $a$. Thus:
If $y=a^{x}$, then $x=\log _{a} y$
For example:
$y=32=2^{5}, \therefore \log _{2} 32=5$
If, $\log _{10} y=3$, then $y=10^{3}$

## Common Logarithms

12. The most commonly used form of logarithms is to the base 10. The abbreviation 'log' is used and unless a base is explicitly stated or otherwise implied then 10 may be assumed. Using index notation any positive integer, N , may be written as:

$$
N=10^{x}, \text { then } \log _{10} N=x
$$

Values of the common log of any number may be found either from tables or from an electronic calculator. Prior to the widespread use of electronic calculators, logs were used as an aid to calculation. As logs are no more than indices, they obey the same rules as indices. Thus if it is necessary to multiply two numbers this can be achieved by finding the logs of the numbers, adding these and then finding the number corresponding to this log. Similarly, division may be accomplished by subtracting logs, the power of a number can be found by multiplying its log by the power, and the root of a number by dividing its log by the root index.

## Naperian, Natural or Hyperbolic Logarithms

13. In many natural processes, the rate of growth or decay of a substance is proportional to the amount of substance present at a given time. It has been found that this relationship can be expressed in terms of a universal constant known as the exponential constant, e. The number, e, is irrational and is the sum of the infinite series:

$$
1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\ldots \ldots
$$

where the symbol! means factorial, i.e. that number multiplied by all of the positive integers less than itself, e.g. $6!=6 \times 5 \times 4 \times 3 \times 2 \times 1$.
14. By taking an appropriate number of terms, e can be calculated to any desired level of accuracy. Note that as the terms have factorials of ever increasing numbers as their denominators then each successive term becomes smaller and the reduction in significance is rapid. As a comparison, the third term is 0.5 , the seventh term is 0.00139 , and the tenth term is 0.00000276 . A value to 4 significant figures can be calculated from the first seven terms as 2.718 .
15. Logarithms with $e$ as the base are known as natural or Naperian (occasionally hyperbolic) logarithms. They are frequently encountered in scientific texts and are the only logarithms used in calculus. The abbreviation Ln is generally used. Whereas natural logarithms follow the same rules as common logarithms and can be used for the same purposes, they are rather more difficult to extract from tables. In any case, the use of logarithms to carry out arithmetic has been superseded by the electronic calculator.

## Decibels

16. An application of logarithms is encountered in the field of amplification or gain, which is often expressed in units of bels or more normally decibels. If $\mathrm{P}(\mathrm{I})$ is the input power into an amplifier and $\mathrm{P}(\mathrm{O})$ is the output power, the gain is given by:

$$
\mathrm{G}=\log \frac{\mathrm{P}(\mathrm{O})}{\mathrm{P}(\mathrm{I})} \text { bels }=10 \log \frac{\mathrm{P}(\mathrm{O})}{\mathrm{P}(\mathrm{I})} \text { decibels }
$$

The ratio of the powers $\frac{\mathrm{P}(\mathrm{O})}{\mathrm{P}(\mathrm{I})}$ can be expressed in terms of output and input voltages as:

$$
\begin{aligned}
\frac{\mathrm{P}(\mathrm{O})}{\mathrm{P}(\mathrm{I})} & =\frac{\mathrm{V}(\mathrm{O})^{2}}{\mathrm{~V}(\mathrm{I})^{2}}=\left(\frac{\mathrm{V}(\mathrm{O})}{\mathrm{V}(\mathrm{I})}\right)^{2} \\
\therefore \mathrm{G} & =10 \log \left(\frac{\mathrm{~V}(\mathrm{O})}{\mathrm{V}(\mathrm{I})}\right)^{2} \\
& =20 \log \frac{\mathrm{~V}(\mathrm{O})}{\mathrm{V}(\mathrm{I})} \text { decibels }
\end{aligned}
$$

Similarly in terms of output and input currents:

$$
\mathrm{G}=20 \log \frac{\mathrm{I}(\mathrm{O})}{\mathrm{I}(\mathrm{I})} \text { decibels }
$$

17. Example. If an amplifier has a gain of 30 decibels, calculate the input voltage required to produce an output of 50 volts.

$$
\text { Using } \begin{aligned}
\mathrm{G} & =20 \log \frac{\mathrm{~V}(\mathrm{O})}{\mathrm{V}(\mathrm{I})} \\
30 & =20 \log \frac{50}{\mathrm{~V}(\mathrm{I})} \\
1.5 & =\log \frac{50}{\mathrm{~V}(\mathrm{I})}
\end{aligned}
$$

Taking antilogs:

$$
\begin{aligned}
31.62 & =\frac{50}{\mathrm{~V}(\mathrm{I})} \\
\mathrm{V}(\mathrm{I}) & =\frac{50}{31.62} \\
& =1.581 \mathrm{~V}
\end{aligned}
$$

## CHAPTER 6 - GRAPHS

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## Introduction

1. The term 'graph' is usually applied to a pictorial representation of how one variable changes in response to changes in another. This chapter will deal with the form of simple graphs, together with the extraction of data from them, and from the 'families of graphs' and carpet graphs that are frequently encountered in aeronautical publications. Although not strictly a 'graph', the Nomogram will also be covered. The pictorial representation of data in such forms as histograms, frequency polygons, and frequency curves will be treated in Volume 13, Chapter 16.

## Coordinate Systems

2. Graphs are often constructed from a table of, say, experimental data which gives the value of one variable, $x$, and the experimentally found value of the corresponding variable, $y$. In order to construct a graph from this data it is necessary to establish a framework or coordinate system on which to plot the information. Two such coordinate systems are commonly used: Cartesian coordinates and Polar coordinates. Both systems will be described below, but the remainder of this chapter will be concerned only with the Cartesian system.
3. Cartesian Coordinates. Cartesian coordinates are the most frequently used system. Two axes are constructed at right angles, their intersection being known as the origin. Conventionally the horizontal ' $x$ ' axis represents the independent variable; the vertical ' $y$ ' axis represents the dependent variable, i.e. the value that is determined for a given value of $x$. Any point on the diagram can now be represented uniquely by a pair of coordinate values written as ( $\mathrm{x}, \mathrm{y}$ ) provided that the axes are suitably scaled. It is not necessary for the axes to have the same scale. Thus, in Fig 1, the point $P$ has the coordinates ( 3,4 ), i.e. it is located by moving 3 units along the x axis and then vertically by 4 ' y ' units. It is sometimes inconvenient to show the origin $(0,0)$ on the diagram when the values of either x or y cover a range which does not include 0 . Fig 2 shows such an arrangement where the $x$-axis is scaled from 0 but the corresponding values of $y$ do not include 0 . The intersection of the axes is the point $(0,200)$. It should be noted from Fig 1 that negative values of $x$ or $y$ can be shown to the left and below the origin respectively.

13-6 Fig 1 Cartesian Coordinates


13-6 Fig 2 Cartesian Coordinates - Displaced Origin

4. Polar Coordinates. Polar coordinates specify a point as a distance and direction from an origin. Polar coordinates are commonly encountered in aircraft position reporting where the position is given as a range and bearing from a ground beacon; they are also used in certain areas of mathematics and physics. As with Cartesian systems it is necessary to define an origin, but only one axis or reference line is required. Any point is then uniquely described by its distance from the origin and by the angle that the line joining the origin to the point makes with the reference line. The coordinates are written in the form $(r, \theta)$, with $\theta$ in either degree or radian measure. Conventionally, angles are measured anti-clockwise
from the reference line as positive and clockwise as negative. Fig 3 illustrates the system. Point $Q$ has the coordinates $\left(3,30^{\circ}\right)$ or $\left(3,-330^{\circ}\right)$ in degree measure; $\left(3, \frac{\pi}{6}\right)$ or $\left(3, \frac{-11 \pi}{6}\right)$ in radian measure.

## 13-6 Fig 3 Polar Coordinates



## The Straight Line Graph

5. Table 1 shows a series of values of $x$ and the corresponding values of $y$. Fig 4 shows these points plotted on a graph.

Table 1 Values of $x$ and $y$

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| y | -6 | -4 | -2 | 0 | 2 | 4 | 6 |

It will be seen that all the points lie on a straight line which passes through the origin. It is clear from the table of values that if the value of $x$ is, say, doubled then the corresponding value of $y$ is also doubled. Such a relationship is known as direct proportion and the graphical representation of direct proportion is always a straight line passing through the origin. In general the value of $y$ corresponding to a value of $x$ may be derived by multiplying $x$ by some constant factor, $m$, ie: $y=m x$. In the example, $m$ has the value 2, i.e. $y=2 x$. Because such a relationship produces a straight-line graph, it is known as a linear relationship and $y=m x$ is known as a linear equation. Such relationships are not uncommon. For example the relationship between distance travelled, (d), speed, ( s ), and time, ( t ) is given by $d=s t$. This would be a straight-line graph with $d$ plotted on the $y$-axis and $t$ on the $x$-axis.

## 13-6 Fig 4 Graph of $\mathbf{y}=\mathbf{2 x}$


6. It is of course possible for a straight line through the origin to slope down to the right rather than up to the right as in the previous example. In this case positive values of $y$ are generated by negative values of $x$ and the equation becomes: $y=-m x$
7. Consider now the values of $x$ and $y$ in Table 2, and the associated graph, Fig 5.

Table 2 Values of $x$ and $y$

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| y | -4 | -2 | 0 | 2 | 4 | 6 | 8 |

## 13-6 Fig 5 Graph of $y=2 x+2$



Clearly the graph is closely related to the previous example of $y=2 x$. In essence the line has been raised up the $y$-axis parallel to the $y=2 x$ line. Investigation of the table of values will reveal that the relationship between x and y is governed by the equation: $\mathrm{y}=2 \mathrm{x}+2$, and in general, a graph of this type has the equation: $y=m x+c$, where $c$ is a constant. It will be apparent that the equation $y=m x$ is identical to the equation $y=m x+c$ if a value of 0 is attributed to the constant $c$. Thus, $y=m x+c$ is the general equation for a straight line, m and c being constants which can be positive, negative or zero. A zero value of $m$ generates a line parallel to the $x$-axis. The value of $c$ is given by the point at which the line crosses the $y$-axis and is known as the intercept.
8. Gradient. Consider Fig 6 which shows two straight-line graphs: $y=2 x$ and $y=4 x$. Both lines pass through the origin and the essential difference between them is their relative steepness. The line $y=4 x$ shows $y$ changing faster for any given change in $x$ than is the case for $y=2 x$. The line $y=4 x$ is said to have a steeper gradient than the line $y=2 x$. The gradient is defined as the change in $y$ divided by the corresponding change in $x$, ie $\frac{y}{x}$. Rearranging the general equation for a straight line $(y=m x)$, to make $m$ the subject gives $m=\frac{y}{x}$, i.e. the constant $m$ is the gradient of the straight line. As the line $y=m x+c$ has been shown to be parallel to $y=m x$, this clearly has the same gradient, given by the value of $m$. In the equation:

$$
\text { distance }=\text { speed } \times \text { time }
$$

'speed' is equivalent to ' $m$ ' in the general equation, and it is apparent that the gradient, speed, represents a rate of change - in this case the rate of change of distance with time. This concept of the gradient representing a rate of change will become important when dealing with calculus in Volume 13, Chapter 13.

## 13-6 Fig 6 Graphs of $y=2 x$ and $y=4 x$



## Non-Linear Graphs

9. Not all relationships result in straight-line graphs, indeed, they are a minority. A body falling to earth under the influence of gravity alone falls a distance $y$ feet in time $t$ seconds governed by the equation:

$$
y=16 t^{2}
$$

Table 3 shows a range of values of $t$ with the corresponding values of $y$, and Fig 7 the associated graph.

## Table 3 Values of $t$ and $y$

| t | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| y | 0 | 16 | 64 | 144 | 256 | 400 |

Although not relevant in this example, notice that negative values of $t$ produce identical positive values of $y$ to their positive counterparts. The graph is therefore symmetrical about the $y$-axis and the shape is known as a parabola. The constant in front of the $t^{2}$ term determines the steepness of the graph.

$$
\text { 13-6 Fig } 7 \text { Graph of } y=16 t^{2}
$$


10. Consider now the problem "How long will it take to travel 120 km at various speeds?" This can be expressed as the equation:

$$
\mathrm{t}=\frac{120}{\mathrm{~s}}
$$

where $t=$ time in hours and $s=s p e e d$ in $k m / h r$. This is an example of inverse proportion, ie an increase in $s$ results in a proportional decrease in $t$. If values are calculated for $s$ and $t$, and a graph is plotted, it will have the form illustrated in Fig 8 known as a hyperbola.

$$
\text { 13-6 Fig } 8 \text { Graph of } t=\frac{120}{s} \text { - Inverse Proportion }
$$


11. Graphs of $y=\sin x$ and $y=\cos x$ will be encountered frequently. The shapes of the graphs are shown below (Fig 9).

## 13-6 Fig 9 Graphs of sin and cos

Fig 9a $y=\sin x$


Fig 9b $\mathbf{y}=\cos \mathbf{x}$


The sine graph is shown with the $x$-axis scaled in degrees while the cosine graph has the $x$-axis scaled in radians. Either is correct; the radian form is frequently encountered in scientific texts. Sketches of these graphs are useful when trying to determine the value and sign of trigonometric functions of angles outside of the normal $0^{\circ}$ to $90^{\circ}$ range. Notice that both graphs repeat themselves after $360^{\circ}$ ( $2 \pi$ radians).
12. Finally, it is worth considering the graph that describes the relationship:

$$
y=e^{a x}
$$

where a is a positive or negative constant. This form of equation is very common in science and mathematics and variants of it can be found in the description of radioactive decay, in compound interest problems, and in the behaviour of capacitors. The irrational number 'e' equates to 2.718 to 4 significant figures. The graph of $y=e^{x}$ is shown in Fig 10a and that of $y=e^{-x}$ in Fig 10b. The significant point about these graphs, which are known as exponential graphs, is that the rate of increase (or decrease) of $y$ increases (or decreases) depending upon the value of $y$. A large value of $y$ exhibits a high rate of change. It is also worth noting that there can be found a fixed interval of $x$ over which the value of $y$ doubles (or halves) its original value no matter what initial value of y is chosen.

This is the basis of the concept of radioactive decay half-life. The interval is equivalent to $\frac{0.693}{a}$ where $a$ is the constant in the equation $y=e^{a x}$.

## 13-6 Fig 10 Exponential Graphs

Fig 10a $y=e^{x}$


Fig 10b $y=e^{-x}$
13. Logarithmic Scales. Clearly plotting and interpreting from exponential graphs can be difficult. The problem can be eased by plotting on a graph where the $x$-axis is scaled linearly while the $y$-axis has a logarithmic scale. This log-linear graph paper reduces the exponential curve to a straight line. A comparison between the linear and log-linear plots of $y=e^{x}$ is shown in Fig 11.

## 13-6 Fig 11 Comparison Between Linear and Log-linear Plots

Fig 11a $y=e^{x}$ (Linear Scales)


Fig 11b $y=e^{x}$ (Log-linear Scales)


## The Presentation and Extraction of Data

14. So far this chapter has been concerned with the mathematical background to simple graphs. More commonly graphs will be encountered and used as a source of data, especially in the field of flight planning and aircraft performance. Whilst occasionally these graphs will be either the simple forms already described or variations on these forms, more often rather complex graphs are used as being the only practical way of displaying complex relations. Two such types of complex graph will be described here in order to establish the method of data extraction. Finally, the nomogram will be discussed.
15. Carpet Graphs. An example of a carpet graph is shown in Fig 12.


The aim of the graph is to indicate the lag in the altimeter experienced in a dive. Unlike the graphs already discussed where one input, 'x', produced one output, 'y', the carpet graph has two inputs for one output. The output is on the conventional ' $y$ ' axis but there is no conventional ' $x$ ' axis, rather there are two input axes. On the right hand edge of the 'carpet' diagram are figures for dive angle whilst on the bottom edge are figures for indicated air speed. To use the graph it is necessary to enter with one parameter, say dive angle, and follow the relevant dive angle line into the diagram until it intersects the appropriate IAS line. Intermediate dive angle and IAS values need to be interpolated, thus in the example values of $17^{\circ}$ and 375 kt have been entered. From the point of intersection a horizontal line is constructed which will give the required lag correction figure where it intersects the ' $y$ ' axis, -118 feet in the example.
16. Families of Graphs. It is often necessary to consider a number of independent factors before coming to an end result. In this situation a family of graphs is frequently used to present the required information. Fig 13 shows such a family designed for the calculation of the aircraft's take-off ground run. Apart from the aircraft configuration which is indicated in the graph title, there are five input parameters. There will very often be a series of related graphs with variations in the title, for example in this case there will be another family of graphs for an aircraft with wing stores. It is clearly important that the correct set is selected. The method of using the graph will be described with reference to the example.

## 13-6 Fig 13 Family of Graphs


17. At the left end is a small carpet graph. Starting with the value of outside air temperature $\left(21^{\circ}\right)$ proceed vertically to intersect the altitude line ( 2,000 feet). Alternatively enter the 'carpet' at the intersection of the altitude and temperature relative to ISA. From this intersection proceed horizontally into the next graph to intersect the vertical reference line, marked RL. From this point parallel the curves until reaching the point representing the value of runway slope as indicated on the bottom scale (1\% uphill). From here construct a horizontal line to the next graph reference line. Repeat the procedure of paralleling the curves for aircraft mass ( 4.8 tonnes) and then proceed horizontally into the last graph for head/tail wind ( 20 kt head) which is used in the same manner. Finally the horizontal line is produced to the right hand scale where the figure for ground run can be read ( 1,900 feet).
18. The Nomogram. The nomogram is not strictly a graph but a diagrammatic way of solving rather complex equations. There are usually two input parameters for which one or two resultant outputs may be derived. Fig 14 shows a nomogram for the determination of aircraft turning performance. The equations involved are:

$$
\begin{aligned}
\text { Rate of turn TAS } & =\frac{\text { TAS }}{\text { Radius of Turn }} \\
& =\text { Acceleration } \times \text { tan Bank Angle }
\end{aligned}
$$

This nomogram consists of four parallel scaled lines. Two known values are joined by a straight line, and the intersection of this line with the other scales gives the unknown values. In the example illustrated, an input TAS of 180 kts with a Rate 1 turn gives a resultant of 1.1 g , and a radius of turn of 1 nm .

## 13-6 Fig 14 A Nomogram



The variables are related by the equation $V_{w}=\frac{V}{T} \quad g \tan \theta$
To use the Nomogram join two known values by a straight line and the intersection of this line on its projection with the other scales give the unknown values

| Example : TAS $(\mathrm{V})$ | $=180 \mathrm{Kt}$ |
| :--- | :--- |
| Rate of Turn $(\mathrm{W})$ | $=1$ |
| Angle of Bank $(\theta)$ | $=25^{\circ}$ |
| Using the Nomogram (see dotted line) |  |
| Resultant Acceleration (g) | $=1.1$ |
| Radius of Turn $(\mathrm{T})$ | $=1 \mathrm{~nm}$ |

## CHAPTER 7 - UNIT CONVERSIONS

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## SI Units

1. The Systeme International (SI) d'Unites is a metric system based upon seven fundamental units which are:

| Length | - | metre (m) |
| :--- | :--- | :--- |
| Mass | - | kilogram (kg) |
| Time | - | second (s) |
| Electric current | - | ampere (A) |
| Luminous intensity | - | candela (cd) |
| Temperature | - | kelvin (K) |
| Amount of substance | - | mole (mol) |

Other SI units in common usage are:

| Frequency | - | hertz (Hz) |
| :--- | :--- | :--- |
| Energy | - | joule (J) |
| Force | - | newton (N) |
| Power | - | watt (W) |
| Electric charge | - | coulomb (C) |
| Potential difference | - | volt (V) |
| Capacitance | - | farad (F) |
| Inductance | - | henry (H) |
| Magnetic field | - | tesla (T) |

## Magnitudes

2. Prefixes used with SI Units to indicate magnitudes:

| Factor | Name of Prefix | Symbol |
| :---: | :---: | :---: |
| $10^{-18}$ | atto | a |
| $10^{-15}$ | femto | f |
| $10^{-12}$ | pico | p |
| $10^{-9}$ | nano | n |
| $10^{-6}$ | micro | $\mu$ |
| $10^{-3}$ | milli | m |
| $10^{-2}$ | centi | c |
| $10^{-1}$ | deci | d |
| 10 | deca | da |
| $10^{2}$ | hecto | h |
| $10^{3}$ | kilo | k |
| $10^{6}$ | mega | M |
| $10^{9}$ | giga | G |
| $10^{12}$ | tera | T |
| $10^{15}$ | peta | P |
| $10^{18}$ | exa | E |

## Conversion Factors

3. The following conversion factors have been selected as those most likely to be of general use.

## Length

4. 

| 0.0394 | inches (in) |
| :---: | :--- |
| 3.2808 | feet $(\mathrm{ft})$ |
| 1.0936 | yards $(\mathrm{yd})$ |
| $5.399 \times 10^{-4}$ | nautical miles $(\mathrm{nm})$ |
| 0.6214 | miles |


| millimetres $(\mathrm{mm})$ | 25.40 |
| :--- | :---: |
| metres $(\mathrm{m})$ | 0.3048 |
| metres $(\mathrm{m})$ | 0.9144 |
| metres $(\mathrm{m})$ | 1852.0 |
| kilometres $(\mathrm{km})$ | 1.6093 |

Area
5.

| 0.1550 | square inches $\left(\mathrm{in}^{2}\right)$ |
| :---: | :--- |
| 10.7639 | square feet $\left(\mathrm{ft}^{2}\right)$ |
| 1.1960 | square yards $\left(\mathrm{yd}^{2}\right)$ |

square centimetres $\left(\mathrm{cm}^{2}\right)$
6.4516
square metres $\left(\mathrm{m}^{2}\right) \quad 0.0929$
square metres $\left(\mathrm{m}^{2}\right)$
0.8361

## Volume

6. 

0.2200
0.2643
0.0353
35.3147
1.3080
0.0610
$1 \times 10^{-3}$
gallons $(\mathrm{UK})$
gallons $(\mathrm{US})$
cubic feet $\left(\mathrm{ft}^{3}\right)$
cubic feet $\left(\mathrm{ft}^{3}\right)$
cubic yards $\left(\mathrm{yd}^{3}\right)$
cubic inches $\left(\mathrm{in}^{3}\right)$
cubic metres $\left(\mathrm{m}^{3}\right)$
litres (I)
4.5460
litres (I) 3.785
litres (I) 28.3161
cubic metres $\left(\mathrm{m}^{3}\right) \quad 0.0283$
cubic metres $\left(\mathrm{m}^{3}\right) \quad 0.7646$
cubic centimetres $\left(\mathrm{cm}^{3}\right) \quad 16.3871$
litres (I) 1000.0
Multiply By $\longleftarrow$
To
To Convert

|  |  | To Convert | To $\longrightarrow$ | Multiply By |
| :---: | :---: | :---: | :---: | :---: |
| Mass |  |  |  |  |
| 7. | 0.0353 | ounces (oz) | grams (g) | 28.3495 |
|  | 2.2046 | pounds (lb) | kilograms (kg) | 0.4536 |
|  | 0.0685 | slugs | kilograms (kg) | 14.5939 |
| Velocity |  |  |  |  |
| 8. | 3.2808 | feet/second (ft/s) | metres/second (m/s) | 0.3048 |
|  | 1.9685 | feet/minute (ft/min) | centimetres/second (cm/s) | 0.5080 |
|  | 0.6214 | miles/hour (mph) | kilometres/hour (km/h) | 1.6093 |
|  | 2.2369 | miles/hour (mph) | metres/second (m/s) | 0.4470 |
|  | 0.5400 | knots (kt) | kilometres/hour(km/h) | 1.8520 |
|  | 0.5921 | knots (kt) | feet/second (fps) | 1.6889 |
|  | 1.9426 | knots (kt) | metres/second (m/s) | 0.5148 |

## Acceleration

9. 

## Force

10. 

| 0.2248 | pounds-force (lbf) |
| :--- | :--- |
| 2.2046 | pounds-force (lbf) |
| 7.2330 | poundals (pdl) |
| 0.1020 | kilograms-force (kgf) |
| 32.174 | poundals (pdl) |


| newtons $(\mathrm{N})$ | 4.4482 |
| :--- | :--- |
| kilograms-force (kgf) | 0.4536 |
| newtons (N) | 0.1383 |
| newtons (N) | 9.8067 |
| pounds-force (lbf) | 0.0311 |

## Torque

11. 

0.7376
8.8507
0.1020
pounds-force feet (lbf ft)
pounds inches ( lb in )
kilograms-force metres (kgf m)
newton metres (Nm)
1.3558
8.8507
kilograms-force metres (kgf m)
newton metres (Nm)
0.1130
0.1020
feet/second ${ }^{2}\left(\mathrm{ft} / \mathrm{s}^{2}\right)$
gravitational acceleration
pounds-force (lbf)
pounds-force (lbf)
poundals (pdl)
kilograms-force (kgf)
poundals (pdl)
metres/second ${ }^{2}\left(\mathrm{~m} / \mathrm{s}^{2}\right)$
0.3048
0.1020
metres/second ${ }^{2}\left(\mathrm{~m} / \mathrm{s}^{2}\right)$
9.8067
9.8067

## Pressure

12. 

$9.869 \times 10^{3}$
0.0680
0.1450
0.0100
10.00
33.86
1.000
25.4
7.493
atmospheres (atm)
atmospheres (atm)
pounds-force/inch ${ }^{2}(\mathrm{psi})$
bars
millibars (mbar)
millibars (mbar)
newtons/metre ${ }^{2}\left(\mathrm{~N} / \mathrm{m}^{2}\right)$
mm mercury $(\mathrm{mm} \mathrm{Hg})$
mm mercury $(\mathrm{mm} \mathrm{Hg})$

| kilopascals $(\mathrm{kPa})$ | 101.30 |
| :--- | :---: |
| pounds-force $/ \mathrm{inch}^{2}$ (psi) | 14.6960 |
| kilopascals $(\mathrm{kPa})$ | 6.8948 |
| kilopascals $(\mathrm{kPa})$ | 100.0 |
| kilopascals $(\mathrm{kPa})$ | 0.1000 |
| inches mercury (in Hg) | 0.0295 |
| pascals (Pa) | 1.000 |
| inches mercury (in Hg) | 0.0394 |
| kilopascals $(\mathrm{kPa})$ | 0.1334 |
| To Convert |  |


|  |  | To Convert $\longrightarrow$ | To $\longrightarrow$ | Multiply By |
| :---: | :---: | :---: | :---: | :---: |
| Density |  |  |  |  |
| 13. | 0.0624 | pounds/foot ${ }^{3}\left(\mathrm{lb} / \mathrm{ft}^{3}\right)$ | kilograms/metre ${ }^{3}\left(\mathrm{~kg} / \mathrm{m}^{3}\right)$ | 16.0185 |
|  | $10^{-3}$ | grams/centimetre ${ }^{3}\left(\mathrm{~g} / \mathrm{cm}^{3}\right)$ | kilograms/metre ${ }^{3}\left(\mathrm{~kg} / \mathrm{m}^{3}\right)$ | 1000.0 |
|  | 0.0100 | pounds/gallon | kilograms/metre ${ }^{3}\left(\mathrm{~kg} / \mathrm{m}^{3}\right)$ | 99.776 |
|  | 10.0221 | pounds/gallon | kilograms/litre (kg/l) | 0.0998 |
| Power |  |  |  |  |
| 14. | 1.3410 | horsepower (hp) | kilowatts (kW) | 0.7457 |
|  | 1.8182 | horsepower (hp) | foot pounds-force/second (ft lbf/s) | 550.0 |
|  | 0.7376 | foot pounds-force/second (ft lbf/s) | watts (W) | 1.3558 |
|  | $0.7376 \times 10^{3}$ | foot pounds-force/second (ft lbf/s) | kilowatts (kW) | $1.3558 \times 10^{-3}$ |
| Energy, Work, Heat |  |  |  |  |
| 15. | 0.7376 | foot pounds-force (ft lbf) | joules (J) | 1.3558 |
|  | 0.2388 | calories (cal) | joules (J) | 4.1868 |
|  | $9.478 \times 10^{-4}$ | British thermal units (Btu) | joules (J) | 1055.1 |
|  | 3412.1 | British thermal units (Btu) | kilowatt hours (kWh) | $2.931 \times 10^{-4}$ |
|  | 0.3725 | horsepower hours (hph) | megajoules (MJ) | 2.6845 |
|  | 1.3410 | horsepower hours (hph) | kilowatt hours (kWh) | 0.7457 |
|  | $9.478 \times 10^{-3}$ | therms | megajoules (MJ) | 105.51 |
|  | ultiply By | To $\longleftarrow$ | To Convert |  |

## CHAPTER 8 - PRINCIPLES AND RULES

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## Introduction and Notation

1. Algebra is that branch of mathematics dealing with the properties of, and relations between, quantities expressed in terms of symbols rather than numbers. The use of symbols allows general mathematical statements to be written down rather than just specific ones. For example, the relationship between ${ }^{\circ} \mathrm{C}$ and ${ }^{\circ} \mathrm{F}$ can be expressed as:

$$
\begin{aligned}
\mathrm{F} & =\frac{9}{5} \mathrm{C}+32 \\
\text { or } \mathrm{C} & =\frac{5}{9}(\mathrm{~F}-32)
\end{aligned}
$$

Thus given a value of temperature in either scale, the corresponding value in the other scale can be calculated. This is a considerably more concise method of relating the two scales than having a table showing the equivalent values, which in practice would have to be limited to a specified range of temperatures and with a specified level of precision. The algebraic relationship is in general more accurate than any representation by graph or nomogram.
2. Normally, when an algebraic expression is written down the conventional multiplication sign is omitted, both for brevity and to avoid confusion with the often-used symbol, x. Sometimes a full stop is used instead. The division sign is usually replaced by the solidus (/), or by separating the expression to be divided and the divisor by a horizontal line, thus for example:

$$
(3 x-6) \div(7 x+3) \text { may be written as }(3 x-6) /(7 x+3) \text { or, more commonly, as } \frac{(3 x-6)}{(7 x+3)}
$$

## The Laws of Algebra

3. There are several laws of algebra which govern how algebraic expressions may be manipulated:
a. Commutative Law. This law states that additions and subtractions within an expression may be performed in any order. So may divisions and multiplications,
e.g. $x+y=y+x ; \quad x+y-z=x-z+y ; \quad x y=y x ; \quad x y z=z x y=y z x$
b. Associative Law. This law states that terms in an algebraic expression may be grouped in any order,
e.g. $x+y+z=(x+y)+z=x+(y+z) ; \quad x y z=x(y z)=(x y) z$
c. Distributive Law. This law states that the product of a compound expression and a single term is the algebraic sum of the products of the single term with all the terms in the expression,
e.g. $\quad x(y+z)=x y+x z ; \quad 4 x(2 y-4 z)=8 x y-16 x z$
d. Laws of Precedence. These laws dictate the order in which algebraic operations should be effected.

First, deal with terms in brackets; then work out multiplications and divisions; finally, work out additions and subtractions.

Operations within brackets are dealt with using the same precedence.
4. Addition and Subtraction. Within an algebraic expression, like terms, e.g. all $x$ terms, all $y$ terms, all $x y$ terms, all $z^{2}$ terms etc, may be collected together and combined into a single term; unlike terms cannot be so combined,
e.g. $\quad 3 x^{2}+6 x-4 y+2 y+5 x y-x^{2}+9 x y=2 x^{2}+6 x-2 y+14 x y$
whereas, $3 x^{2}+6 x-4 y+5 x y$ cannot be simplified any further by addition or subtraction of terms.
5. Multiplication and Division. If two expressions which are to be multiplied together (or one divided by the other) have the same sign, the result is positive; while if their signs are different, the result is negative. The rules of indices (Volume 13, Chapter 5) similarly apply to algebraic expressions. Thus, for example: $4 x y^{2} \times 12 x^{-3} y^{4}=48 x^{-2} y^{6} ; \quad 25 a^{4} b^{6} \div 5 a^{2} b=5 a^{2} b^{5}$
6. When multiplying an expression within brackets then all of the terms within the bracket must be multiplied,
e.g. $\quad 5\left(3 x^{2}-4 y+5\right)=15 x^{2}-20 y+25$
7. When two bracketed expressions are to be multiplied together then all of the terms within one set of brackets must be multiplied by all the terms within the other set,
e.g. $\left(3 x^{2}+6\right)(2 x-4)=6 x^{3}-12 x^{2}+12 x-24$

## Factorization

8. A factor is a term by which an expression may be divided without leaving a remainder; a common factor is a term which is common to all of the terms of the expression. Thus for example in the expression $b x+b y, b$ is a common factor and the expression may be rewritten $a s b(x+y)$. Similarly in the expression: $24 a^{3}+6 a^{2}-12 a$
$6 a$ is common to all the terms and thus it may be rewritten as:

$$
6 a\left(4 a^{2}+a-2\right)
$$

9. Often an expression can be arranged into groups of terms where each group has its own factor, e.g. $\quad a x+b x+a y+b y$ can be regarded as two pairs of terms, thus:

$$
(a x+b x)+(a y+b y)
$$

then each pair has its own common factor, $x$ and $y$ respectively, and so can be rewritten as:

$$
x(a+b)+y(a+b)
$$

$(a+b)$ now appears as a common factor and so the expression can be further factorized as:

$$
(a+b)(x+y)
$$

## CHAPTER 9 - EQUATIONS

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## Introduction

1. An equation is a mathematical statement expressing an equality, i.e. it equates one algebraic expression with another. Equations may range in complexity from simple linear equations which contain only one unknown quantity and whose graphical representation is a straight line, to complex equations containing elements of calculus. This chapter will deal with simple linear, simultaneous and quadratic equations.

## Transposition

2. Perhaps the most common use of an equation is in the determination of the value of one parameter given the values of other terms. Thus, for example, given the equation for temperature conversion:

$$
\mathrm{F}=\frac{9}{5} \mathrm{C}+32
$$

if values for ${ }^{\circ} \mathrm{C}$ are given then ${ }^{\circ} \mathrm{F}$ can be determined by substituting the value in the equation, eg for $20^{\circ} \mathrm{C}$ :

$$
\begin{aligned}
F & =\frac{9 \times 20}{5}+32 \\
& =36+32 \\
& =68^{\circ} \mathrm{F}
\end{aligned}
$$

3. However, suppose that it is necessary to find the Celsius equivalent of a Fahrenheit temperature. Clearly the equation needs to be rearranged so that $C$ becomes the subject. In order to achieve this it is important to remember that operations may be carried out on the equation provided that the same process is applied to both sides of the ' $=$ ' sign. The only forbidden operation is division by zero; multiplication by zero is not forbidden but of course gives the trivial result $0=0$.
4. In the example, the first step is to subtract 32 from both sides of the equation:

$$
\mathrm{F}-32=\frac{9}{5} \mathrm{C}+32-32
$$

Next, both sides of the equation can be multiplied by $5 / 9$, remembering that all terms must be multiplied:

$$
\begin{aligned}
\frac{5}{9}(F-32) & =\frac{5}{9} \times \frac{9}{5} \mathrm{C} \\
\text { Thus } \quad C & =\frac{5}{9}(\mathrm{~F}-32)
\end{aligned}
$$

5. Frequently, equations will be encountered which contain powers and/or roots. These can be dealt with in an analogous fashion provided again that the same operations are carried out on both sides of the equation.
6. As an example, the periodic time of a simple pendulum, of length $L$, is given by the equation:

$$
\mathrm{T}=2 \pi \sqrt{\frac{\mathrm{~L}}{\mathrm{~g}}} \text { seconds }
$$

Suppose it is required to make 'L' the subject of the equation. The first operation is to square both sides to remove the square root:

$$
\begin{aligned}
T^{2} & =(2 \pi)^{2} \frac{L}{g} \\
& =4 \pi^{2} \frac{L}{g}
\end{aligned}
$$

Then divide both sides by $4 \pi^{2}$ :

$$
\frac{\mathrm{T}^{2}}{4 \pi^{2}}=\frac{\mathrm{L}}{\mathrm{~g}}
$$

Finally, multiply both sides by $g$ (and conventionally the subject term is taken to the left side).

$$
\text { Thus } \mathrm{L}=\frac{\mathrm{g} \mathrm{~T}^{2}}{4 \pi^{2}}
$$

7. Sometimes the way to proceed is not immediately obvious; the relationship between the distance of an object from a lens, (u), the distance of its image, (v), and the focal length of the lens, (f), is given by:

$$
\frac{1}{\mathrm{u}}+\frac{1}{\mathrm{v}}=\frac{1}{\mathrm{f}}
$$

Suppose it is necessary to make $f$ the subject of this equation. One technique is to initially multiply by the product of the denominators of the left-hand side, ie by uv.

$$
\begin{aligned}
& \frac{u v}{u}+\frac{u v}{v}=\frac{u v}{f} \\
& \text { ie } \quad v+u=\frac{u v}{f}
\end{aligned}
$$

Next, taking the 'f" term to the left and inverting both sides.

$$
\frac{f}{u v}=\frac{1}{v+u}
$$

Finally, multiply again by uv:

$$
f=\frac{u v}{v+u}
$$

8. However, there are often alternative methods. For example, taking the optical equation again:

$$
\frac{1}{u}+\frac{1}{v}=\frac{1}{f}
$$

The left-hand side can be combined into one term by using a common denominator, uv, thus:

$$
\frac{\mathrm{v}+\mathrm{u}}{\mathrm{uv}}=\frac{1}{\mathrm{f}}
$$

then inverting both sides

$$
f=\frac{u v}{v+u}
$$

Click here to open a short quiz to test your understanding of Transposition.

## Simple Linear Equations

9. A simple linear equation is one which has only one unknown quantity and in which the unknown quantity has no power other than 1, e.g. $4 x+6=30$. The solution, ie finding the value of $x$ which satisfies the conditions of the equation, is accomplished using the transposition techniques discussed above. Thus in the example:

$$
4 x+6=30
$$

Subtract 6 from both sides

$$
4 x=30-6=24
$$

Divide both sides by 4

$$
x=\frac{24}{4}=6
$$

Notice that addition and subtraction is equivalent to transferring the term to the other side of the equation accompanied by a change of sign, eg

Taking

$$
4 x+9=3 x-6
$$

The ' $3 x$ ' term can be transferred to the left-hand side if its sign is changed to ' - ', and similarly the ' 9 ' may be transferred to the right-hand with a sign change, thus:

$$
\begin{aligned}
4 x-3 x & =-6-9 \\
\text { ie } x & =-15
\end{aligned}
$$

Click here to open a short quiz to test your understanding of Linear Equations.

## Linear Simultaneous Equations

10. Linear simultaneous equations are independent equations, with no powers other than 1 , relating to more than one unknown. All of the equations must be true at the same time. For example:

$$
\begin{array}{ll}
x+3 y & =20 \\
9 x-y & =12 \tag{2}
\end{array}
$$

In general, to find values of all the unknowns which satisfy the equations then it is necessary to have as many independent equations as there are unknowns, for example if there are 5 unknowns then 5 independent equations would be required. For a pair of simultaneous equations with two unknowns there are two methods of solution; elimination and substitution.
11. Solution by Elimination. In this method one or both of the equations are manipulated so that the coefficient of one of the unknowns is identical in each equation. One equation is then subtracted from the other to eliminate one unknown resulting in a simple equation in the other unknown which can be solved readily. This value is then substituted back into one of the original equations to generate another readily soluble simple equation. Taking the examples from para 10:

Multiply equation (1) by 9 :

$$
\begin{array}{rlr}
9 x+27 y & =180 \\
9 x-y & =12 \\
\hline 28 y & =168 \\
y & =6
\end{array}
$$

$$
\text { Subtract equation (2) } \quad \begin{aligned}
& 9 x-y \\
& \hline 28 y \\
& =168
\end{aligned}
$$

Divide by 28
Substitute this value for y in equation (2)

$$
\begin{aligned}
9 x-6 & =12 \\
9 x & =18 \\
x & =2
\end{aligned}
$$

12. Solution by Substitution. In this method one equation is rearranged to express one unknown in terms of the other. This 'value' is then substituted into the other equation, which reduces to a simple equation in one unknown. After solution of this simple equation the value is substituted into either equation to find the other unknown value. Taking the same example equations:

$$
\begin{align*}
& x+3 y=20  \tag{1}\\
& 9 x-y=12 \tag{2}
\end{align*}
$$

Rearrange equation (1) to make ' $x$ ' the subject

$$
x=20-3 y
$$

Substitute this into equation (2) and solve for y :

$$
\begin{aligned}
9(20-3 y)-y & =12 \\
180-27 y-y & =12 \\
28 y & =168 \\
y & =6
\end{aligned}
$$

Substitute this value into equation (1):

$$
\begin{aligned}
x+18 & =20 \\
x & =2
\end{aligned}
$$

## Quadratic Equations

13. A quadratic equation contains the square of the unknown quantity but no higher power. The simplest type has the form $\mathrm{x}^{2}=\mathrm{n}$ where n is a positive number. The solution is a simple matter of finding the square root of the positive number ie $x=\sqrt{n}$, remembering that the result can have a negative or positive value. More commonly, a quadratic equation has the form:

$$
a x^{2}+b x+c=0, \text { where } a, b \text { and } c \text { are numbers. }
$$

14. It is instructive to examine the graphs representative of quadratic functions; their shape is parabolic. The solutions, or roots, of a quadratic equation are where $y=0$ on the graph, i.e. where the graph crosses the x-axis. Four examples are illustrated in Fig 1.

13-9 Fig 1 Graphs of Quadratic Equations

15. Fig 1a shows the graph of $y=x^{2}-4 x+3$. It will be seen that there are two positive roots; where $x=$ 1 and where $x=3$, ie where the graph crosses the $x$-axis. If the function had been $x^{2}+4 x+3$ then the graph would have crossed the $x$-axis to the left of the origin, ie giving two negative roots, -1 and -3 .
16. Fig 1 b shows the graph of $\mathrm{y}=\mathrm{x}^{2}-\mathrm{x}-2$. Here the graph crosses the axis at two points, one to the left and one to the right of the origin, thus there is one positive root and one negative root.
17. Fig 1 c shows the graph of $\mathrm{y}=\mathrm{x}^{2}-4 \mathrm{x}+4$. Here the graph does not cross the x -axis, rather the $\mathrm{x}-$ axis is a tangent to the curve at the value $x=2$. Here the two roots are said to be coincident or equal.
18. Fig 1 d shows the graph of $\mathrm{y}=\mathrm{x}^{2}-2 \mathrm{x}+2$. In this case the graph does not cross the x -axis at all therefore there are no real roots.

## Solving Quadratic Equations

19. Clearly, a quadratic equation can be solved as illustrated above by drawing the graph of the function and establishing where the curve crosses the x-axis. However this method is tedious and often inaccurate. There are two other methods of solution in common use: factorization and formula.
20. Factorization. Some quadratic equations, but not all, are readily solved by the factorization method. The equation must first be arranged so that all the terms are on the left-hand side with just a zero to the right of the ' $=$ ' sign. The problem then is to find factors of the expression on the left-hand side, remembering that it will not always be possible. Consider the equation:

$$
x^{2}-4 x+3=0
$$

The left-hand side can be factorized as:

$$
(x-1)(x-3)=0
$$

As the left-hand side is now comprised of the product of two factors equal to zero, then one of the factors at least must equal zero, ie either:

$$
\begin{array}{r}
x-1=0, \therefore x=1 \\
r \quad x-3=0, \therefore x=3
\end{array}
$$

21. Formula. The formula method can be used to solve all quadratic equations that have a solution and, unless the factors are readily apparent, is normally the preferred method of solution. The equation must first be arranged into the form:

$$
a x^{2}+b x+c=0
$$

The formula to be used is then:

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

As an example take the equation:

$$
2 x^{2}+3 x-2=0
$$

In terms of the standard form, $a=2, b=3, c=-2$. Thus putting these values into the formula:

$$
\begin{array}{ll}
x=\frac{-3 \pm \sqrt{9-(-16)}}{4} \\
x=\frac{-3+5}{4} & \therefore x=\frac{1}{2} \\
\text { or } x=\frac{-3-5}{4} & \therefore x=-2
\end{array}
$$

22. The part of the formula ' $b^{2}-4 a c$ ' is known as the discriminant and gives information about the roots of the equation. There are three possible cases:
a. $\quad b^{2}>4 \mathrm{ac}$. This generates a positive term and so will have two real square roots. Thus there will be two real roots to the equation. This is the situation shown by Figs 1a and 1b.
b. $\quad b^{2}=4 a c$. This is the case illustrated by Fig 1c. $b^{2}-4 a c=0$ and the roots are coincident and equal to -b/2a.
c. $b^{2}<4 a c$. This makes $b^{2}-4 a c$ negative. There are no real square roots to a negative number and therefore the equation has no real roots. This is the case illustrated in Fig 1d where the graph does not cross the x-axis. Although there are no real solutions in this case, this form of equation has many applications in, for example, control systems and aerodynamics.

## CHAPTER 10 - PLANE TRIGONOMETRY

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## Definitions and Axioms

1. Angles. An angle is formed by the intersection of two lines. In Fig 1 AOB is an angle which is formed by a line which starts from the position OA and rotates about $O$ in an anti-clockwise direction to the position OB. In this case $O$ is the 'vertex' of the angle while OB and OA are the 'arms' of the angle. With anti-clockwise rotation, the angle is regarded as positive; if clockwise, the angle is regarded as negative. Unless otherwise stated, it is assumed that rotation will be anti-clockwise.

## 13-10 Fig 1 An Angle


2. Measurement of Angles. One complete revolution is divided into 360 degrees $\left({ }^{\circ}\right)$. The degree is sub-divided into 60 minutes ('), and the minute is sub-divided into 60 seconds (''). Ninety degrees constitutes one right-angle.
3. Acute, Obtuse and Reflex Angles. An acute angle is one which is less than $90^{\circ}$, an obtuse angle is greater than $90^{\circ}$ but less than $180^{\circ}$ and a reflex angle is greater than $180^{\circ}$ (see Fig 2).

13-10 Fig 2 Acute, Obtuse, and Reflex Angles

4. Complementary and Supplementary Angles. If two angles together make up $90^{\circ}$ they are said to be complementary angles and each is the complement of the other. If two angles together make up $180^{\circ}$ they are said to be supplementary and each is the supplement of the other.
5. Slope and Gradient. The slope of the line CA in Fig 3 is the angle ACB. The gradient of the line $C A$ is the ratio $A B / C B=1 / 5$.

## 13-10 Fig 3 Slope and Gradient


6. Angles Formed by Two Intersecting Straight Lines. When two straight lines intersect, as in Fig 4, the sum of the adjacent angles is $180^{\circ}$, and the vertically opposite angles are equal.

## 13-10 Fig 4 Two Intersecting Straight Lines

$A+B=B+A_{1}=A_{1}+B_{1}=B_{1}+A=180^{\circ}$
$A=A_{1}$
$B=B_{1}$

## A

$B_{1}$
B
7. Parallel Lines Cut by a Transversal. When a transversal intersects two parallel straight lines, as in Fig 5:
a. The corresponding angles are equal.
( $\angle \mathrm{B}=\angle \mathrm{D}, \angle \mathrm{F}=\angle \mathrm{H}, \angle \mathrm{A}=\angle \mathrm{C}, \angle \mathrm{G}=\angle \mathrm{E}$ )
b. The alternate angles are equal.
$(\angle \mathrm{F}=\angle \mathrm{B}, \angle \mathrm{C}=\angle \mathrm{G})$
c. The sum of the interior angles on the same side of the transversal is equal to $180^{\circ}$.
$\left(\angle \mathrm{C}+\angle \mathrm{B}=180^{\circ}, \angle \mathrm{F}+\angle \mathrm{G}=180^{\circ}\right)$

## 13-10 Fig 5 Parallel Straight Lines and Transversal


8. Angle Properties of a Triangle. The sum of the angles of a triangle is $180^{\circ}$. When one side of a triangle is produced, as in Fig 6, the exterior angle thus formed is equal to the sum of the two interior opposite angles.

## 13-10 Fig 6 Angle Properties of a Triangle


9. Congruency of Triangles. Two triangles are congruent if one can be superimposed on the other, so that they exactly coincide with regard to their vertices and their sides. Their areas must consequently be equal. Thus the three sides of one triangle must have the same lengths as the three sides of the other and the angles opposite to the equal sides must be equal. Triangles can be proved congruent when:
a. The three sides of one are equal to the corresponding sides of the other.
b. They have two sides and the included angle of one, equal to the corresponding sides and included angle of the other.
c. They have two angles and a corresponding side equal.
10. The Theorem of Pythagoras. The conventional notation used for the solution of triangles is to denote the three angles by the capital letters $A, B, C$ and the sides opposite these angles by the small letters $a, b, c$. Pythagoras' theorem states that, in any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides, ie in Fig 7, where the angle at $C=90^{\circ}, c 2=b 2+a 2$. This theorem is of considerable importance and can be used to find one side of a right-angled triangle when the other two are known. For example, if a 12-metre ladder rests against a house so that its foot is 4 metres from the wall, it is possible to calculate how far up the side of the house the ladder will reach (see Fig 8).

## 13-10 Fig 7 Pythagoras' Theorem



13-10 Fig 8 A Practical Application of Pythagoras' Theorem


From Pythagoras, it is known that

$$
\begin{aligned}
\mathrm{a}^{2} & =\mathrm{b}^{2}+\mathrm{c}^{2} \\
\therefore \mathrm{~b}^{2} & =\mathrm{a}^{2}-\mathrm{c}^{2} \\
\text { ie } \quad \mathrm{b} & =\sqrt{144-16} \\
& =\sqrt{128} \\
& =11.3
\end{aligned}
$$

thus, the ladder reaches 11.3 metres up the wall.
11. Similar Triangles. If, in two triangles, the three angles of one are equal to the three angles of the other, it does not necessarily follow that they are congruent. Consider Fig 9 in which angle $A$ is common to the three triangles AFG, ADE and ACB.

## 13-10 Fig 9 Similar Triangles



$$
\begin{aligned}
& \angle \mathrm{AFG}=\angle \mathrm{ADE}=\angle \mathrm{ACB} \text { (corresponding angles) } \\
& \angle \mathrm{AGF}=\angle \mathrm{AED}=\angle \mathrm{ABC} \text { (corresponding angles) }
\end{aligned}
$$

Such triangles are said to be similar. When triangles are equiangular, the ratios of corresponding sides are also equal.

$$
\text { Thus, } \frac{\mathrm{AF}}{\mathrm{AG}}=\frac{\mathrm{AD}}{\mathrm{AE}}=\frac{\mathrm{AC}}{\mathrm{AB}}
$$

and it follows that $\frac{\mathrm{AF}}{\mathrm{AD}}=\frac{\mathrm{AG}}{\mathrm{AE}}=\frac{\mathrm{FG}}{\mathrm{DE}}$
Note: A similar relation holds good for two polygons which are equiangular.
12. The Relationship Between Sides and Areas of Similar Triangles. Triangles $A_{1} B_{1} C_{1}$ and $\mathrm{A}_{2} \mathrm{~B}_{2} \mathrm{C}_{2}$ are similar triangles with heights h 1 and h 2 respectively (see Fig 10).

## 13-10 Fig 10 Areas of Similar Triangles

a
b

Then $\frac{\mathrm{h}_{1}}{\mathrm{a}_{1}}=\frac{\mathrm{h}_{2}}{\mathrm{a}_{2}} \quad \therefore \mathrm{~h}_{1}=\frac{\mathrm{a}_{1} \mathrm{~h}_{2}}{\mathrm{a}_{2}}$

$$
\text { Also, } \frac{\text { Area of } A_{1} B_{1} C_{1}}{\text { Area of } A_{2} B_{2} C_{2}}=\frac{\frac{1}{2} a_{1} h_{1}}{\frac{1}{2} a_{2} h_{2}}
$$

Substituting for h1

$$
\frac{\text { Area of } A_{1} B_{1} C_{1}}{\text { Area of } A_{2} B_{2} C_{2}}=\frac{\frac{1}{2} a_{1} \times \frac{a_{1} h_{2}}{a_{2}}}{\frac{1}{2} a_{2} h_{2}}=\frac{a_{1}^{2}}{a_{2}^{2}}
$$

Similarly the areas can be proved proportional to the squares of b1, b2 and c1, c2. Hence the areas of similar triangles are proportional to the squares of the corresponding sides.

## Trigonometrical Ratios

13. In any right-angled triangle, the side opposite to the right-angle is called the hypotenuse, and the other two sides are called the opposite and adjacent according to their position relative to the angle under consideration. Fig 11 shows a right-angled triangle $A B C$ in which, relative to angle $A$, the side $B C$ is opposite and the side $A C$ is adjacent. The reverse is true relative to angle $B$. There are six trigonometrical ratios:

## 13-10 Fig 11 Right-angled Triangle



The sine of an angle (sin)

$$
=\frac{\text { opposite }}{\text { hypotenuse }}
$$

The cosine of an angle (cos) $\quad=\frac{\text { adjacent }}{\text { hypotenuse }}$
The tangent of an angle (tan) $=\frac{\text { opposite }}{\text { adjacent }}$
The cosecant of an angle (cosec) $=\frac{1}{\sin A}$
The secant of an angle (sec)

$$
=\frac{1}{\cos \mathrm{~A}}
$$

The cotangent of an angle (cot)

$$
=\frac{1}{\tan \mathrm{~A}}
$$

14. The trigonometric ratios, in terms of the triangle in Fig 11 are:

$$
\begin{aligned}
\sin A & =\frac{a}{c} & \sin B & =\frac{b}{c} \\
\cos A & =\frac{b}{c} & \cos B & =\frac{a}{c} \\
\tan A & =\frac{a}{b} & \tan B & =\frac{b}{a} \\
\operatorname{cosec} A & =\frac{c}{a} & \operatorname{cosec} B & =\frac{c}{b} \\
\sec A & =\frac{c}{b} & \sec B & =\frac{c}{a} \\
\cot A & =\frac{b}{a} & \cot B & =\frac{a}{b}
\end{aligned}
$$

and it may be deduced that:

$$
\begin{array}{ll}
\tan A=\frac{\sin A}{\cos A} & \cot A=\frac{\cos A}{\sin A} \\
\sin A=\cos (90-A) ; & \cos A=\sin (90-A) ; \quad \tan A=\cot (90-A) ; \quad \cot A=\tan (90-A)
\end{array}
$$

15. The Trigonometric Ratios for Angles of any Magnitude. So far, only acute angles have been considered but it is also necessary to be able to find the trigonometrical ratios of obtuse, reflex and sometimes negative angles. Consider a set of rectangular axes OX, OY (see Fig 12). To determine any trigonometrical ratio of any angle, the angle is set up on this system of axes as follows. A radius vector, $O P$, initially along $O X$, is considered to turn about $O$ in a counter-clockwise sense through the required angle, A . For a negative angle it turns in the clockwise sense. From P , drop a perpendicular, PN, on to the $x$-axis. Any trigonometrical ratio of $A$ is then referred to the right-angled triangle OPN and the acute angle $\alpha$ which OP makes with the $x$-axis. OP is always taken to be +ve, but ON and PN take the signs which would be attached to them when regarded as the coordinates of the point $P$.

## 13-10 Fig 12 The Four Quadrants



Thus, for the angles A and -A in the figure:

$$
\begin{array}{llr}
\sin \mathrm{A} & =\frac{\mathrm{PN}}{\mathrm{OP}}, \text { and is }+\mathrm{ve} ; & \sin (-\mathrm{A})=\frac{-\mathrm{P}^{\prime} \mathrm{N}}{\mathrm{OP}^{\prime}}, \text { and is }-\mathrm{ve} \\
\cos \mathrm{~A} & =\frac{-\mathrm{ON}}{\mathrm{OP}}, \text { and is }-\mathrm{ve} ; & \cos (-\mathrm{A})=\frac{-\mathrm{ON}}{\mathrm{OP}^{\prime}}, \text { and is }-\mathrm{ve} \\
\tan \mathrm{~A} & =\frac{\mathrm{PN}}{-\mathrm{ON}}, \text { and is }-\mathrm{ve} ; & \tan (-\mathrm{A})=\frac{-\mathrm{P}^{\prime} \mathrm{N}}{-\mathrm{ON}}, \text { and is }+\mathrm{ve}
\end{array}
$$

The reciprocal ratios, cosec, sec and cot, have the same sign respectively as sin, cos and tan. These more general definitions of the trigonometrical ratios, which apply to all angles of any magnitude and sign, are consistent with the former definitions which applied only to acute angles, since, for an acute angle, the radius vector, OP would lie in the first quadrant and ON and OP would then both be +ve. It has been shown that, for angles in the second quadrant, sin is +ve while cos and tan are -ve. Similarly it can be shown that, for angles in the third quadrant, tan is +ve while sin and cos are -ve, and for angles in the fourth quadrant, cos is +ve while sin and tan are -ve. Hence the 'all, sin, tan, cos' rule for determining the sign of a trigonometrical ratio. Fig 13 indicates which functions are positive in each of the quadrants.

13-10 Fig 13 Signs of Trig Functions by Quadrant


## The Sine Rule

16. The three sides and three angles of a triangle are sometimes called its 'elements'. When given a sufficient number of these elements it is possible to find the remainder. Thus for example if two sides and one angle or one side and two angles are known, and in each case an angle and the side opposite to it are included, the Sine Rule can be used to evaluate the unknown sides and angles.
17. In the triangle $A B C$ at Fig 14, draw $A D$ perpendicular to $B C$ and let $A D=p$.

## 13-10 Fig 14 The Sine Rule


$\frac{\mathrm{p}}{\mathrm{c}}=\sin \mathrm{B} \quad \therefore \mathrm{p}=\mathrm{c} \sin \mathrm{B} ; \quad \frac{\mathrm{p}}{\mathrm{b}}=\sin \mathrm{C} \quad \therefore \mathrm{p}=\mathrm{b} \sin \mathrm{C}$

$$
\therefore c \sin B=b \sin C \text { or } \quad \frac{c}{\sin C}=\frac{b}{\sin B}
$$

In a similar way it can be proved that:

$$
\frac{c}{\sin C}=\frac{a}{\sin A}
$$

and the Sine Formula is:

$$
\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}
$$

18. In the triangle $A B C$ at Fig 15, where angle $C$ is obtuse, draw $A D$ perpendicular to $B C$ produced and let $A D=p$.

## 13-10 Fig 15 Sine Rule - Obtuse Angled Triangle


$\frac{\mathrm{p}}{\mathrm{c}}=\sin \mathrm{B} \quad \therefore \mathrm{p}=\mathrm{c} \sin \mathrm{B} ;$ and $\frac{\mathrm{p}}{\mathrm{b}}=\sin \mathrm{ACD} \therefore \mathrm{p}=\mathrm{b} \sin \mathrm{ACD}$
but, $\sin A C D=\sin (180-A C D)=\sin C$
$\therefore \mathrm{p}=\mathrm{c} \sin \mathrm{B}=\mathrm{b} \sin \mathrm{C}$, as before, or $\frac{\mathrm{b}}{\sin \mathrm{B}}=\frac{\mathrm{c}}{\sin \mathrm{C}}$

## The Cosine Rule

19. When the Sine Rule is not applicable, as, for instance, when two sides and an included angle are given, the Cosine Rule may be used. Consider the triangle ABC in Fig 16:

## 13-10 Fig 16 The Cosine Rule



From $A$ draw $A D(=p)$ perpendicular to $B C$. Let $D C=n$ and $B D=a-n$. Then, by the principle of Pythagoras,

$$
\begin{array}{lll} 
& \mathrm{p}^{2} & =\mathrm{b}^{2}-\mathrm{n}^{2} \\
\text { and } & \mathrm{p}^{2} & =\mathrm{c}^{2}-(\mathrm{a}-\mathrm{n})^{2} \\
\therefore & \mathrm{~b}^{2}-\mathrm{n}^{2} & =\mathrm{c}^{2}-\left(\mathrm{a}^{2}-2 \mathrm{an}+\mathrm{n}^{2}\right) \\
\therefore & \mathrm{b}^{2}-\mathrm{n}^{2} & =\mathrm{c}^{2}-\mathrm{a}^{2}+2 \mathrm{an}-\mathrm{n}^{2} \\
\therefore & \mathrm{~b}^{2} & =\mathrm{c}^{2}-\mathrm{a}^{2}+2 \mathrm{an} \\
\text { but } & \mathrm{n} & =\mathrm{b} \cos \mathrm{C} \\
\therefore & \mathrm{~b}^{2} & =\mathrm{c}^{2}-\mathrm{a}^{2}+2 \mathrm{ab} \cos C \\
\text { and } & \mathrm{c}^{2} & =\mathrm{a}^{2}+\mathrm{b}^{2}-2 \mathrm{ab} \cos C
\end{array}
$$

20. In the case of triangle $A B C$, where $C$ is an obtuse angle, as in Fig 17:

## 13-10 Fig 17 Cosine Rule - Obtuse Angled Triangle


which is identical to the previous formula.
21. By the same method, it can be shown that

$$
b^{2}=a^{2}+c^{2}-2 \mathrm{ac} \cos B ; \quad \text { and } \mathrm{a}^{2}=\mathrm{b}^{2}+\mathrm{c}^{2}-2 \mathrm{bc} \cos \mathrm{~A}
$$

Thus, given any two sides and their included angle, the third side can be found. It may then be more convenient to apply the Sine Rule to find any other unknown elements.
22. When three sides $a, b$, and $c$ are given, the cosine formula may be re-arranged as follows:

$$
\begin{aligned}
& c^{2}=a^{2}+b^{2}-2 a b \cos C \\
& 2 a b \cos C=a^{2}+b^{2}-c^{2} \\
& \cos C=\frac{a^{2}+b^{2}-c^{2}}{2 a b}
\end{aligned}
$$

Similarly $\cos A$ and $\cos B$ may be found.

## CHAPTER 11 - THE CIRCLE

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## Definitions

1. Some important definitions relating to the circle are explained with reference to Fig 1.

## 13-11 Fig 1 The Properties of a Circle


Page
a. The Chord of a circle is any straight line which divides the circle into two parts, and is terminated at each end by the circumference. AB in Fig 1 is a chord.
b. A Segment of a circle is a figure bounded by a chord and the arc which it cuts off. In Fig 1 the chord $A B$ divides the circle into two segments:
(1) The minor segment is $A B C$.
(2) The major segment is ADB.
c. An Arc of a circle is a portion of the circumference. EF in Fig 1 is an arc.
d. A Sector of a circle is a figure which is bounded by two radii and the arc between them. OEF in Fig 1 is a sector.
e. A Tangent to a circle is a straight line which meets the circle in one point called the point of contact, and does not cut the circle when produced. A tangent is at right angles to the radius drawn from the point of contact. GHI in Fig 1 is a tangent.
f. The Angle in a Segment is the angle subtended at a point on the arc of a segment by the chord of the segment. In Fig 1 the angle in the major segment is $\angle A D B$ and the angle in the minor segment is $\angle A C B$.
g. The Angle at the Centre is the angle subtended at the centre of a circle by a chord or by an arc. $\angle E O F$ in Fig 1 is the angle at the centre subtended by EF.

## Theorems

2. The angle at the centre of a circle subtended by an arc is double the angle at the circumference subtended by the same arc. In Fig $2 \angle A O B=2 \angle A D B$ and the reflex $\angle A O B=2 \angle A C B$. Some important results follow from this theorem:

## 13-11 Fig 2 The Angles in a Circle


a. All angles in the same segment of a circle are equal.
b. The opposite angles of a quadrilateral inscribed in a circle are together equal to $180^{\circ}$; that is, they are supplementary.
c. The angle in a semi-circle is a right angle.
d. In equal circles, arcs which subtend equal angles either at the centres or at the circumference are equal.
e. In equal circles, chords which cut off equal arcs are equal.
3. The angle between a tangent and a chord drawn through the point of contact is equal to the angle in the alternate segment. In Fig $3 \angle \mathrm{DBC}=\angle \mathrm{DEB}$ and $\angle \mathrm{ABD}=\angle \mathrm{DFB}$.

## 13-11 Fig 3 The Angle between Tangent and Chord


4. The ratio of the circumference of a circle to its diameter is denoted by $\pi$, so that $\mathrm{C} / \mathrm{d}=\pi$ or $\mathrm{C}=\pi \mathrm{d}$. Since diameter is equal to two times the radius, $C=2 \pi r$.

## Circular Measure

5. The magnitude of an angle is commonly expressed in degrees which are obtained by the division of a right angle into 90 parts. There is another method which is of great practical importance and in which the unit employed is an absolute one. Consider Fig 4. Suppose the line OA in Fig 4 rotated about the point $O$ to the position $O B$, so that the length of the arc $A B$ is equal to the radius of the circle. The angle $A O B$ subtended by the arc $A B$ is called a radian. The radian is the unit of measurement in circular measure. Hence a radian may be defined as the angle subtended at the centre of a circle by an arc equal in length to the radius.

## 13-11 Fig 4 The Radian


6. The length of an arc, when the angle is given in radians, can be calculated as follows:

Length of an arc for 1 radian = $r$
Length of arc for $\sigma$ radians $=r \sigma$
Arc $=r \sigma$
7. The Relationship Between Radians and Degrees. Since an arc of $r$ units in length subtends an angle of one radian, the number of radians subtended by the circumference of a circle is given by
the number of times the radius is contained in the circumference, ie $C=2 \pi r$. The number of radians for one revolution $=\frac{2 \pi r}{r}=2 \pi$ radians. From this, $\pi$ radians $=180^{\circ}$ and 1 radian $=180 / \pi=57.3^{\circ}$.
Radians may be converted to degrees by multiplying by $\pi$ and dividing by 180 .

## Examples:

Convert to radians $45^{\circ}, 30^{\circ}$ taking $\pi$ as 3.1416
a. $\quad \frac{45 \times 3.1416}{180}=\frac{3.1416}{4}=0.7854 \mathrm{rad}$
b. $\quad \frac{30 \times 3.1416}{180}=\frac{3.1416}{6}=0.5236 \mathrm{rad}$

Conversions are easily carried out if an electronic calculator is available which will enter an accurate value for $\pi$ at the touch of a button.

## Angular Rotation

8. A straight strip of tape stuck on the face of a gear wheel from the axis to perimeter, will enable the observation that, in one revolution of a gear wheel, an individual tooth is rotated through $360^{\circ}$. Since $360^{\circ}=2 \pi$ radians, then one revolution is also $2 \pi$ radians. In circular measure, therefore, all even multiples of $\pi$ correspond to complete revolutions. For example, $4 \pi$ radians will be 2 revolutions, $6 \pi$ radians will be 3 revolutions, etc.
9. If a shaft or pulley is rotating at 3 revolutions per second, then the angular rotation must be $3 \times 2 \pi$ radians per second. In general terms, a rotation of $n$ revs per second will give an angular velocity of $2 \pi n$ radians per second.
10. The Relationship Between Angular and Linear Velocity. Let QMN of Fig 5 represent a flywheel which has an angular velocity of $\omega$ radians per sec. This means that any radius OQ rotates through an angle of $\omega$ radians in 1 sec . Any point $P$ on $O Q$ will also have the same angular velocity. Since arc $=r \sigma$, the arc traced out by $Q$ in $1 \mathrm{sec}=\omega \times O Q$, and the arc traced out by $P$ in $1 \mathrm{sec}=\omega \times O P$. In general, if the point is at a distance $r$ from the centre of rotation, the linear velocity of that point will be $\omega$. If $V$ is the linear velocity of a point, then $\mathrm{V}=\omega$. Although all points on the flywheel have the same angular velocity, the linear velocity of any point will depend on its distance from the centre of rotation.

13-11 Fig 5 Angular and Linear Velocity


## CHAPTER 12 - SPHERICAL TRIGONOMETRY

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## Introduction

1. Many of the problems encountered in navigation require the solution of triangles on the Earth's surface. In all but very small triangles, the effect of curvature must be taken into account.

## GEOMETRY OF THE SPHERE

2. The following properties and definitions apply to the geometrical sphere and, although the shape of the Earth is not truly spherical, it is considered to be so for most practical navigational purposes.
a. The section of the surface of a sphere made by any intersecting plane is a circle.
b. The axis of a circle on a sphere is the diameter of the sphere at right angles to the plane of the circle ( $\mathrm{PP}^{\prime}$ in Figs 1 and 2).
c. The two points at which the axis of the circle intersects the surface of the sphere are called the poles of the circle ( P and $\mathrm{P}^{\prime}$ in Figs 1 and 2).
d. If the plane passes through the centre of the sphere the section is called a great circle. All other sections are called small circles.
e. A quadrant is the great circle arc drawn from any point on a great circle to its pole (Fig 1).
f. Only one great circle passes through two points which are not at opposite ends of a diameter of a sphere. The shorter arc of this great circle is the shortest distance between these two points, measured over the surface of the sphere.

## 13-12 Fig 1 Great Circle



13-12 Fig 2 Small Circle


## Spherical Distance

3. The spherical distance between two points on the surface of a sphere is the length of the shorter great circle arc joining them. It is measured by the angle which that arc subtends at the centre of the sphere, expressed in degrees, minutes and seconds, or in radians. In Fig 3, the spherical distance $B C$ is measured as $B O \hat{C}$, e.g. $B C=42^{\circ} 27^{\prime}$ means that the arc $B C$ subtends an angle of $42^{\circ} 27$ 'at the centre of the sphere.

## 13-12 Fig 3 Spherical Distance


4. Angular measurement of spherical distance is convenient for the following reasons:
a. The actual length of the arc is readily obtained, given the radius of the sphere, since the length of the arc = radius of the sphere $\times$ the angle subtended at the centre (in radians). In the
case of the Earth, by definition, an arc of length 1 nautical mile subtends an angle of $1^{\prime}$. Thus, if $B$ and $C$ are two points on the surface of the Earth, the length of $B C$ is given directly by converting BÔC to minutes, e.g.:

$$
\mathrm{BC}=42^{\circ} 27^{\prime}=2,547 \mathrm{~nm}
$$

b. Angular measurement of spherical distance can be made without reference to the size of the sphere; this is of value when dealing with an abstract quantity such as the celestial sphere.
c. The use of angular measurement of the sides simplifies the solution of the various formulae used in spherical trigonometry.

## Spherical Angle

5. A spherical angle is formed at a point where two great circles intersect and is measured by the angle between the great circles at that point. This is the equivalent of measuring the angle between the planes of the two great circles in a plane mutually perpendicular to them both.
6. In Fig 4, BÂC is the spherical angle formed by the great circles $A B$ and $A C$ at the point $A$ and is measured by bÂc = BÔC = arc BC. BOC is a plane perpendicular to the planes of the great circles $A B D$ and $A C D$, and is contained in the plane of the great circle XBCY. OA is perpendicular to the plane of $X B C Y$ and $A$ is a pole of that great circle. By definition, BOC is a measure of the spherical angle at $A$, from which it follows that the length of $\operatorname{arc} B C$ is also a measure of $A$. Thus the spherical angle formed at a point may be measured by the arc intercepted between those great circles along the great circle to which that point is a pole.

13-12 Fig 4 Spherical Angle


## The Spherical Triangle

7. A spherical triangle is formed by the intersection of three great circles on the surface of a sphere. Fig 5 shows such a triangle, $A B C$. The arcs $B C, C A$ and $A B$ form the sides of the triangle and are denoted by $a, b$ and $c$ respectively. The angles $B \bar{A} C, A \bar{B} C$, and $B \bar{C} A$ form the angles of the triangle
and are denoted by $A, B$ and $C$. Note that by this convention, $a$ is the side opposite angle $A, b$ is opposite angle $B$ and $c$ is opposite angle $C$.

## 13-12 Fig 5 Spherical Triangle



## The Polar Triangle

8. Consider the spherical triangle $A B C$ in Fig 6. Point $A^{\prime}$ is a pole of the great circle of which side a is a part and is that particular pole which lies on the same side of the great circle as angle A. Similarly, $B^{\prime}$ and $C^{\prime}$ are the poles appropriate to sides $b$ and $c$ respectively.

## 13-12 Fig 6 Polar Triangle


9. The 3 points so obtained are joined by arcs of great circles $B^{\prime} A^{\prime}, A^{\prime} C^{\prime}$ and $C^{\prime} B^{\prime}$ giving a second spherical triangle $A^{\prime} B^{\prime} C^{\prime}$. The original triangle is called the primitive and the second, the polar triangle. It should be noted that in many cases the shape of the polar triangle might bear little resemblance to that of its primitive.
10. From para $2 c$, if $A^{\prime}$ is the pole of arc a then the arc $B A^{\prime}$ is a quadrant. But point $A^{\prime}$ lies on the arc $b^{\prime}$, therefore $B$ is also a pole of arc $b^{\prime}$. Similarly, $A$ and $C$ are poles of arcs $a^{\prime}$ and $c^{\prime}$ respectively; thus triangle $A B C$ is the polar triangle of $A^{\prime} B^{\prime} C^{\prime}$. So, if one triangle should be a polar triangle of another, the latter will be the polar triangle of the former.

## Relationship between the Primitive and Polar Triangles

11. In Fig 6, let $D$ and $E$ be the points where the arc $a^{\prime}$ is intercepted by the arcs $c$ and $b$ respectively. Then, from para 6 , since $A$ is a pole of $a^{\prime}$, the spherical angle $A$ is measured by the arc DE. But $B^{\prime} E$ and $C^{\prime} \mathrm{D}$ are both quadrants:

$$
\begin{array}{lr}
\therefore & B^{\prime} E+C^{\prime} D \\
\text { also } & B^{\prime} E+C^{\prime} D \quad=B^{\prime} C^{\prime}+D E \\
\therefore & D E=180^{\circ}-B^{\prime} C^{\prime} \\
& =180^{\circ}-a^{\prime}
\end{array}
$$

or, angle $A$ is the supplement of the angle subtended by arc $a^{\prime}$. Similarly it can be proved that:

$$
\begin{aligned}
& A^{\prime}=180^{\circ}-a, B^{\prime}=180^{\circ}-b, C^{\prime}=180^{\circ}-c \\
& \text { and } a^{\prime}=180^{\circ}-A, b^{\prime}=180^{\circ}-B, c^{\prime}=180^{\circ}-C
\end{aligned}
$$

## Properties of Spherical Triangles

12. The following statements are rules of spherical trigonometry and are stated without the proof which may be found in any basic spherical trigonometry primer. For a spherical triangle:
a. By convention, each side is less than $180^{\circ}$.
b. From a above it follows that each angle must be less than $180^{\circ}$.
c. Any two sides are together greater than the third side.
d. If two sides are equal, the angles opposite those sides are equal; conversely, if two angles are equal, then the sides opposite those angles are equal.
e. The greater side is opposite the greater angle; conversely, the greater angle is opposite the greater side.
f. The sum of all the sides is less than $360^{\circ}$.
g. The sum of all the angles lies between $180^{\circ}$ and $540^{\circ}$.

## Determination of Spherical Triangles

13. The spherical triangle has 6 parts, i.e. 3 sides and 3 angles. In general, if any 3 parts are known the triangle is fixed. The following combination of 3 parts all determine unique triangles:
a. Two sides and the included angle.
b. Two angles and the included side.
c. Three sides.
d. Three angles.

In the following cases, there may be 1 or 2 solutions:
e. Two sides and a non-included angle.
f. Two angles and a non-included side.

For example, in Fig 7, given $c$, $b$ and $B$, there are 2 possible triangles, $A B C_{1}$ and $A B C_{2}$. In Fig 8, given $B, C$ and $c$, there are again 2 possible triangles, $A B C_{1}$ and $A B C_{2}$.

## 13-12 Fig 7 Two Possible Triangles given Two Sides and a Non-included Angle



13-12 Fig 8 Two Possible Triangles given Two Angles and a Non-included Side


## Symmetrical Equality

14. The 2 triangles in Fig 9 obey all the normal rules of congruency, i.e. the sides and angles of one are equal to the corresponding sides and angles of the other. However, triangle ABC cannot be superimposed on $A B^{\prime} \mathrm{C}^{\prime}$ since its curvature is in the opposite sense. Hence, the 2 triangles cannot be truly congruent since they fail in this respect. They are, therefore, said to be symmetrically equal.

13-12 Fig 9 Symmetrically Equal Triangles

15. Points to Note. The following points of difference between plane and spherical triangles should be noted:
a. Given 2 angles of a spherical triangle, the third angle is still undetermined since, from para 12 g , the sum of the three angles may be anywhere between $180^{\circ}$ and $540^{\circ}$. This is in contrast to the plane triangle in which the third angle may be obtained by subtracting the sum of the two known angles from $180^{\circ}$.
b. Since 3 angles determine a unique spherical triangle (para 13d), it follows that similar triangles do not occur on the same sphere.
c. In plane trigonometry, the angles of an equilateral triangle are all $60^{\circ}$. In an equilateral spherical triangle, however, whilst the 3 angles are all equal, their value is not restricted to $60^{\circ}$.

## FORMULAE FOR THE SOLUTION OF SPHERICAL TRIANGLES

16. Spherical triangles may be solved by various formulae as listed below.

## The Sine Formula

17. From C (Fig 10) drop a perpendicular to the plane OAB, meeting the plane in H . Draw a perpendicular from C to OA , meeting that line in J . Then JH will be perpendicular to OA. Draw a perpendicular from C to OB , meeting that line in K . Then KH will be perpendicular to OB .

## 13-12 Fig 10 Proof of the Sine Formula


18. From Fig 10:

$$
\begin{aligned}
C J & =O C \sin C O ̂ J \\
O J & =O C \cos C O ̂ J \\
J H & =C J \cos C J \hat{H} \\
& =O C \sin C O \hat{J} \cos C J \hat{H} \\
C H & =O C \sin C O ̂ J \sin C J H H
\end{aligned}
$$

Let $O C=1$ unit
CÔJ $=\mathrm{b}$ (angle subtended at the centre)
CĴH = A (CJ is in the plane of $\operatorname{arc} \mathrm{b}, \mathrm{JH}$ is in the plane of arc c, CĴH is the angle between the planes)
$\therefore \quad \mathrm{OJ}=\cos \mathrm{b}$
and $C J=\sin b$
$J H=\sin b \cos A$
$C H=\sin b \sin A$
Similarly,
OK = cos a
and $\mathrm{CK}=\sin \mathrm{a}$
$K H=\sin a \cos B$
$C H=\sin a \sin B$

Equating (1) and (2):
$\sin b \sin A=\sin a \sin B$.
By repeating this construction in the plane OCB it may be shown that:
$\sin b \sin C=\sin c \sin B$
From (3) and (4) it follows that:

$$
\begin{equation*}
\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin C}{\sin C} . \tag{5}
\end{equation*}
$$

Or, the sines of the angles are proportional to the sines of the sides opposite. This expression is known as the Sine Formula

## The Cosine Formula

19. Fig 11 shows the plane $O A B$ of Fig 10. By definition $B O A=c$ (angle subtended at the centre). Draw JL perpendicular to OB; draw HM perpendicular to JL.

Then $O K=O J \cos c+J H \sin M J ̂ H$.

But $\mathrm{M} \hat{\mathrm{H}} \mathrm{H}=90^{\circ}-\mathrm{M} \hat{\mathrm{J} O}=\mathrm{BÔA}=\mathrm{c}$.
$O K=O J \cos c+J H \sin c$

## 13-12 Fig 11 The Plane OAB of Fig 10



Substituting the values for OK, OJ, and JH in (6)
$\cos \mathbf{a}=\cos b \cos \mathbf{c}+\sin b \sin \mathbf{c} \cos A$.

By repeating the construction in the planes of OCB and OCA, 2 further expressions are obtained, viz:
$\cos b=\cos a \cos c+\sin a \sin c \cos B$
$\cos c=\cos a \cos b+\sin a \sin b \cos C$.

Equations 7, 8 and 9 are of the same form and permit the determination of 1 side knowing the other 2 sides and the included angle.

## Examples of the Use of the Sine Formula

20. The Sine formula can be used in the following cases:
a. To find a non-included angle, given 2 sides and a non-included angle.
b. To find a non-included side, given 2 angles and a non-included side.

In accordance with para 13, these cases are ambiguous and may yield 2 possible results, as demonstrated below.
21. Example 1, Sine Formula. From Fig 12, find C, given $A=38^{\circ} 42^{\prime}$, $a=76^{\circ} 18^{\prime}, c=57^{\circ} 25^{\prime}$.

## 13-12 Fig 12 Example 1 - Sine Formula



From the Sine formula

$$
\begin{aligned}
& \frac{\sin \mathrm{C}}{\sin \mathrm{c}}=\frac{\sin \mathrm{A}}{\sin \mathrm{a}} \\
& \sin \mathrm{C}=\frac{\sin \mathrm{A} \sin \mathrm{c}}{\sin \mathrm{a}} \\
& \sin \mathrm{C}=\frac{\sin 38^{\circ} 42^{\prime} \sin 57^{\circ} 25^{\prime}}{\sin 76^{\circ} 18^{\prime}}
\end{aligned}
$$

Using calculator or logarithms, log $\sin C=1.73422$.

$$
\therefore \quad C=32^{\circ} 50^{\prime} \text { or } 147^{\circ} 10^{\prime}
$$

In this particular case, one result may be eliminated by applying the rule that the greater side must be opposite the greater angle (para 12e): a is greater than c hence $A$ must be greater than C ; therefore C cannot have a value $147^{\circ} 10^{\prime}$. The requirements are satisfied by 1 triangle only and, therefore, $\mathrm{C}=32^{\circ} 50^{\prime}$.
22. Example 2, Sine Formula. From Fig 13, find b, given $c=28^{\circ} 25^{\prime}, B=62^{\circ} 07$ ' and $C=33^{\circ} 42^{\prime}$.

## 13-12 Fig 13 Example 2 - Sine Formula



$$
\begin{aligned}
\frac{\sin b}{\sin B} & =\frac{\sin c}{\sin \mathrm{C}} \\
\sin \mathrm{~b} & =\frac{\sin \mathrm{B} \sin \mathrm{c}}{\sin \mathrm{C}} \\
& =\frac{\sin 62^{\circ} 07^{\prime} \sin 28^{\circ} 25^{\prime}}{\sin 33^{\circ} 42^{\prime}}
\end{aligned}
$$

So, using calculator or logarithms, $\log \sin b=\overline{1} .87973$.
$\therefore \mathrm{b}=49^{\circ} 18^{\prime}$ or $130^{\circ} 42^{\prime}$
In this case, $B$ is greater than $C$, thus $b$ must be greater than $c$. Both values of $b$ satisfy this requirement, hence the ambiguity is unresolved and both results must be accepted.

## Examples of the Use of the Cosine Formula

23. The Cosine formula may be used as follows:
a. To find the third side, given 2 sides and the included angle.
b. To find the third angle, given 2 angles and the included side (by transposition to the polar triangle).

Both these cases determine unique triangles; hence no ambiguity will arise.
24. Example 1, Cosine Formula. From Fig 14, find c, given $a=47^{\circ} 15^{\prime}, b=115^{\circ} 20^{\prime}$ and $\mathrm{C}=82^{\circ} 38^{\prime}$.

## 13-12 Fig 14 Example 1 - Cosine Formula



From the Cosine formula:

```
cos c= cos a cos b + sin a sin b cos C
cos c = cos 47 }\mp@subsup{}{}{\circ}1\mp@subsup{5}{}{\prime}\operatorname{cos}11\mp@subsup{5}{}{\circ}2\mp@subsup{0}{}{\prime}+\operatorname{sin}4\mp@subsup{7}{}{\circ}1\mp@subsup{5}{}{\prime}\operatorname{sin}11\mp@subsup{5}{}{\circ}2\mp@subsup{0}{}{\prime}\operatorname{cos}8\mp@subsup{2}{}{\circ}3\mp@subsup{8}{}{\prime
```

Now $115^{\circ} 20^{\prime}$ is in the second quadrant. It may thus be written that:

```
cos 115 20' = - cos 64'40'
sin 115' 20' = sin 64* 40'
\therefore cos c = - cos 47 }1\mp@subsup{7}{}{\circ}1\mp@subsup{5}{}{\prime}\operatorname{cos}6\mp@subsup{4}{}{\circ}4\mp@subsup{0}{}{\prime}+\operatorname{sin}4\mp@subsup{7}{}{\circ}1\mp@subsup{5}{}{\prime}\operatorname{sin}6\mp@subsup{4}{}{\circ}4\mp@subsup{0}{}{\prime}\operatorname{cos}8\mp@subsup{2}{}{\circ}3\mp@subsup{8}{}{\prime
            =-0.29044 +0.08510
\therefore cosc = - 0.20534
    c = 180 - 78 0}0\mp@subsup{9}{}{\prime}=10\mp@subsup{1}{}{\circ}5\mp@subsup{1}{}{\prime
```

25. Example 2, Cosine Formula. From triangle $A B C$ in Fig 15a, find $C$, given $A=47^{\circ} 15^{\prime}, B=115^{\circ}$ $20^{\prime}, \mathrm{c}=82^{\circ} 38^{\prime}$. In this case, two angles and the included side are given and the Cosine formula is not directly applicable. However, the polar triangle may be derived thus; from the rules of para 11:

13-12 Fig 15 Example 2 - Cosine Formula Indirect Application

a Primitive Triangle

b Polar Triangle

$$
\begin{aligned}
& \mathrm{a}^{\prime}=180^{\circ}-\mathrm{A}=180^{\circ}-47^{\circ} 15^{\prime}=132^{\circ} 45^{\prime} \\
& \mathrm{b}^{\prime}=180^{\circ}-\mathrm{B}=180^{\circ}-115^{\circ} 20^{\prime}=64^{\circ} 40^{\prime} \\
& \mathrm{C}^{\prime}=180^{\circ}-\mathrm{c}=180^{\circ}-82^{\circ} 38^{\prime}=97^{\circ} 22^{\prime} \\
& \mathrm{c}^{\prime}=180^{\circ}-\mathrm{C}^{\prime} .
\end{aligned}
$$

The given quantities are now in terms of 2 sides and an included angle and:

$$
\begin{aligned}
\cos c^{\prime} & =\cos a^{\prime} \cos b^{\prime}+\sin a^{\prime} \sin b^{\prime} \cos C^{\prime} \\
\cos c^{\prime} & =\cos 132^{\circ} 45^{\prime} \cos 64^{\circ} 40^{\prime}+\sin 132^{\circ} 45^{\prime} \sin 64^{\circ} 40^{\prime} \cos 97^{\circ} 22^{\prime} \\
\cos c^{\prime} & =-\cos 47^{\circ} 15^{\prime} \cos 64^{\circ} 40^{\prime}-\sin 47^{\circ} 15^{\prime} \sin 64^{\circ} 40^{\prime} \cos 82^{\circ} 38^{\prime} \\
& =-0.29044-0.08510 \\
& =-0.37554 \\
c^{\prime} & =112^{\circ} 03^{1} / 2^{\prime} \\
\therefore C^{\prime} & =180^{\circ}-112^{\circ} 03^{1} 1^{\prime}=67^{\circ} 56^{1} 1^{\prime}
\end{aligned}
$$

## The Haversine Formula

26. In the preceding examples it has been necessary to consider the sign of the various functions. This is an inconvenience and calculations would be much simplified if only positive values occurred. The Haversine (half-reverse-sine) formula may be used to achieve this object.
27. The expression $\frac{1-\cos A}{2}$ is known as the haversine of an angle $A$, written hav $A$ and has special properties. Thus:

When $A=0^{\circ} \quad \cos A=1$ and hav $A=0$
$A=90^{\circ} \quad \cos A=0 \quad$ and hav $A=1 / 2$
$A=180^{\circ} \quad \cos A=-1 \quad$ and $\operatorname{hav} A=1$
$A=270^{\circ} \quad \cos A=0 \quad$ and hav $A=1 / 2$
$A=-90^{\circ} \quad \cos A=0 \quad$ and $\operatorname{hav} A=1 / 2$

$$
\begin{array}{lll}
A=-180^{\circ} & \cos A=-1 & \text { and hav } A=1 \\
A=-270^{\circ} & \cos A=0 & \text { and hav } A=1 / 2 \\
A=-360^{\circ} & \cos A=1 & \text { and hav } A=0
\end{array}
$$

The value of the haversine never exceeds 1 and is always positive irrespective of whether the angle is positive or negative and since:
hav $\mathrm{a}=\frac{1-\cos \mathrm{a}}{2}\left(\right.$ and hav $\left.\mathrm{A}=\frac{1-\cos \mathrm{A}}{2}\right)$
$\cos a=1-2$ hav $a($ and $\cos A=1-2$ hav $A)$
Substituting for $\cos$ a and $\cos A$ in the Cosine formula (para 19):

$$
\begin{gathered}
\cos a=\cos b \cos c+\sin b \sin c \cos A \\
1-2 \text { hav } a=\cos b \cos c+\sin b \sin c(1-2 \text { hav } A) \\
\text { or } 1-2 \text { hav } a=\cos b \cos c+\sin b \sin c-2 \sin b \sin c \text { hav } A
\end{gathered}
$$

Now: $\quad \cos b \cos c+\sin b \sin c=\cos (b-c)$
and $\cos (b-c)=1-2 \operatorname{hav}(b-c)$
$\therefore 1-2$ hav $a=1-2$ hav $(b-c)-2 \sin b \sin c$ hav $A$
From which: hav $a=\operatorname{hav}(b-c)+\sin b \sin c$ hav $A$
28. Since the haversine is always positive, the value of hav $(b-c)$ is positive no matter what values are assigned to $b$ and $c$ and the equation may be simplified by writing $(b \sim c)$, meaning the difference between $b$ and $c$. So:

```
hav a = hav (b ~ c) + sin b sin c hav A
```

Similarly it may be shown that:
hav $b=$ hav $(a \sim c)+\sin a \sin c$ hav $B$
and hav $c=$ hav $(a \sim b)+\sin a \sin b$ hav $C$

By convention, only angles and sides up to $180^{\circ}$ are considered. Terms $\sin a$, sin $b$ and $\sin c$ will, therefore, always be positive; hence every term in the Haversine formula is positive.

## Example of the Use of the Haversine Formula

29. The Haversine formula is used to:
a. Find the third side, given 2 sides and the included angle.
b. Find the third angle, given 2 angles and the included side (by transposition to the polar triangle).

From triangle $A B C$ in Fig 16, find $c$ given $a=47^{\circ} 15^{\prime}, b=115^{\circ} 20^{\prime}$ and $C=82^{\circ} 38^{\prime}$.

## 13-12 Fig 16 Application of the Haversine Formula



```
hav c = hav (47* 15' ~ 1150}2\mp@subsup{5}{}{\prime})+\operatorname{sin}4\mp@subsup{7}{}{\circ}15'\operatorname{sin}11\mp@subsup{5}{}{\circ}20'\mathrm{ hav 82}38
hav c = hav 68* 05' + sin 47' 15' sin 64* 40' hav 82 }3\mp@subsup{0}{}{\circ}3\mp@subsup{8}{}{\prime
    = 0.31337 + 0.28931
\therefore c= 1010}51
```


## The Cosecant Formula

30. Given the 3 sides of a spherical triangle, the 3 angles may be determined by substitution in the Haversine formula. This is transposed for convenience to give the Cosecant formula:
$\operatorname{hav} a=\operatorname{hav}(b \sim c)+\sin b \sin c h a v A$
$\operatorname{hav} \mathrm{A}=\frac{\operatorname{hav} \mathrm{a}-\operatorname{hav}(\mathrm{b} \sim \mathrm{c})}{\sin \mathrm{b} \sin \mathrm{c}}$
$\therefore \quad$ hav $\mathbf{A}=$ hav $\mathbf{a}$ - hav $(\mathbf{b} \sim \mathbf{c}) \operatorname{cosec} \mathbf{b} \operatorname{cosec} c$.
The other 2 forms are:
hav $B=$ hav $b-h a v(a \sim c) \operatorname{cosec} a \operatorname{cosec} c$, and
hav $C=$ hav $c-h a v(a \sim b) \operatorname{cosec} a \operatorname{cosec} b$

From these equations, all 3 angles may be determined. Alternatively, given 3 angles, the 3 sides can be obtained by transposition to the polar triangle.

## Example of the Use of the Cosecant Formula

31. From the triangle $A B C$ in Fig 17a, find side a given $A=82^{\circ} 30^{\prime}, B=60^{\circ} 52^{\prime}$ and $C=45^{\circ} 02^{\prime}$. The corresponding sides of the polar triangle (Fig 17b) are:

$$
\begin{aligned}
& a^{\prime}=180^{\circ}-82^{\circ} 30^{\prime}=97^{\circ} 30^{\prime} \\
& b^{\prime}=180^{\circ}-60^{\circ} 52^{\prime}=119^{\circ} 08^{\prime} \\
& c^{\prime}=180^{\circ}-45^{\circ} 02^{\prime}=134^{\circ} 58^{\prime} \\
& A^{\prime}=180^{\circ}-a
\end{aligned}
$$

13-12 Fig 17 Application of the Cosecant Formula

a

b

Then:

$$
\text { hav } \begin{aligned}
\mathrm{A}^{\prime} & =\left[\text { hav } \mathrm{a}^{\prime}-\text { hav }\left(\mathrm{b}^{\prime} \sim \mathrm{c}^{\prime}\right)\right] \operatorname{cosec} \mathrm{b}^{\prime} \operatorname{cosec} \mathrm{c}^{\prime} \\
& =\left[\text { hav } 97^{\circ} 30^{\prime}-\operatorname{hav}\left(119^{\circ} 08^{\prime} \sim 134^{\circ} 58^{\prime}\right)\right] \times\left(\operatorname{cosec} 119^{\circ} 08^{\prime} \operatorname{cosec} 134^{\circ} 58^{\prime}\right) \\
& =\left[\operatorname{hav} 97^{\circ} 30^{\prime}-\operatorname{hav} 15^{\circ} 50^{\prime}\right] \times\left(\operatorname{cosec} 60^{\circ} 52^{\prime} \operatorname{cosec} 45^{\circ} 02^{\prime}\right)
\end{aligned}
$$

$$
\left(\because 60^{\circ} 52^{\prime} \text { is } 180^{\circ}-119^{\circ} 08^{\prime} \text { and } 45^{\circ} 02^{\prime} \text { is } 180^{\circ}-134^{\circ} 58^{\prime}\right)
$$

$$
=[0.56526-0.01897] \times\left(\operatorname{cosec} 60^{\circ} 52^{\prime} \operatorname{cosec} 45^{\circ} 02^{\prime}\right)
$$

Using calculator or logarithms, $\log$ hav $\mathrm{A}^{\prime}=\overline{1} .94642$

$$
\begin{aligned}
& \therefore A^{\prime}=140^{\circ} 10^{\prime} \\
& \text { and } \quad a=180^{\circ}-140^{\circ} 10^{\prime}=39^{\circ} 50^{\prime}
\end{aligned}
$$

Sides b and c can be found in a similar manner.

## The Half-log Haversine Formula

32. By rewriting equation (11), substituting $\frac{1-\cos a}{2}$ for hav $a$ and $\frac{1-\cos (b \sim c)}{2}$ for hav $(b \sim c)$, the following expression is obtained:
hav $A=\frac{1}{2}[1-\cos a-(1-\cos (b \sim c))](\operatorname{cosec} b \operatorname{cosec} c)$

$$
=\frac{1}{2} \cos [(b \sim c)-\cos a](\operatorname{cosec} b \operatorname{cosec} c)
$$

Now: $\quad \cos (\mathrm{b} \sim \mathrm{c})-\cos \mathrm{a}=-2 \sin \left[\frac{(\mathrm{~b} \sim \mathrm{c})+\mathrm{a}}{2}\right] \sin \left[\frac{(\mathrm{b} \sim \mathrm{c})-\mathrm{a}}{2}\right]$

$$
=2 \sin \left[\frac{\mathrm{a}+(\mathrm{b} \sim \mathrm{c})}{2}\right] \sin \left[\frac{\mathrm{a}-(\mathrm{b} \sim \mathrm{c})}{2}\right]
$$

But: hav $\theta=\frac{1-\cos \theta}{2}=\frac{2 \sin ^{2} \frac{\theta}{2}}{2}=\sin ^{2} \frac{\theta}{2}$
So: $\quad \sin \frac{\theta}{2}=\sqrt{\operatorname{hav} \theta}$
Therefore:

$$
\sin \left[\frac{a+(b \sim c)}{2}\right]=\sqrt{\operatorname{hav}[a+(b \sim c)]}
$$

and $\sin \left[\frac{\mathrm{a}-(\mathrm{b} \sim \mathrm{c})}{2}\right]=\sqrt{\operatorname{hav}[\mathrm{a}-(\mathrm{b} \sim \mathrm{c})]}$

Therefore:

$$
\operatorname{hav} A=\sqrt{\operatorname{hav}[a+(b \sim c)]} \times \sqrt{\operatorname{hav}[a-(b \sim c)]} \times(\operatorname{cosec} b \operatorname{cosec} c)
$$

Similarly:

$$
\begin{aligned}
& \operatorname{hav} B=\sqrt{\operatorname{hav}[b+(a \sim c)]} \times \sqrt{\operatorname{hav}[b-(a \sim c)]} \times(\operatorname{cosec} a \operatorname{cosec} c) \\
& \operatorname{hav} C=\sqrt{\operatorname{hav}[c+(a \sim b)]} \times \sqrt{\operatorname{hav}[c-(a \sim b)]} \times(\operatorname{cosec} a \operatorname{cosec} b)
\end{aligned}
$$

This equation is easier to manipulate than the Cosecant formula since a straight multiplication is the only operation required. A calculator or logarithms will produce the relevant results. It should be noted that:

$$
\log \sqrt{\operatorname{hav}[\mathrm{a}+(\mathrm{b} \sim \mathrm{c})]}=\frac{1}{2} \log \operatorname{hav}[\mathrm{a}+(\mathrm{b} \sim \mathrm{c})]
$$

An alternative to this formula is called the All Natural Haversine formula where:

$$
\operatorname{hav} \mathrm{A}=\frac{\operatorname{hav} \mathrm{a}-\operatorname{hav}(\mathrm{b} \sim \mathrm{c})}{\operatorname{hav}(\mathrm{b}+\mathrm{c})-\operatorname{hav}(\mathrm{b} \sim \mathrm{c})}
$$

## Example of the Use of the Half-log Haversine Formula

33. Using triangle $A B C$ in Fig 17 again, find side a given $A=82^{\circ} 30^{\prime}, B=60^{\circ} 52^{\prime}$ and $C=45^{\circ} 02^{\prime}$. The corresponding sides of the polar triangle are:

$$
\begin{aligned}
\mathrm{a}^{\prime}= & 180^{\circ}-82^{\circ} 30^{\prime}=97^{\circ} 30^{\prime} \\
\mathrm{b}^{\prime}= & 180^{\circ}-60^{\circ} 52^{\prime}=11^{\circ} 08^{\prime} \\
\mathrm{c}^{\prime}= & 180^{\circ}-45^{\circ} 02^{\prime}=134^{\circ} 58^{\prime} \\
\mathrm{b}^{\prime} \sim & \mathrm{c}^{\prime}=15^{\circ} 50^{\prime} \\
\text { havA }= & \sqrt{\operatorname{hav}\left(97^{\circ} 30^{\prime}+15^{\circ} 50^{\prime}\right)} \times \sqrt{\operatorname{hav}\left(97^{\circ} 30^{\prime}-15^{\circ} 50^{\prime}\right)} \times\left(\operatorname{cosec} 119^{\circ} 08^{\prime} \operatorname{cosec} 134^{\circ} 58^{\prime}\right) \\
= & \sqrt{\operatorname{hav}\left(113^{\circ} 20^{\prime}\right)} \times \sqrt{\operatorname{hav}\left(81^{\circ} 40^{\prime}\right)} \times\left(\operatorname{cosec} 60^{\circ} 52^{\prime} \operatorname{cosec} 45^{\circ} 02^{\prime}\right) \\
& \left(\because 60^{\circ} 52^{\prime} \text { is } 180^{\circ}-119^{\circ} 08^{\prime} \text { and } 45^{\circ} 02^{\prime} \text { is } 180^{\circ}-134^{\circ} 58^{\prime}\right)
\end{aligned}
$$

Using calculator or logarithms, $\log$ hav $A^{\prime}=1.94642$

$$
\begin{aligned}
& \therefore A^{\prime}=140^{\circ} 10^{\prime} \\
& \text { and } \\
& a=180^{\circ}-140^{\circ} 10^{\prime} \\
& =39^{\circ} 50^{\prime}
\end{aligned}
$$

sides $b$ and $c$ can be found in similar manner.

## The Four Parts Formula

34. A case that arises frequently in practice is that in which, given 3 consecutive parts of a spherical triangle (say 2 sides and an included angle), it is required to know a further angle. This could be solved by using the Haversine and Cosecant formulae in succession, but such a method is laborious.
35. Consider any spherical triangle $A B C$ in which $a, c$ and $B$ are known.

Then: $\boldsymbol{c o s} \mathbf{a}=\boldsymbol{\operatorname { c o s }} \mathbf{b} \cos \mathbf{c}+\boldsymbol{\operatorname { s i n }} \mathbf{b} \sin \mathbf{c} \cos \mathbf{A}$

$$
\begin{aligned}
& \cos b=\cos a \cos c+\sin a \sin c \cos B \\
& \sin b=\frac{\sin a}{\sin A} \sin B
\end{aligned}
$$

Substituting for $\sin b$ and $\cos b$ in equation (12):

```
\(\cos a=\cos c(\cos a \cos c+\sin a \sin c \cos B)+\sin c \cos A \frac{\sin a}{\sin A} \sin B\)
    \(=\cos a \cos ^{2} c+\cos c \sin a \sin c \cos B+\sin a \sin c \sin B \cot A\)
\(\cos a\left(1-\cos ^{2} c\right)=\sin a \sin c(\cos c \cos B+\sin B \cot A)\)
```

Now: $1-\cos ^{2} c=\sin ^{2} c$
So: $\cos a \sin ^{2} c=\sin a \sin c(\cos c \cos B+\sin B \cot A)$
$\therefore \quad \cot a \sin c \quad=\cos c \cos B+\sin B \cot$

Likewise: $\quad \cot a \sin b=\cos b \cos C+\sin C \cot A$ $\cot b \sin c=\cos c \cos A+\sin A \cot B$ $\cot c \sin a=\cos a \cos B+\sin B \cot C$ $\cot b \sin a=\cos a \cos C+\sin C \cot B$ $\cot c \sin b=\cos b \cos A+\sin A \cot C$
36. The following rules, using Fig 18 as an example, may assist in memorizing these formulae:
a. The four parts follow consecutively around the spherical triangle, eg A, c, B, a in the example.
b. $\quad c$ and $B$ are known as inner parts.
c. A (to be found) and a are known as outer parts.
d. The equations always follow the form cot $\sin \cos , \cos \sin \cot$.
e. The sides always appear in the first 3 terms, the angles always appear in the last 3 terms.
f. Each inner part appears twice in the equation.
g. Each outer part appears only once.
h. The outer parts appear at the each end of the equation.

Applying these rules to Fig 18:

The sequence is: $\cot \sin =\cos \cos +\sin \cot$
$B$ and $c$ must both appear twice, a appears only once and at the left-hand end since it is a side and must be in the first 3 terms; Therefore, A must be at the right hand end. Thus cot a $\sin \mathrm{c}=$ $\cos c \cos B+\sin B \cot A$ which matches equation (13) and is correct.

## 13-12 Fig 18 Illustration of Four Parts Formula



## Example of the Use of the Four Parts Formula

37. The Four Parts formula can be used as follows:
a. To find an angle, given 2 sides and the included angle.
b. To find a side, given 2 angles and the included side.
38. In the spherical triangle $A B C$ at Fig 19, find $B$ given that $A=18^{\circ} 55^{\prime}, b=123^{\circ} 59^{\prime}, c=36^{\circ} 58^{\prime}$.

13-12 Fig 19 The Four Parts Formula - Example

$B$ and $b$ are the outer parts.
$\therefore \cot b \sin c=\cos c \cos A+\sin A \cot B$
$\cot B=\frac{\cot \mathrm{b} \sin \mathrm{c}-\cos \mathrm{c} \cos \mathrm{A}}{\sin \mathrm{A}}=\frac{\cot 56^{\circ} 01^{\prime} \sin 36^{\circ} 58^{\prime}-\cos 36^{\circ} 58^{\prime} \cos 18^{\circ} 55^{\prime}}{\sin 18^{\circ} 55^{\prime}}$

Using calculator or logarithms
$\cot B=\frac{-0.40536-0.75584}{\sin 18^{\circ} 55^{\prime}}=\frac{-1.16120}{\sin 18^{\circ} 55^{\prime}}=-0.55411$
$\therefore B=-15^{\circ} 36^{\prime}$
$\therefore B=180^{\circ}-15^{\circ} 36^{\prime}=164^{\circ} 24^{\prime}$

## Tangent Formula or Napier's Analogies

39. Although the lengthy working is omitted, from the Sine and Cosine formulae it can be proven that for any spherical triangle $A B C$ :

$$
\begin{equation*}
\tan \frac{1}{2}(A-B)=\cot \frac{1}{2} C \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \tag{14}
\end{equation*}
$$

Also

$$
\begin{equation*}
\tan \frac{1}{2}(A+B)=\cot \frac{1}{2} C \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \tag{15}
\end{equation*}
$$

40. From any polar triangle $\mathrm{A}^{\prime} \mathrm{B}^{\prime} \mathrm{C}^{\prime}$ :

$$
C=\left(180^{\circ}-c^{\prime}\right)
$$

$\therefore \cot 1 / 2 C=\cot \left(90^{\circ}-1 / 2 C^{\prime}\right)=\tan 1 / 2 C^{\prime}$

$$
(A-B)=\left(180^{\circ}-a^{\prime}\right)-\left(180^{\circ}-b^{\prime}\right)=-\left(a^{\prime}-b^{\prime}\right)
$$

$$
\tan [1 / 2(\mathrm{~A}-\mathrm{B})]=\tan \left[-1 / 2\left(\mathrm{a}^{\prime}-\mathrm{b}^{\prime}\right)\right]=-\tan 1 / 2\left(\mathrm{a}^{\prime}-\mathrm{b}^{\prime}\right)
$$

Similarly, $(a-b)=-\left(A^{\prime}-B^{\prime}\right)$

$$
\begin{aligned}
& \sin [1 / 2(a-b)]=\sin \left[-1 / 2\left(A^{\prime}-B^{\prime}\right)\right]=-\sin 1 / 2\left(A^{\prime}-B^{\prime}\right) \\
& \quad 1 / 2(a+b)=1 / 2\left(180^{\circ}-A^{\prime}+180^{\circ}-B^{\prime}\right)=180^{\circ}-1 / 2\left(A^{\prime}+B^{\prime}\right) \\
& \sin [1 / 2(a+b)]=\sin \left[180^{\circ}-1 / 2\left(A^{\prime}+B^{\prime}\right)\right]=\sin 1 / 2\left(A^{\prime}+B^{\prime}\right)
\end{aligned}
$$

Substituting these values in equation (14):

$$
\begin{aligned}
& -\tan \frac{1}{2}\left(\mathrm{a}^{\prime}-\mathrm{b}^{\prime}\right)=\tan \frac{1}{2} \mathrm{c}^{\prime}-\frac{\sin \frac{1}{2}\left(\mathrm{~A}^{\prime}-\mathrm{B}^{\prime}\right)}{\sin \frac{1}{2}\left(\mathrm{~A}^{\prime}+\mathrm{B}^{\prime}\right)} \\
& \therefore \quad \tan \frac{1}{2}\left(\mathrm{a}^{\prime}-\mathrm{b}^{\prime}\right)=\tan \frac{1}{2} \mathrm{c}^{\prime} \frac{\sin \frac{1}{2}\left(\mathrm{~A}^{\prime}-\mathrm{B}^{\prime}\right)}{\sin \frac{1}{2}\left(\mathrm{~A}^{\prime}+\mathrm{B}^{\prime}\right)}
\end{aligned}
$$

Thus in any spherical triangle ABC :

$$
\begin{equation*}
\tan \frac{1}{2}(a-b)=\tan \frac{1}{2} c \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tag{16}
\end{equation*}
$$

Similarly, from equation (15):

$$
\begin{equation*}
\tan \frac{1}{2}(a+b)=\tan \frac{1}{2} c \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tag{17}
\end{equation*}
$$

## Examples of the Use of the Tangent Formulae

41. The tangent formulae are used:
a. Given 2 sides and the included angle, to find the other 2 angles.
b. Given 2 angles and the included side, to find the other 2 sides.
c. Given 2 sides and the angles opposite, to find the other unknowns.
42. Example 1, Tangent Formula. In Fig 20 (a repeat of Fig 16), find $A$ and $B$, given $a=47^{\circ} 15^{\prime}, b=$ $115^{\circ} 20^{\prime}$ and $\mathrm{C}=82^{\circ} 38^{\prime}$.

## 13-12 Fig 20 Example 1 of the Tangent Formulae



$$
\tan \frac{1}{2}(\mathrm{~A}-\mathrm{B})=\cot \frac{82^{\circ} 38^{\prime}}{2} \frac{\sin \frac{1}{2}\left(47^{\circ} 15^{\prime}-115^{\circ} 20^{\prime}\right)}{\sin \frac{1}{2}\left(47^{\circ} 15^{\prime}+115^{\circ} 20^{\prime}\right)}
$$

A useful feature of the tangent formulae emerges here. Since $b>a, \sin \frac{1}{2}(a-b)$ is negative. However, with $b>a$, it follows that $B>A$ and $\tan (A-B)$ is also negative. The negative sign appears on both sides of the equation and may, thus, be disregarded. It is therefore permissible to write:

$$
\tan \frac{1}{2}(B-A)=\cot \frac{1}{2} C \frac{\sin \frac{1}{2}(b-a)}{\sin \frac{1}{2}(b+a)}
$$

That is to say, in the application of the tangent formulae to any example the order may be changed so that the smaller quantity is subtracted from the larger and negative angles do not occur. This operation must be performed throughout all terms in the equation. Then:

$$
\begin{aligned}
& \tan \frac{1}{2}(B-A)=\cot 41^{\circ} 19^{\prime} \frac{\sin 34^{\circ} 02 \frac{1}{2}^{\prime}}{\sin 81^{\circ} 17 \frac{1}{2}^{\prime}} \\
& \tan \frac{1}{2}(B+A)=\cot 41^{\circ} 19^{\prime} \frac{\cos 34^{\circ} 02 \frac{1}{2}^{\prime}}{\cos 81^{\circ} 17 \frac{1}{2}^{\prime}}
\end{aligned}
$$

Using a calculator or logarithms:

$$
\begin{aligned}
& \frac{1}{2}(\mathrm{~B}-\mathrm{A})=32^{\circ} 47 \frac{1}{2}^{\prime} \\
& \frac{1}{2}(\mathrm{~B}+\mathrm{A})=80^{\circ} 52 \frac{1}{2}^{\prime} \\
& B=\frac{1}{2}(B-A)+\frac{1}{2}(B+A) \\
& A=\frac{1}{2}(B+A)-\frac{1}{2}(B-A) \\
& \therefore \quad A=113^{\circ} 40^{\prime} \text { and } B=48^{\circ} 05^{\prime}
\end{aligned}
$$

43. Example 2, Tangent Formula. In Fig 21, find B, given $A=38^{\circ} 42^{\prime}$, $a=76^{\circ} 18^{\prime}, C=32^{\circ} 50^{\prime}$ and $\mathrm{c}=57^{\circ} 25^{\prime}$.

13-12 Fig 21 Example 2 of the Tangent Formulae


$$
\begin{aligned}
\tan \frac{1}{2}(A-C) & =\cot B \frac{\sin \frac{1}{2}(a-c)}{\sin \frac{1}{2}(a+c)} \\
\cot \frac{1}{2} B & =\frac{\tan \frac{1}{2}(A-C) \sin \frac{1}{2}(a+c)}{\sin \frac{1}{2}(a-c)}=\frac{\tan \frac{1}{2}\left(38^{\circ} 42^{\prime}-32^{\circ} 50^{\prime}\right) \sin \frac{1}{2}\left(76^{\circ} 18^{\prime}+57^{\circ} 25^{\prime}\right)}{\sin \frac{1}{2}\left(76^{\circ} 18^{\prime}-57^{\circ} 25^{\prime}\right)}
\end{aligned}
$$

$$
=\frac{\tan 2^{\circ} 56^{\prime} \sin 66^{\circ} 51 \frac{1}{2}^{\prime}}{\sin \frac{1}{2}\left(76^{\circ} 18^{\prime}-57^{\circ} 25^{\prime}\right)}
$$

From which, using a calculator or logarithms:

$$
B=147^{\circ} 57^{\prime}
$$

## Right-angled Spherical Triangles

44. Consider a triangle $A B C$ in Fig 22 in which the spherical angle $A=90^{\circ}$.

## 13-12 Fig 22 Right-angled Spherical Triangle



Then, by the Four Parts formula:

$$
\begin{align*}
\cot a \sin c & =\cos c \cos B+\sin B \cot A \\
\text { but: } \cot A & =\cot 90^{\circ}=0 \\
\therefore \quad \cot a \sin c & =\cos c \cos B \\
\text { and: } \cos B & =\cot a \tan c \\
\text { Now: } \cos B & =\sin \left(90^{\circ}-B\right) \\
\text { and: } \quad \cot a & =\tan \left(90^{\circ}-a\right) \\
\therefore \quad \sin \left(90^{\circ}-B\right) & =\tan \left(90^{\circ}-a\right) \tan c \ldots \ldots . . \tag{18}
\end{align*}
$$

From the Cosine formula:
$\cos a=\cos b \cos c+\sin b \sin c \cos A$
but: $\cos \mathrm{A}=0$
so: $\cos a=\cos b \cos c$
$\therefore \quad \sin \left(90^{\circ}-a\right)=\cos b \cos c$

By taking each form of the Cosine and Four Parts formulae in turn a series of expressions can be obtained as follows:

```
\(\operatorname{Sin}\left(90^{\circ}-C\right)=\tan \left(90^{\circ}-a\right) \tan b\)
\(\sin \left(90^{\circ}-C\right)=\cos \left(90^{\circ}-B\right) \cos C\)
\(\sin \left(90^{\circ}-B\right)=\cos \left(90^{\circ}-C\right) \cos b\)
\(\sin \left(90^{\circ}-A\right)=\tan \left(90^{\circ}-B\right) \tan \left(90^{\circ}-C\right)\)
\(\operatorname{Sin} c=\quad=\tan \left(90^{\circ}-B\right) \tan b\)
\(\operatorname{Sin} c \quad=\cos \left(90^{\circ}-a\right) \cos \left(90^{\circ}-C\right)\)
\(\operatorname{Sin} \mathrm{b} \quad=\tan \left(90^{\circ}-\mathrm{C}\right) \tan \mathrm{c}\)
\(\sin b \quad=\cos \left(90^{\circ}-B\right) \cos \left(90^{\circ}-a\right)\)
```


## Napier's Rules of Circular Parts

46. The foregoing rules are difficult to memorize and are conveniently summarized in Napier's Rules of Circular Parts. Fig 23 shows a right-angled spherical triangle with the appropriate circular parts written alongside. Note that:
a. The parts are written down in the order in which they appear in the triangle.
b. The right angle is not counted as a circular part and is represented in the diagram by the double line.
c. The circular parts corresponding to the other 2 angles are the complements of those angles.
d. The circular part corresponding to the side opposite the right angle is the complement of that side.

## 13-12 Fig 23 Diagram for Napier's Rules of Circular Parts for a Right-angled Spherical Triangle


47. Provided that the circular parts are written down in accordance with the above principles, any one of the formulae in para 45 may be derived on sight from the following 2 rules:
a. The sine of the middle part is equal to the product of the tangents of the adjacent parts.
b. The sine of the middle part is equal to the product of the cosines of the opposite parts.

For example, select any part as the middle part. Let this be c in Fig 23.

Then: $\quad b$ and $\left(90^{\circ}-B\right)$ are the adjacent parts

$$
\therefore \quad \sin \mathrm{c}=\tan \mathrm{b} \tan \left(90^{\circ}-\mathrm{B}\right)
$$

$\left(90^{\circ}-\mathrm{a}\right)$ and $\left(90^{\circ}-\mathrm{C}\right)$ are the opposite parts

$$
\therefore \quad \sin c=\cos \left(90^{\circ}-a\right) \cos \left(90^{\circ}-C\right)
$$

## Examples of the Use of Napier's Rules.

48 Napier's Rules are especially useful when it is required to solve the spherical triangle for any other part, given:
a. Two sides and a non-included angle.
b. Two angles and a non-included side.

This is done by dividing the triangle into 2 right-angled triangles and applying the preceding rules. In para 22, an attempt was made to solve such a case by use of the Sine formula and it was apparent that ambiguity could arise. This, unfortunately, is also possible using Napier's Rules.
49. Example Using Napier's Rules. In the spherical triangle ABC in Fig 24, find side b, given that $B=62^{\circ} 07^{\prime}, C=33^{\circ} 42^{\prime}$ and $c=28^{\circ} 25^{\prime}$. To simplify the calculation, first construct AD, a perpendicular drawn from A to $\operatorname{arc} \mathrm{BC}$. Let this be designated d . In order to obtain b , side d is required.

## 13-12 Fig 24 Example of Napier's Rules



From Napier's Rules, using the circular parts diagram for triangle ABD at Fig 25:

## 13-12 Fig 25 Circular Parts for Triangle ABD



$$
\begin{aligned}
\sin d & =\cos \left(90^{\circ}-B\right) \cos \left(90^{\circ}-c\right) \\
& =\cos 27^{\circ} 53^{\prime} \cos 61^{\circ} 35^{\prime}
\end{aligned}
$$

Using a calculator or logarithms

$$
\mathrm{d} \quad=24^{\circ} 521^{1} 2^{\prime} \text { or } 155^{\circ} 071^{\prime} 2^{\prime}
$$

But c is opposite the right-angle so d cannot possibly be larger than c.
Hence: d $=24^{\circ} 52 \frac{1}{2}{ }^{2}$

Continuing, from Napier's Rules using the circular parts diagram for triangle ADC at Fig 26:

```
sin d= cos(90
    = sin C sin b
sin}b=\operatorname{sin}d\operatorname{cosec}
    = sin 244}52\mp@subsup{2}{}{1/2
```

Using a calculator or logarithms:
b $\quad=49^{\circ} 18^{\prime}$ or $130^{\circ} 42^{\prime}$, both of which are valid.

## 13-12 Fig 26 Circular Parts for Triangle ADC



## Right-sided Spherical Triangles

50. A spherical triangle (Fig 27) in which 1 side has a value of $90^{\circ}$ (sometimes called a 'quadrantal triangle') may be solved by Napier's Rules because, if a triangle is right-sided, it follows that its polar triangle is right-angled. To save the labour of conversion to the polar form in such cases, the following rules for the circular parts of right-sided triangles are stated without proof:
a. The parts are written down in the order in which they appear in the spherical triangle.
b. The right side is not counted as a circular part.
c. The circular parts corresponding to the other 2 sides are the complements of those sides.
d. The circular part corresponding to the angle opposite the right side is the complement of that angle.

13-12 Fig 27 Right-sided Spherical Triangle


Fig 28 shows the circular parts diagram in this case.

## 13-12 Fig 28 Circular Parts Diagram for Right-sided Triangle


51. Napier's Rules for right-sided spherical triangles are the same as those given in para 47, viz:
a. The sine of the middle part is equal to the product of the tangents of the adjacent parts.
b. The sine of the middle part is equal to the product of the cosines of the opposite parts.

The exception is that when the adjacent or opposite parts are both sides or both angles, a negative sign is added to the equation.
e.g. $\sin \left(90^{\circ}-A\right)=-\tan \left(90^{\circ}-b\right) \tan \left(90^{\circ}-c\right)$, but $\sin \left(90^{\circ}-b\right)=+\cos \left(90^{\circ}-c\right) \cos B$
52. Example of a Right-sided Spherical Triangle. In Fig 29, find A, given a $=90^{\circ}$, $\mathrm{c}=73^{\circ} 19^{\prime}$ and $\mathrm{b}=54^{\circ} 32^{\prime}$.

## 13-12 Fig 29 Right-sided Triangle Example



By Napier's Rules:

$$
\sin \left(90^{\circ}-A\right)=-\tan \left(90^{\circ}-b\right) \tan \left(90^{\circ}-c\right)
$$

$$
\therefore \quad \cos \mathrm{A}=-\tan 35^{\circ} 28^{\prime} \tan 16^{\circ} 41^{\prime}
$$

Using a calculator or logarithms:

$$
A=102^{\circ} 19.7^{\prime}
$$

## Summary of Formulae

53. Table 1 summarizes the use of the various formulae covered in this chapter.

Table 1 Summary of Formulae

| Formulae to Use | Giving |
| :--- | :--- | :--- |
| Haversine | Third side <br> Four Parts <br> Tangent |
| Both angles angle |  |

## CHAPTER 13 - FUNCTIONS AND LIMITS

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## Functions

1. In Volume 13, Chapter 6, it was shown that the relationship between two variables, $x$ and $y$ say, can be expressed in an equation such as $y=m x+c$. The principle is not confined to the linear relationship but may also be extended to such equations as:

$$
y=\sin x, y=e^{x}, \text { etc. }
$$

Since values are attributed to $x$ it is known as the independent variable; the corresponding values of $y$ may then be determined, and $y$ is therefore known as the dependent variable.
2. The dependence of $y$ upon $x$ is expressed mathematically in the phrase ' $y$ is a function of $x$ ' and is usually written as $y=f(x)$, in which $f(x)$ is a shorthand way of indicating some expression in terms of $x$. Thus, in the expression $y=x^{2}-4 x+3, f(x)$ is $x^{2}-4 x+3$; similarly, in $y=\sin 2 x, f(x)$ is $\sin 2 x$; and in $y=e^{2 x}, f(x)$ is $e^{2 x}$. In each case by plotting the graphs of these functions a smooth curve is obtained whose shape depends upon the nature of $f(x)$.
3. In each of the above examples an explicit statement has been made, i.e. $y$ is equal to some function of $x$. Such functions are known as explicit functions.
4. It is however possible to write a function, such as $9 x+6 x y+4 y^{2}=1$, in which, although there is no direct statement of $y$ in terms of $x$, it is evident nevertheless that corresponding values of $y$ could be determined by giving values to $x$. Such a function is known as an implicit function.

## Gradients

5. Suppose that an object is moving in a straight line in such a way that its distance, s metres, from a fixed point on the line at any time, $t$ seconds after it started moving, is governed by the equation:

$$
s=12+10 t-t^{2} \text {, i.e. } s=f(t)
$$

By giving a series of values to $t$ and calculating the corresponding values of $s$ then a graph of the function can be plotted showing how s changes as $t$ changes. Such a graph is shown in Fig 1.
6. Information about the speed at which the object is moving can be obtained from this graph by constructing chords. For example, over the period of 5 seconds, the increase in $s$ is indicated by PF $=25$ metres and the object's average speed over the period is therefore $25 / 5 \mathrm{~m} / \mathrm{sec}=5 \mathrm{~m} / \mathrm{sec}$. Letting $\angle \mathrm{FAP}$ be called $\theta$, then $\tan \theta=\mathrm{PF} / \mathrm{AP}=25 / 5$. So, the average speed during the 5 secs is given by the slope, or gradient, of the chord AF. Similarly the object's average speed over the first 4 secs is given by the gradient of the chord $A E=24 / 4=6 \mathrm{~m} / \mathrm{sec}$. Thus, it can be inferred that the average speed over any
selected period of time will be given by the gradient of the chord spanning that part of the curve. For example, the average speed of the object during the third second of its movement is given by the gradient of the chord CD, i.e. $5 \mathrm{~m} / \mathrm{sec}$.

## 13-13 Fig 1 Distance/Time Graph $s=12+10 t-t^{2}$

| t | 0 | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| s | 12 | 21 | 28 | 33 | 36 | 37 |


7. Whereas, for reasonably long intervals of time, it is possible to measure the gradient of the chord directly from the graph, if it becomes necessary to determine the gradient over a short period, such as KL, then this method will be difficult and inaccurate. However, it is possible to obtain the desired result by using the actual function:

$$
s=12+10 t-t^{2}
$$

As an example, suppose that it is required to find the average speed of the object over the period of time from $t=3$ secs to $t=3.1$ secs.

$$
\begin{array}{ll}
\text { After } 3.1 \text { secs, } \quad \mathrm{s} & =12+10(3.1)-(3.1)^{2} \mathrm{~m} \\
& =(43-9.61) \mathrm{m}=33.39 \mathrm{~m} \\
\text { After } 3 \text { secs, } \quad \mathrm{s} & =12+30-9 \mathrm{~m}=33 \mathrm{~m}
\end{array}
$$

Therefore, in 0.1 secs the object covered 0.39 m at an average speed of $3.9 \mathrm{~m} / \mathrm{sec}$.
8. By shortening the interval of time to 0.01 secs, i.e. from 3 to 3.01 secs, and substituting these figures in the function, the average speed becomes $3.99 \mathrm{~m} / \mathrm{sec}$. Taking an even shorter interval from 3 secs to 3.001 secs yields an average speed of $3.999 \mathrm{~m} / \mathrm{sec}$. If the same exercise is repeated for time intervals just prior to 3 secs, the following results are obtained:
a. From 2.9 to 3 secs: $4.1 \mathrm{~m} / \mathrm{sec}$
b. From 2.99 to 3 secs: $4.01 \mathrm{~m} / \mathrm{sec}$
c. From 2.999 to 3 secs: $4.001 \mathrm{~m} / \mathrm{sec}$

From the figures, it can be inferred that at the precise time of 3 secs the actual or instantaneous speed was $4 \mathrm{~m} / \mathrm{sec}$.
9. Fig 2 shows a magnified section of the graph with just two of the chords drawn. The gradient of the chord PD represents the average speed between 2.9 and 3.0 secs; the gradient of DQ represents the average speed between 3.0 and 3.1 secs. The chord PD has been extended to M and the chord QD projected back to L. Between 2.9 and 3.1 secs, the chord PM rotates about an axis through D until it is aligned with LQ. At some instant during this rotation, the chord will take up the position of the tangent to the curve at $D$. It can be inferred that this will occur at the time $t=3$ secs; thus the gradient of the tangent at a point on a distance/time graph measures the actual speed at that instant, i.e. the rate of change of $s$ compared with the rate of change of $t$ at that instant.

## 13-13 Fig 2 Two Chords on Magnified Section of $s=12+10 t-t^{2}$



## Infinitesimals and Limits

10. A shorter method of arriving at this conclusion without using specified intervals was devised by Newton. He suggested that a small increase in any quantity like s might be indicated by a special symbol $\delta$ (delta s) which has no specified size but represents a minutely small change in s . A similar small change in $t$ would be denoted by $\delta t$, and in $x$ by $\delta x$, etc.
11. Fig 3 shows a section of the curve

$$
s=12+10 t-t^{2}
$$

PM represents the distance covered, s , at time $\mathrm{OM}(\mathrm{t})$. QN represents the distance ( $\mathrm{s}+\delta \mathrm{s}$ ) covered in time $\mathrm{ON}(\mathrm{t}+\delta \mathrm{t})$. In both cases, $\delta \mathrm{s}$ and $\delta \mathrm{t}$ are very small. Thus, the graph shown is a greatly magnified portion of a very small arc of the curve. The gradient of the chord PQ represents the average speed between time $t$ and $(t+\delta t)$ and can be measured as $Q R / P R=\delta s / \delta t$.

## 13-13 Fig 3 Section of the Curve $s=12+10 t-\mathbf{t}^{2}$


12. Since $Q$ is on the curve:

$$
\begin{align*}
s+\delta s & =12+10(t+\delta t)-(t+\delta t)^{2} \\
& =12+10 t+10 \delta t-t^{2}-2 t \delta t-(\delta t)^{2} . \tag{1}
\end{align*}
$$

and for $P$

$$
\begin{equation*}
s=12+10 t-t^{2} \tag{2}
\end{equation*}
$$

Subtracting (2) from (1)

$$
\delta s=10 \delta t-2 t \delta t-(\delta t)^{2}
$$

Dividing by $\delta$ t

$$
\delta s / \delta t=10-2 t-\delta t
$$

Thus, a formula has been derived for calculating the average speed over any period of time however small.

For example, between $t=3$ and $t=3.0001$ secs, ie $\delta t=0.0001$ secs:

$$
\delta \mathrm{s} / \delta \mathrm{t}=10-6-0.0001=3.9999 \mathrm{~m} / \mathrm{sec}
$$

If a value of 0.000001 secs had been used in the formula, then $\delta s / \delta t$ would have been 3.999999 .
13. Thus, it will be seen that in the expression:

$$
\delta s / \delta t=10-2 t-\delta t
$$

if the value of $\delta t$ is allowed to grow smaller and smaller, i.e. approaches zero, then $\delta s / \delta t$ approaches the value 10-2t. This is written as:

$$
\operatorname{Lim}_{\delta t \rightarrow 0} \frac{\delta s}{\delta t}=10-2 t
$$

This is read as 'The limit of delta s by delta $t$, as delta $t$ tends to zero, equals $10-2 t$ '.
14. If the value of 3 secs is now substituted into this expression, then $10-2 t=4$, which is the value for the gradient that was deduced earlier, i.e. the actual speed at the instant of $t=3$ secs. To indicate that this is the actual gradient at an instant then:

$$
\operatorname{Lim}_{\delta \mathrm{t} \rightarrow 0} \frac{\delta \mathrm{~s}}{\delta \mathrm{t}} \text { is replaced by } \frac{\mathrm{ds}}{\mathrm{dt}}
$$

Thus, in summary, if an object is moving so that the distance s metres covered in time $t$ seconds is a function of the time, (i.e. $s=f(t)$ and $f(t)=12+10 t-t^{2}$ ), then its speed at any time, $t$, is equal to the gradient of the tangent to the distance/time graph at time $t$ and is defined by $\mathrm{ds} / \mathrm{dt}$ which may be calculated from the expression $\mathrm{ds} / \mathrm{dt}=10-2 \mathrm{t}$. The notation $\mathrm{ds} / \mathrm{dt}$ is, therefore, a measure of the rate at which $s$ is changing compared with the rate at which $t$ is changing at an instant of time ' t '.

## CHAPTER 14 - DIFFERENTIATION

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## Gradients

1. In Volume 13, Chapter 13 it was shown that given the relationship between distance gone and time it was possible to find the rate of change of distance with time, i.e. speed, either over a specified interval or at a particular instant, by determining the gradient of the appropriate chord or tangent. The technique is not restricted to the distance/time problem but has a general applicability whenever one parameter is changing in response to changes in another.
2. For example, when we say that a train passes us at 60 mph , we do not mean that it has travelled 60 miles in the last hour nor that it will travel 60 miles in the next hour. We mean that it will travel about 1 mile in the next minute or, better still, about half a mile in the next 30 seconds or, with still greater probability, about 88 feet in the next second. To find the speed of the train at the instant at which it passes us, we must measure the distance it goes in as small an interval of time as possible and then work out the average speed for this short interval. The shorter the interval, the closer will our answer be to the train's actual speed at that instant. In practice, there will always be an error in the measurement of instantaneous speed but we can calculate instantaneous speed with complete accuracy by means of differentiation provided we know enough about the motion.
3. In moving along the straight line $A B$ (Fig 1), starting at $P(x, y)$, an increase $N M$ (= PR) in $x$ produces an increase $R Q$ in $y$. The ratio of the increase in $y$ to the increase in $x$ (ie $R Q / P R$ ) is called the gradient of the slope of the line $A B$. Clearly, the gradient is equal to tan $\theta$.

## 13-14 Fig 1 Gradient of a Straight Line


4. A small interval in the $x$-axis, like NM, is usually denoted by $\delta x$, pronounced delta- $x$ and must be thought of as a single symbol, the $x$ never being separated from the variable. If $y$ is given in terms of, or as a function of, $x$, i.e. $y=f(x)$, then any change in the value of $x$ produces a change in the value of $y$. The symbol $\delta y$ is used to denote the increment in $y$ caused by the increment $\delta x$. Notice the difference in the definitions of $\delta x$ and $\delta y$ because $x$ is the independent and $y$ the dependent variable. In Fig $1, \delta x=N M=P R, \delta y=R Q$ and the gradient of $A B$ is $\delta y / \delta x$.
5. In Fig 2, $\delta x=N M=P R$ and $\delta y$, the corresponding increase in $y$, is $-R Q$. $\delta y$ is negative because an increase in $x$ causes a decrease in $y$. The resulting gradient $\delta y / \delta x$ is, therefore, negative.

## 13-14 Fig 2 Negative Gradient


6. Gradient of a Curve. The gradient of a curve at any point is defined as the gradient of the tangent to the curve at that point. In Fig 3, let $P(x, y)$ be any point on the curve. Let $N M=\delta x$, then the corresponding increase in $y$ is $R Q$ and so $\delta y=R Q$. Then $\delta y / \delta x=$ the gradient of the chord $P Q$ and represents the average gradient of the curve between the points $P$ and $Q$. As $\delta x \rightarrow 0$ (meaning $\delta x$ tends towards 0 , i.e. becomes closer to 0 ), the value of $\delta y / \delta x$ changes and, at the same time, the chord PQ approaches its limiting position, namely the tangent to the curve at $P$. Hence the gradient of the curve at $P$ is described as:

$$
\mathrm{P}=\lim _{\delta \mathrm{x} \rightarrow 0} \frac{\delta \mathrm{y}}{\delta \mathrm{x}}
$$

and this limit can be denoted as $\frac{d y}{d x}$, pronounced "dee- $y$ by dee-x". The value is, of course, given also by $\tan \theta$, where $\theta$ is the angle between the tangent and OX in Fig 3.

13-14 Fig 3 Gradient of a Curve

7. The Gradient of $\mathbf{y}=\mathbf{x}^{3}$. As an example, Fig 4 shows the graph representing $y=x^{3}$. Let $P(x, y$, be any point on the curve. NM represents a small change $\delta x$, in $x$, and $R Q$ represents the consequential change $\delta y$, in $y$. Thus, $Q$ is the point $(x+\delta x, y+\delta y)$. As both $P$ and $Q$ lie on the line then:

$$
\begin{equation*}
\text { for } P, \quad y=x^{3} \tag{1}
\end{equation*}
$$

and for $Q, \quad y+\delta y=(x+\delta x)^{3}$

$$
\begin{equation*}
=x^{3}+3 x^{2} \delta x+3 x(\delta x)^{2}+\delta x^{3} \tag{2}
\end{equation*}
$$

Subtracting (2) - (1)

$$
\delta y=3 x^{2} \delta x+3 x(\delta x)^{2}+\delta x^{3}
$$

Dividing by $\delta x$

$$
\frac{\delta y}{\delta x}=3 x^{2}+3 x \delta x+\delta x^{2}
$$

Then by definition (para 6) the gradient of the tangent to the curve:

$$
\frac{\mathrm{dy}}{\mathrm{dx}}=\lim _{\delta \mathrm{x} \rightarrow 0} \frac{\delta \mathrm{y}}{\delta \mathrm{x}}
$$

i.e. $\frac{d y}{d x}=3 x^{2}$ as the $3 x \delta x$ and $\delta x^{2}$ terms are eliminated (because $\delta x$ becomes 0 ).

## 13-14 Fig 4 Fig $1 y=f(x)=x^{3}$



Clearly, the gradient varies from point to point, as can be seen from the graph. The way in which the gradient varies is given by $\frac{d y}{d x}$ i.e. by the function $3 x^{2}$. Thus the value of the gradient can be determined by substituting the appropriate value of $x$ into the expression $3 x^{2}$.
e.g. gradient at $x=0$, is 0
gradient at $x=1$, is 3
gradient at $x=2$, is 12
8. The Gradient of $\mathbf{y}=\mathbf{x}^{2}$. Fig 5 shows the graph representing $\mathrm{y}=\mathrm{x}^{2}$.

13-14 Fig 5 Fig $2 y=f(x)=x^{2}$


If $(x, y)$ is any point on the curve then:

$$
\begin{equation*}
y=x^{2} \tag{3}
\end{equation*}
$$

If $x$ increases by $\delta x$ so that $y$ increases by $\delta y$ then

$$
\begin{align*}
y+\delta y & =(x+\delta x)^{2} \\
& =x^{2}+2 x \delta x+(\delta x)^{2} \tag{4}
\end{align*}
$$

Subtracting (4) - (3) gives

$$
\begin{aligned}
\delta y & =2 x \delta x+(\delta x)^{2} \\
\text { and } \quad \frac{\delta y}{\delta x} & =2 x+\delta x
\end{aligned}
$$

from which the gradient of $y=x^{2}$ at $(x, y)$, ie $\frac{d y}{d x}$, is $2 x$.

## The Differential Coefficient (Derivative)

9. $\frac{d y}{d x}$ is called the differential coefficient of y with respect to x , or the derivative of y with respect to x . The process of obtaining $\frac{d y}{d x}$ is called differentiating $y$ with respect to $x$. Sometimes $\frac{d y}{d x}$ is written $\frac{d}{d x}(y)$ in which $\frac{d}{d x}$ is an operator, like the symbol $\sqrt{ }$, and means simply "the derivative of". Thus, the expressions:

$$
\begin{aligned}
& \frac{d}{d x}\left(x^{2}+5 x\right) \\
& \text { or } \frac{d\left(x^{2}+5 x\right)}{d x} \\
& \text { or } \frac{d y}{d x} \text { where } y=x^{2}+5 x,
\end{aligned}
$$

all mean the same thing, namely:

$$
\lim _{\delta x \rightarrow 0} \frac{\delta y}{\delta x} \text { when } y=x^{2}+5 x
$$

## Differentials

10. Although $\frac{\delta y}{\delta x}$ is a quotient (i.e. it stands for $\delta y \div \delta x$ ), $\frac{d y}{d x}$ is not. Once the quotient $\frac{\delta y}{\delta x}$ has been obtained, $\frac{d y}{d x}$ can be found as the limiting value of this quotient. Strictly speaking $\frac{d y}{d x}$ ought to be regarded as a single symbol like $\delta y$ or $\delta x$. However, although it is important to remember that $\frac{d y}{d x}$ is obtained as a limit and is not strictly a quotient, it is often convenient to treat it as if it is. For example having found that when $y=x^{2}, \frac{d y}{d x}=2 x$, this result could be written as $d y=2 x d x$ or $d\left(x^{2}\right)=2 x d x$. In this notation, $d y, d x$ and $d\left(x^{2}\right)$ are best regarded as infinitesimal increments in $y, x$, and $x^{2}$ and are called the differentials of those quantities.

## The General Case

11. The arguments in paras 7 and 8 to obtain the gradient or differential coefficients of $y=x^{3}$ and $y=$ $x^{2}$, can be generalized for the case of $y=f(x)$. Remembering that if $y$ is given as a function of $x$, i.e. $y=f(x)$, then any change in the value of $x$ produces a change in the value of $y$. So, if $(x, y)$ is any point on a curve then:

$$
\begin{equation*}
y=f(x) \tag{5}
\end{equation*}
$$

and an increase in $x$, i.e. $\delta x$ causes an increase in $y$, i.e. $\delta y$, such that:

$$
\begin{equation*}
y+\delta y=f(x+\delta x) \tag{6}
\end{equation*}
$$

Subtracting (6) $-(5)$ :

$$
\delta y=f(x+\delta x)-f(x)
$$

Hence, $\quad \frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$
and so $\quad \frac{d y}{d x}=\lim _{\delta x \rightarrow 0} \frac{f(x-\delta x)-f(x)}{\delta x}$

## Successive Differentiation

12. When $y=f(x)=x^{3}$ was differentiated the result was

$$
\frac{\mathrm{dy}}{\mathrm{dx}}=\mathrm{or}\left[\frac{\mathrm{~d}}{\mathrm{dx}}\left(\mathrm{x}^{3}\right)\right]=3 \mathrm{x}^{2}
$$

which is itself a function of $x$, sometimes expressed as $f^{\prime}(x)$. This can itself be differentiated and can be shown to be:

$$
\frac{d}{d x}\left(\frac{d y}{d x}\right)=6 x
$$

The expression $\frac{d}{d x}\left(\frac{d y}{d x}\right)$ is usually written as $\frac{d^{2} y}{d x^{2}}$ or as $f "(x)$. Similarly the result of further differentiation of $6 x$ would be written as $f^{\prime \prime \prime}(x)$ or $\frac{d^{3} y}{d x^{3}}=6$, and $f^{\prime \prime \prime \prime}(x)$ or $\frac{d^{4} y}{d x^{4}}=0$.

## Standard Derivatives

13. Examination of the successive differentiation above, known as differentiation from first principles, reveals a pattern from which a general rule can be derived. In practice, there are a number of rules which allow the derivatives of certain functions to be determined without recourse to formal working. For example, the results in para 12 show that:

$$
\frac{\mathrm{d}}{\mathrm{dx}} \mathrm{ax}^{\mathrm{n}}=\operatorname{nax}^{\mathrm{n}-1}
$$

where $a$ and $n$ are constants which may be positive or negative, fractions or integers. This formula may be applied to each level of differentiation to reach the result. For example, using the formula:

Where $y=x^{2}, \frac{d y}{d x}=2 x\left(\right.$ i.e. $\left.2 \times 1 \times x^{1}\right)$ and $\frac{d}{d x}(2 x)=2\left(\right.$ i.e. $\left.1 \times 2 \times x^{0}\right)$
14. Sum of Terms. Where $f(x)$ is the sum of a number of terms eg:

$$
y=a x^{3}+b x^{2}+c x+d
$$

where $a, b, c$, and $d$ are constants, then $f^{\prime}(x)$ is the sum of the derivatives of each individual term. Thus, in this case:

$$
f^{\prime}(x)=3 a x^{2}+2 b x+c
$$

Note that the derivative of a constant $=0$.
15. Product Rule. It may be that it becomes necessary to differentiate an expression which is the product of two functions of the same variable, e.g.:

$$
y=(x+1)\left(x^{2}-3\right)
$$

In this case it would be possible to multiply out the expression without much difficulty and then differentiate the sum of the terms as outlined in para 14. However this may not be convenient, especially if there are several functions rather than just two. In this situation the product rule can be used. Let one function be $u$ and the other $v$, so $y=u v$. Then, by the product rule:

$$
\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{\mathrm{udv}}{\mathrm{dx}}+\frac{\mathrm{vdu}}{\mathrm{dx}}
$$

i.e. the result is the first function multiplied by the derivative of the second function, plus the second function multiplied by the derivative of the first function. Thus in the example:

$$
y=(x+1)\left(x^{2}-3\right)
$$

Let, $(x+1)=u$ and $\left(x^{2}-3\right)=v$
Then: $\frac{d y}{d x}=\frac{u d v}{d x}+\frac{v d u}{d x}$
Thus: $\quad \frac{d y}{d x}=(x+1) \times 2 x+\left(x^{2}-3\right) \times 1$

$$
\begin{aligned}
& =2 x^{2}+2 x+x^{2}-3 \\
& =3 x^{2}+2 x-3
\end{aligned}
$$

This method can be extended to cover more than two factors.
Thus, $\frac{d(u v w)}{d x}=u v \frac{d w}{d x}+w u \frac{d v}{d x}+w v \frac{d u}{d x}$.
For example:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dx}}(x+1)(x+2)(x+3) & =(x+1)(x+2)+(x+1)(x+3)+(x+2)(x+3) \\
& =x^{2}+3 x+2+x^{2}+4 x+3+x^{2}+5 x+6 \\
& =3 x^{2}+12 x+11
\end{aligned}
$$

16. Function of a Function - The Chain Rule. Consider an expression such as:

$$
y=\left(3 x^{2}+2\right)^{2}
$$

Here, $y$ is a function of $\left(3 x^{2}+2\right)$ and $\left(3 x^{2}+2\right)$ is a function of $x$. As with the product rule, in some cases the function may be simplified into the sum of several functions which may then be differentiated individually. However this will be tedious if the power is greater than, say, 2. The chain rule can be used to solve this problem as follows:

Let, $3 x^{2}+2=u$
Then, $\mathrm{y}=\mathrm{u}^{2}$
$\therefore \frac{\mathrm{dy}}{\mathrm{du}}=2 \mathrm{u}$
and as

$$
u=3 x^{2}+2
$$

$$
\frac{\mathrm{du}}{\mathrm{dx}}=6 x
$$

$$
\frac{d y}{d u} \times \frac{d u}{d x}=\frac{d y}{d x}=2 u \times 6 x
$$

Then substituting back for $u$ :

$$
\begin{aligned}
\frac{d y}{d x} & =2\left(3 x^{2}+2\right) \times 6 x \\
& =\left(6 x^{2}+4\right) \times 6 x \\
& =36 x^{3}+24 x
\end{aligned}
$$

17. Quotient of $\mathbf{2}$ Functions. If $y=u / v$ where $u$ and $v$ are functions in $x$ then:

$$
\frac{d y}{d x}=\frac{v \frac{d u}{d x}-u \frac{d v}{d x}}{v^{2}}
$$

As an example consider the function:

$$
y=\frac{\left(x^{2}+1\right)}{(3 x+2)}
$$

Then, $\mathrm{u}=\left(\mathrm{x}^{2}+1\right) \quad \therefore \frac{\mathrm{du}}{\mathrm{dx}}=2 \mathrm{x}$
And, $v=(3 x+2) \therefore \frac{d v}{d x}=3$
Thus, $\frac{d y}{d x}=\frac{(3 x+2) \times 2 x-\left(x^{2}+1\right) \times 3}{(3 x+2)^{2}}$

$$
\begin{aligned}
& =\frac{6 x^{2}+4 x-3 x^{2}-3}{(3 x+2)^{2}} \\
& =\frac{3 x^{2}+4 x-3}{(3 x+2)^{2}}
\end{aligned}
$$

## Summary

18. A summary of the results derived above together with other standard derivatives is shown in Table 1.

Table 1 Standard Differential Coefficients

| Type of Function | Standard Type | Standard Differential Coefficient | Comments |
| :---: | :---: | :---: | :---: |
| Standard | $y=f(x)$ | $\frac{d y}{d x}$ |  |
| Algebraic | $y=a x^{n}$ | nax ${ }^{\text {n-1 }}$ | Reduce the power by 1 and multiply by the original power. |
| Trigonometric | $\begin{aligned} & y=\sin x \\ & y=\cos x \\ & y=\tan x \end{aligned}$ | $\begin{gathered} \cos x \\ -\sin x \\ \sec ^{2} x \end{gathered}$ |  |
| Logarithmic | $y=\log _{\mathrm{e}} \mathrm{x}$ | $\frac{1}{x}$ |  |
| Exponential | $y=e^{k x}$ | $k e^{k x}$ | Multiply the original function by the differential coefficient of its index. |
| Sum of two or more functions | u + v | $\frac{\mathrm{du}}{\mathrm{dx}}+\frac{\mathrm{dv}}{\mathrm{dx}}$ | The differential coefficient of a sum is the sum of the differential coefficients. |
| Product of two functions | uv | $u \frac{d v}{d x}+v \frac{d u}{d x}$ | Multiply each function by the differential coefficient of the other and add the results. |
| Quotient of two functions | $\frac{\mathrm{u}}{\mathrm{v}}$ | $\frac{\mathrm{v} \frac{\mathrm{du}}{\mathrm{dx}}-\mathrm{u} \frac{\mathrm{dv}}{\mathrm{dx}}}{\mathrm{v}^{2}}$ |  |
| Function of a function | $\mathrm{F}[\mathrm{f}(\mathrm{x})$ ] | $\frac{\mathrm{df}}{\mathrm{~d}[\mathrm{f}(\mathrm{x})]} \times \frac{\mathrm{df}(\mathrm{x})}{\mathrm{dx}}$ | Use the chain rule. |

## CHAPTER 15 - INTEGRATION

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## PRINCIPLES OF INTEGRATION

## Introduction

1. Differentiating means solving the problem:

$$
\text { Given } y=f(x) \text {, find } \frac{d y}{d x}
$$

The reverse problem is:

$$
\text { Given } \frac{d y}{d x}=f(x) \text {, find } y
$$

This reverse process is called integration.

## Indefinite Integrals

2. Consider the following:

$$
\begin{equation*}
\frac{d y}{d x}=a x^{n} \tag{1}
\end{equation*}
$$

From the discussion on differentiation in Volume 13, Chapter 14, it will be apparent that the function whose derivative with respect to x is axn is of the form:

$$
\begin{equation*}
y=b x^{n+1} \tag{2}
\end{equation*}
$$

If $(2)$ is differentiated with respect to $x$ the result is:

$$
\begin{equation*}
\frac{\mathrm{dy}}{\mathrm{dx}}=(\mathrm{n}+1) \mathrm{b} x^{\mathrm{n}} \tag{3}
\end{equation*}
$$

Comparing (1) and (3), they will be the same if $(n+1) b=a$, ie if $b=\frac{a}{n+1}$. Substituting this value of $b$ in (2):

$$
\begin{equation*}
\mathrm{y}=\frac{\mathrm{ax} \mathrm{x}^{\mathrm{n}+1}}{\mathrm{n}+1} \tag{4}
\end{equation*}
$$

Although (4) is certainly one solution to the problem, it is not a unique solution. Since the derivative of any constant is zero, the derivative of (4) will be unchanged if any constant, c , is added to the righthand side of the equation. Therefore, the general solution is:

$$
\begin{equation*}
\mathrm{y}=\frac{\mathrm{ax} \mathrm{x}^{\mathrm{n}+1}}{\mathrm{n}+1}+\mathrm{c} \tag{5}
\end{equation*}
$$

Because of the presence of the arbitrary constant, $c$, (5) is known as the indefinite integral of (1).
3. Integration Symbol. It is convenient to have a symbol to denote the indefinite integral of a function, thus (5) may be rewritten as:

$$
\begin{equation*}
\int a x^{n} d x=\frac{a x^{n+1}}{n+1}+c \tag{6}
\end{equation*}
$$

In this notation the $\int$ and the $d x$ are used as brackets to denote that everything between them is to be integrated with respect to $x$. The quantity so bracketed is known as the integrand. Thus $a x^{n}$ is the integrand of $\int a x^{n} d x$, while the right-hand side of (6) is the integral. Formula (6) holds for all values of $n$, integral and fractional, positive and negative, with the single exception of $n=-1$. This case will be dealt with later.
4. The Constant of Integration. When a function is differentiated, the result represents the gradient of the graph of that function. Consequently as the integration process is the reverse of differentiation, an integral represents a function with the given gradient. Recalling that the equation of
a straight line is $y=m x+c$ it will be remembered that the coefficient of $x$, i.e. $m$, equates to the gradient of the line. There is, however, an infinite family of parallel lines, all with the same gradient, $m$, varying in the value of the constant c. Thus the knowledge of the gradient is insufficient to describe uniquely a particular straight line. So when a function is integrated an arbitrary constant must be included to take account of the infinite number of 'parallel' functions. As an example consider the following function:

$$
y=\int 2 x d x
$$

To perform the integration the power of $x$ has to be increased by 1 , and then the integrand has to be divided by the new power. Finally the arbitrary constant must be added. Thus:

$$
y=x^{2}+c
$$

Fig 1 shows the graphs of $y=x^{2}+c$ for a variety of values of $c$.

## 13-15 Fig 1 Graphs of $y=x^{2}+c$



For any given value of $x$ all of the curves have the same gradient, ie they all satisfy the condition $d y / d x=2 x$. In order to determine which graph is the solution to the particular problem then more information is required. For example, it may be known that $y=3$ when $x=0$, hence $3=0+c$, thus $c=3$. Therefore, the required solution is $\mathrm{y}=\mathrm{x}^{2}+3$.
5. As a practical example suppose that a body moves with an acceleration of $3 \mathrm{~ms}^{-2}$ and it is necessary to find an expression for its velocity after $t$ seconds:

$$
\frac{d v}{d t}=3 \text {, ie } \quad v=\int 3 d t=3 t+c
$$

The reason that this is an inadequate description of the velocity is that, although acceleration information was provided, no information was given concerning the initial velocity of the body. Therefore, no definite value for the velocity at any given time can be deduced. If the initial velocity was, say, $2 \mathrm{~ms}^{-1}$ then the velocity at any time, t , becomes $3 \mathrm{t}+2 \mathrm{~ms}^{-1}$ (i.e. $\mathrm{c}=2$ ).

## Standard Integrals

6. Just as with differentiation, there are a number of standard integrals which are used. In general an unfamiliar expression must be converted into a standard form, or a variation or a combination of standard forms, before the integration can be accomplished. Similar rules to those used in differentiating apply in integrating; thus the integral of a sum of a set of functions becomes the sum of the integrals of each individual function. If an integrand has a constant then this is taken out before integration is performed, thus:

$$
\int 3 x^{2} d x=3 \int x^{2} d x
$$

Usually, products or quotients must be simplified into simple functions before integration can take place. Thus, for example:

$$
\begin{aligned}
& \int(x+2)(x-3) d x \\
= & \int\left(x^{2}-x-6\right) d x \\
= & \frac{x^{3}}{3}-\frac{x^{2}}{2}-6 x+c
\end{aligned}
$$

and

$$
\begin{aligned}
& \int \frac{x(x-1)}{x^{\frac{1}{2}}} d x \\
= & \int \frac{x^{2}-x}{x^{\frac{1}{2}}} d x \\
= & \int\left[x^{\frac{3}{2}}-x^{\frac{1}{2}}\right] d x \\
= & \frac{2 x^{\frac{5}{2}}}{5}-\frac{2 x^{\frac{3}{2}}}{3}+c
\end{aligned}
$$

7. In paragraph 3 it was shown that the integral of a simple function in $x, a x^{n}$ is given by:

$$
\frac{\mathrm{ax}^{\mathrm{n}+1}}{\mathrm{n}+1}
$$

However, it was stated that this formula did not apply when $\mathrm{n}=-1$. This is because the denominator of the expression would become $-1+1=0$, and dividing by zero has no real meaning. The paradox can be resolved by recalling that differentiating $\log _{\mathrm{e}} \mathrm{x}$ yields $\frac{1}{\mathrm{x}}$, therefore the converse means:

$$
\int \frac{1}{x} d x=\log _{e} x
$$

8. A list of the more common standard integrals is shown in Table 1.

Table 1 Some Standard Integrals

| $\frac{d y}{d x}$ | $y=\int \frac{d y}{d x} d x$ | Comments |
| :---: | :---: | :--- |
| $\mathrm{x}^{\mathrm{n}}$ | $\frac{\mathrm{x}^{\mathrm{n}+1}}{\mathrm{n}+1}$ | Increase the index by 1, and divide by the new index. |
| $\cos \mathrm{x}$ | $\sin \mathrm{x}$ | Inverse of differentiation. |
| $\sin \mathrm{x}$ | $-\cos \mathrm{x}$ | Inverse of differentiation. |
| $\sec ^{2 \mathrm{x}}$ | $\tan \mathrm{x}$ |  |
| $\mathrm{e}^{\mathrm{kx}}$ | $\frac{e^{k x}}{k}$ | Write down the function $\mathrm{e}^{\mathrm{kx}}$ and divide it by the differential <br> coefficient of the index of e. <br> $\frac{1}{\mathrm{x}}$ $\log _{\mathrm{e}} \mathrm{x}$ |$\quad$|  |
| :--- |

## Definite Integrals

9. Fig 2 shows an arc, $Z B$, of the curve $y=f(x)$. $P$ is the point $(x, y)$ and $Q$ the point $[(x+\delta x),(y+\delta y)]$, where $\delta x$ and $\delta y$ are very small quantities.

## 13-15 Fig 2 Area under the Curve of $f(x)$


10. The elemental strip LPQM is part of the area (A) between the curve and the axes of $x$ and $y$. Let the area, LPQM, be denoted as $\delta A$. The mean height of the arc PQ lies between $y$ and $y+\delta y$. Suppose it equals $y+K \delta y$, where $K<1$, then the area LPQM $=\delta A=\delta x(y+K \delta y)$

$$
\therefore(\delta \mathrm{A} / \delta \mathrm{x})=\mathrm{y}+\mathrm{K} \delta \mathrm{y}
$$

In the limit as $\delta x \rightarrow 0$ then $(\delta A / \delta x) \rightarrow(d A / d x)$ and $\delta y \rightarrow 0$

$$
\therefore \frac{\mathrm{dA}}{\mathrm{dx}}=\mathrm{y} \text {, and } \int \frac{\mathrm{dA}}{\mathrm{dx}} \mathrm{dx}=\int \mathrm{ydx}
$$

i.e. $A=\int y d x=\int f(x) d x$

Suppose, $\int f(x) d x=F(x)+c$, then, $A=F(x)+C$
11. If it is required to find the area between the curve, the $x$ axis, and the ordinates at $x=a$ and $x=b$, i.e. the area DCBA in Fig 3 then:

For the ordinate at $\mathrm{x}=\mathrm{b}$,

$$
A_{1}=F(b)+C
$$

and at $x=a$,

$$
\mathrm{A}_{2}=\mathrm{F}(\mathrm{a})+\mathrm{C}
$$

Subtracting these, $D C B A=A_{1}-A_{2}=F(b)-F(a)$

## 13-15 Fig 3 Area under the Curve - The Definite Integral



This is written as:

$$
\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f}(\mathrm{x}) \mathrm{dx}=\mathrm{F}(\mathrm{~b})-\mathrm{F}(\mathrm{a})
$$

or, in words, "the integral $f(x) d x$ between the limits $x=a$ and $x=b$ ". ' $a$ ' and ' $b$ ' are called, respectively, the lower and upper limits of the value of $x$. Notice that the constant of integration has disappeared; this is because it would appear in both $F(b)$ and in $F(a)$ and is thus cancelled in the subtraction. Because such integrals are evaluated between defined limits, they are called definite integrals.
12. In summary, the method is as follows:
a. Integrate the function, omitting the constant of integration.
b. Substitute the value of the upper limit for $x$; repeat for the value of the lower limit. Subtract the results to give $F(b)-F(a)$.

Example:

$$
\int_{2}^{2} x^{3} d x=\left[\frac{x^{4}}{4}\right]_{1}^{2}=\left(\frac{2^{4}}{4}\right)-\left[\frac{1^{4}}{4}\right]=3.75
$$

## APPROXIMATE NUMERICAL INTEGRATION

## Introduction

13. The integration process is often complex, but, provided that the requirement is for a numerical answer to definite integration, then there are a number of methods available which yield approximate results, most of which are suitable for computer implementation if necessary. Two such methods, the trapezoidal rule and Simpson's rule, will be described.

## Trapezoidal Rule

14. In the trapezoidal rule, the $x$ axis is divided into equal intervals, $h$, and the top of each arc section is approximated by the chord, as in Fig 4. Thus, a series of trapezia are formed whose top coordinates have the values $\mathrm{y}_{1}, \mathrm{y}_{2}$, etc.

## 13-15 Fig 4 The Trapezoidal Rule


15. The area of the first trapezium is: $1 / 2 h\left(y_{1}+y_{2}\right)$, and so:

$$
\begin{aligned}
\int_{x_{1}}^{x_{n}} \mathrm{ydx} & =\frac{1}{2} \mathrm{~h}\left(\mathrm{y}_{1}+\mathrm{y}_{2}\right)+\frac{1}{2} \mathrm{~h}\left(\mathrm{y}_{2}+\mathrm{y}_{3}\right)+\ldots . \cdot \frac{1}{2} \mathrm{~h}\left(\mathrm{y}_{\mathrm{n}-1}+\mathrm{y}_{\mathrm{n}}\right) \\
& =\mathrm{h}\left(1 / 2 \mathrm{y}_{1}+\mathrm{y}_{2}+\mathrm{y}_{3} \ldots \ldots \mathrm{y}_{\mathrm{n}-1}+1 / 2 \mathrm{y}_{\mathrm{n}}\right)
\end{aligned}
$$

In general, a small value of $h$ will give a better solution than a large one, but the best procedure is to repeat the computation with successively smaller values of $h$ until two results agree within the required level of precision.
16. Example. Compute $\int_{0.5}^{1} \mathrm{x}^{\frac{1}{2}} \mathrm{dx}$ using the trapezoidal rule.

First compute with $\mathrm{h}=0.1$, say:

$$
\begin{array}{rlrl}
\mathrm{X}_{1} & =0.5 ; & 1 / 2 \mathrm{y}_{1} & =0.3535 \\
\mathrm{X}_{2} & =0.6 ; & \mathrm{y} 2 & =0.7746 \\
\mathrm{X}_{3} & =0.7 ; & \mathrm{y}_{3} & =0.8367 \\
\mathrm{X}_{4} & =0.8 ; & \mathrm{y}_{4} & =0.8944 \\
\mathrm{X}_{5} & =0.9 ; & \mathrm{y}_{5} & =0.9487 \\
\mathrm{X}_{6} & =1.0 ; & 1 / 2 \mathrm{y}_{6} & =\underline{0.5000} \\
& \text { Sum } & =4.3079 \\
\int_{0.5}^{1} \mathrm{x}^{\frac{1}{2}} \mathrm{dx} & =0.1 \times 4.3079 & =0.43079
\end{array}
$$

Repeat with $\mathrm{h}=0.05$

$$
\begin{array}{lrl}
\mathrm{X}_{1}=0.5 ; & 1 / 2 \mathrm{y}_{1}=0.3535 \\
\mathrm{X}_{2}=0.55 ; & \mathrm{y}_{2}=0.7416 \\
\mathrm{X}_{3}=0.6 ; & \mathrm{y}_{3}=0.7746 \\
\mathrm{X}_{4}=0.65 ; & \mathrm{y}_{4}=0.8062 \\
\mathrm{X}_{5}=0.7 ; & \mathrm{y}_{5}=0.8367 \\
\mathrm{X}_{6}=0.75 ; & \mathrm{y}_{6}=0.8660 \\
\mathrm{X}_{7}=0.80 ; & \mathrm{y}_{7}=0.8944 \\
\mathrm{X}_{8}=0.85 ; & \mathrm{y}_{8}=0.9220 \\
\mathrm{X}_{9}=0.9 ; & \mathrm{y}_{9}=0.9487 \\
\mathrm{X}_{10}=0.95 ; & \mathrm{y}_{10}=0.9747 \\
\mathrm{X}_{11}=1.0 ; & 1 / 2 \mathrm{y}_{11}=\underline{0.5000} \\
& \text { Sum }=8.6184
\end{array}
$$

$$
\int_{0.5}^{1} x^{\frac{1}{2}} d x=0.05 \times 8.6184=0.43092
$$

To three decimal places, the result is 0.431 which compares very well with the correct value of 0.43096 .

## Simpson's Rule

17. In the trapezoidal rule the curve $y=f(x)$ is approximated by a series of straight lines. It can, however, be approximated by any suitable curve and in the case of Simpson's rule, a parabola is used. Rather than joining pairs of points, a parabola is traced through three points on the line as shown in Fig 5.

## 13-15 Fig 5 Simpson's Rule - Fitting a Parabola through Three Points


18. The result for an integration interval divided into 2 parts with 3 ordinates is:

$$
\int_{x_{1}}^{x_{3}} \mathrm{ydx}=\frac{1}{3} \mathrm{~h}\left(\mathrm{y}_{1}+4 \mathrm{y}_{2}+\mathrm{y}_{3}\right)
$$

19. Example. Compute $\int_{0.5}^{1} \mathrm{x}^{\frac{1}{2}} \mathrm{dx}$ using Simpson's rule with $\mathrm{h}=0.25$.

$$
\begin{array}{ll}
\mathrm{x}_{1}=0.5 ; & \mathrm{y}_{1}=0.7071 \\
\mathrm{x}_{2}=0.75 ; & \mathrm{y}_{2}=0.8660 \\
\mathrm{x}_{3}=1.00 ; & \mathrm{y}_{3}=1.0000
\end{array}
$$

and the integral is given by:

$$
1 / 3 \times 0.25(0.7071+4 \times 0.8660+1.0000)=0.4309
$$

which is a better result than that given by the trapezoidal rule with $\mathrm{h}=0.10$.
20. Simpson's rule will usually give a more accurate result than the trapezoidal rule for the same interval, $h$, but it is often necessary to sub-divide the curve into more than one set of three ordinates. A different parabola is fitted over each section. For example, in Fig 6, seven ordinates are used, the parabolas are $P_{1}, P_{2}, P_{3}$ and the integral is given by:
$1 / 3 h\left(y_{1}+4 y_{2}+y_{3}\right)+1 / 3 h\left(y_{3}+4 y_{4}+y_{5}\right)+1 / 3 h\left(y_{5}+4 y_{6}+y_{7}\right)=1 / 3 h\left(y_{1}+4 y_{2}+2 y_{3}+4 y_{4}+2 y_{5}+4 y_{6}+y_{7}\right)$

The principle is capable of extension to any ODD number of ordinates.

13-15 Fig 6 Simpon's Rule - Seven Ordinates


## CHAPTER 16 - THE SCOPE OF STATISTICAL METHOD

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## Introduction

1. The word statistics is used in two distinct ways. It is used to mean either sets of figures, usually tabulated, in which sense it is short for statistical data, or to mean the methods whereby the significant details may be extracted from such sets of figures. In this sense, it is short for statistical method, and it is with this meaning that this section is concerned.
2. A good definition of the subject is: a body of methods for making wise decisions in the face of uncertainty. Defined in this way the subject may be regarded as an extension of the idea of common sense, which is the name each person gives to his own method of making wise decisions in everyday matters. The fact that the answers provided by common sense on the one hand and by statistical method on the other often seem to be poles apart, is attributable either to the inadequacies of common sense or to an incorrect use of statistical method.
3. As a broad generalization, it may be stated that statistics takes over from common sense where the complexity of the problem warrants it, and where the quantities involved can be expressed in the form of numbers. If these conditions are fulfilled then the use of statistical method will give an economy of effort and a precision which is unobtainable in any other way.
4. There are very few aspects of human activity in which uncertainty does not play a part, so that the potential uses of statistical method are very large in number. In general, practical problems have solutions which are more or less probable, and probability theory forms the basis of statistics. An understanding of at least the elementary ideas of probability is, therefore, a prerequisite for the understanding of statistical method.
5. It is important to notice that the word, "wise" appears in the definition of statistics, and not the word "right". That the latter word is inadmissible follows, of course, from the fact that we are accepting uncertainty as a basic ingredient of the problem. The wise decision that is made will be based on the most probable occurrence, but the most probable occurrence is not bound to occur. Our decisions, therefore, will sometimes turn out to be wrong, no matter how elegant the mathematics used in the solution of the problem, and this fact must be accepted.
6. Quite frequently, decisions will prove to be wrong because they were based on inadequate data, and the point must be made that statistical analysis does not bring anything out of the data that is not already there. Statistics provides an objective way of testing the data and of obtaining answers free from personal prejudice and preconceived notions. The use of statistical method, in other words, makes it possible to interpret correctly the influence of chance in the evidence available.
7. It is clearly important that any experiment or series of trials should be designed to provide relevant evidence in sufficient quantity to do what is required. The only safe way to ensure this is to
bring the statistician into the enquiry from the very beginning, so that he will not only analyse the data but in fact will also state what data should be collected. Far too often the statistician is called in too late, so that the investigator is faced with the invidious choice of either giving incomplete answers or of repeating all or part of the investigation in order to obtain adequate data. This procedure is likely to be far more costly in the long run than a properly planned attack on the problem in the first place. The best results are invariably obtained by the statistician and subject specialist working together from the beginning of the enquiry.

## The General Method of Statistics

8. Although there are many distinct statistical techniques, they all have certain features in common. The following remarks are of general application.
9. The Presentation of Data. The raw data of statistics consists of a more or less haphazard collection of numerical data. An important first step before any statistical analysis can be started is to tabulate the data in some way which is meaningful for the investigation in hand. Suitable tabulation will, in fact, probably suggest the profitable line of attack, a process which may be aided by a graphical presentation of the data.
10. Population. Any collection of data which is to be analysed by a statistical process represents a finite selection of values from a practically if not theoretically, infinite population. In this context, the word population is part of statistical jargon, and refers to the totality of values which it is possible to conceive within the restrictions laid down for the data. It is important to note that the population may be one of people, of measurements, of bombs dropped on a target under specified conditions, or of any data whatsoever that may be given numerical values.
11. Sample. The concept of population is always to some extent an abstraction, since it is not possible to obtain access to all members of it (if it were, statistical analysis would be unnecessary). The data that is available is a sample taken from the population being studied. A problem will always be posed in terms of populations and not of samples. The solution to the problem, however, must proceed through an analysis of samples. For this to be a valid procedure it is clearly necessary that the sample should be a fair representation of the population that is being studied. It may then be assumed that parameters calculated from the sample are reliable estimates of the corresponding parameters of the population, and that conclusions based on the sample will be valid for the population also.
12. Sampling. It will be clear from para 11 that the technique of sampling, whereby the sample to be used in the investigation is to be chosen, plays a vital part in statistical work. Every collection of data is a fair sample from some population, but the important thing is to ensure that the collection selected is a fair sample of a specified population. This is by no means an easy thing to ensure. Broadly, two different sampling techniques are used, the choice in a particular case being dependent upon the extent of one's knowledge of the system studied. If a great deal is known about the population it may be possible to select a sample which conforms to the same pattern as the population; this technique is used extensively in public opinion polls. If it is not possible to do this, or if one is uncertain about the completeness of one's knowledge, then a technique of random selection, in which every member of the population has an equal chance of being selected for the sample, will be used. This has the extreme merit, if done correctly, of removing unsuspected bias, which may arise in any subjective method of sampling. A common method of ensuring randomness in the sample is through the use of special tables of random numbers, which have been thoroughly tested and found free from bias.
13. Statistical Significance. It is not unusual in statistical work to find that when a random sample is taken with the aim of providing a hypothesis it turns out that the sample data does not wholly support that hypothesis. The difference could be due to:
a. The hypothesis being wrong, or
b. The sample being biased.

Clearly, tests are needed to determine which is the more likely possibility. Tests of significance are very important to statisticians but are outside the scope of the chapter.
14. Proof and Disproof. It is perhaps clear already from the foregoing discussion that statistical method never provides a definite proof of any hypothesis, though it may provide very strong evidence indeed in favour of it. Any process which involves extrapolation from sample to population, and usually from past to future time, must involve uncertainty, and no matter how improbable an event there is always the possibility that it will happen. The gibe that "you can prove anything by statistics" shows a complete misunderstanding of the methods of statistical inference. Nothing is "proved" or "disproved" by statistics.

## Some Uses of Statistics

15. The point has been made that the opportunities for the application of statistical method are virtually unlimited, so that any list of uses will necessarily be incomplete. The following selection of topics, however, gives a good idea of the great scope of statistical method.
16. The Measurement of the Inexplicable. Many problems are concerned with quantities which do not take unique values under the conditions which it is possible to specify. In such cases, residual uncertainties may be very important, and it is necessary to be able to assess their likely magnitude. Probability theory provides a way of doing this. We may distinguish two important types of problem:
a. Those in which the uncertainties give rise to a random departure from a true value or from a desired result. We are here concerned with the determination of errors.
b. Other cases, in which there is no true or desired value, but in which random variability is an essential feature of the system. Measurements of biological quantities and of human attainment come into this category.

The emphasis here is on the use of statistics to calculate meaningful parameters which may be used to describe the population. Some distribution of values is obtained, and, in practice, the essential features of this distribution are established by comparison with a similar theoretical distribution.
17. The Identification of Important Factors. The problem here is to determine which of several factors that might be expected to affect performance do in fact have a significant effect. In some cases, the important factors may be seen without the aid of statistical analysis. In other cases, because of interactions between factors, the issue may be by no means clear, and statistical techniques then provide a way of separating the effects of individual factors and thus establishing their relative importance.

## Some Misuses of Statistics

18. No discussion of the uses of statistics would be complete unless accompanied by a discussion of misuses, for abuse of statistical method occurs all too frequently. It is because of this that statistics are viewed with such suspicion by so many people.
19. In general we may lay down the principle that from a given set of data one set of conclusions relative to a particular issue will be more likely than any other set. But the fact is that other conclusions, incompatible with the first, will frequently be drawn, often because a certain conclusion is desired and sometimes, as in misleading advertising, because no other conclusion is acceptable. It is often difficult or impossible for a person not intimately engaged in an investigation to trace invalid reasoning, and hence the belief arises that a judicious use of statistics will enable conflicting conclusions to be drawn from the same set of data. Again, the possibilities are legion, but the following examples illustrate some of them.
20. Deceptive Presentation. Cases of presentation with intent to deceive are common features of everyday life, and very often take the form of the omission of relevant data. Thus, a poster supporting an anti-immunization campaign announced boldly that in a certain period of time 5,000 cases of diphtheria occurred among immunized children. The public was expected to infer that immunization failed, but the poster did not disclose the highly relevant information that in the same period 75,000 cases occurred among non-immunized children, nor that non-immunized children were 6 times as likely to get diphtheria and 30 times as likely to die from it.
21. Sampling Errors. The importance of sampling has already been emphasized, and it must be obvious that bad sampling will lead to invalid conclusions. For example, taking the announcement of births from the columns of the "Times" gave a sex ratio of 1,089 males per 1,000 females. However, the Registrar General found during the same period a ratio of 1,050 females per 1,000 males. The reason for the discrepancy is no doubt that the sample from the "Times" was not a fair sample of the whole population, perhaps because parents are more inclined to announce the births of their sons and heirs than of their daughters.
22. False Correlations. The technique of correlation is very easily misapplied, and correlation between two variables should not be sought unless there are reasons, stemming from knowledge of the system studied, to expect it. It is easy to think of variables which, though unrelated, will show strong correlation, often because both happen to vary in a certain way with the passage of time. Such correlations are called nonsense correlations.
23. Statistics versus Experience. It is the subject expert who will make the decision, making full use both of his own specialist knowledge and of the results of the statistical analysis. If a statistical inference runs counter to what the specialist expects, then he should query it. The inference may be correct, but a little probing will be most worthwhile.

## CHAPTER 17 - DESCRIPTIVE STATISTICS

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## Introduction

1. Statistics is concerned with the mathematical analysis of numerical data. The numerical data is in the form of a set of observations of the variable (or variables) under consideration. A variable (or variate) is a quantity which assumes different measurable values, e.g. height, weight, examination marks, temperature, intelligence, length of life, etc. Any set of observations of the variable is
considered, for statistical purpose, as a sample drawn from some infinitely large population. When each and every member of a population has an equal chance of being selected for a sample, the sample is called a random sample. The principle task of statistical analysis is to deduce the properties of the population from those of a random sample. In this chapter, we discuss how samples and populations can be described; in particular, we will look at averages and the bunching of the samples and population about these averages.

## AVERAGES

## Types of Average

2. The Arithmetic Mean. The arithmetic mean is commonly referred to as 'the average'. The mean is the sum of all the values of a variable divided by the number of variables. The algebraic form of expression is:

$$
\overline{\mathrm{X}}=\frac{\mathrm{X}_{1}+\mathrm{X}_{2}+\mathrm{X}_{\mathrm{N}}}{\mathrm{~N}}=\frac{\Sigma \mathrm{X}}{\mathrm{~N}}
$$

where: $\overline{\mathrm{X}} \quad=$ the mean
$X_{1} \ldots X_{N}=$ the values of the different variables in a distribution
$\mathrm{N} \quad=$ the number of variables
$\Sigma \quad=$ the symbol instructing the addition of all of the values
Example:
A sample consists of the data $5,8,9,6,12,14$.
The mean is $\overline{\mathrm{X}}=\frac{5+8+9+6+12+14}{6}=\frac{54}{6}=9$
3. The Median. If the sample observations are arranged in order from the smallest to the largest, the median is the middle observation. If there are two middle observations, as in the case of an even number of observations, the median is halfway between them.

Examples:
a. Given sample $1,14,9,6,12$. Arranged in order $1,6,9,12,14$, the median is 9 .
b. Given sample $20,7,11,10,13,17$. Arranged in order $7,10,11,13,17,20$, the median is 12 .
4. The Mode. The mode is the observation which occurs most frequently in a distribution. If each observation occurs the same number of times, there is no mode. If two or more observations occur the same number of times, and more frequently than any other observations, then the sample is said to be multi-modal.

## Examples:

a. Given sample $16,13,18,16,17,16$, the mode is 16 .
b. Given sample $4,7,4,9,3,7$, then the modes are 4 and 7 .
c. Given sample $3,7,12,11,16,20$ there is no mode.

The mode is seldom used but has been included for completeness.

## MEASURES OF DISPERSION

## Introduction

5. Knowledge of the average of a distribution provides no information about whether figures in a distribution are clustered together or well spread out. For example, two groups of students have examination marks of $64 \%, 66 \%, 70 \%$, and $80 \%$, for the first group and $37 \%, 61 \%, 88 \%$ and $94 \%$ for the second group. Both groups have a mean mark of $70 \%$ but the marks of the second group have a much greater dispersion than those of the first group. Clearly, it would be useful to have a way of measuring dispersion (or variance) and expressing it as a simple figure. The most commonly used measures are:
a. Range.
b. Quartile Deviation.
c. Standard Deviation.
d. CEP (used in particular applications).

## Range

6. Range is the difference between the highest and lowest values. Unfortunately, range is too much influenced by extreme values so that one value differing widely from the remainder in a group could give a distorted picture of the distribution. Range also fails to indicate the clustering of values into particular groups or areas.

## Quartile Deviation

7. Quartiles are the values of the items one quarter and three quarters of the way through a distribution. If the top and bottom quarters are cut off, extreme values are discarded and a major disadvantage of range as a measure of dispersion is avoided.

$$
\text { Quartile Deviation }=\frac{\text { Third Quartile - First Quartile }}{2}
$$

As with Range, this method fails to indicate clustering.

## Standard Deviation

8. Standard deviation is the most important of the measures of dispersion. The standard deviation $(\sigma)$ is found by adding the square of the deviations of the individual values from the mean of the distribution, dividing the sum by the number of items in the distribution, and then finding the square root of the quotient. (Scientific calculators include a facility for finding $\sigma$ and other statistical parameters from sets of data.)

$$
\sigma=\sqrt{\frac{\Sigma(\mathrm{x}-\overline{\mathrm{x}})^{2}}{\mathrm{~N}}}
$$

The more that values of individual items differ from the mean, the greater will be the square of these differences, giving rise to a large measure of dispersion. The main disadvantage of using standard deviation as a measure of dispersion therefore is that it can give disproportionate weight to extreme values because it squares the deviations eg a value twice as far from the mean as another is weighted by a factor of $4,\left(2^{2}\right)$. Nevertheless, standard deviation is the best and most useful measure of dispersion within a set of observations.

## Circular Error Probable (CEP)

9. A term commonly encountered in weapon effects planning is Circular Error Probable (CEP). The CEP is the radius of the circle, centered on the mean impact point, within which $50 \%$ of weapons fall. Strictly, CEP should only be used when the distribution of impacts is known to be circular, but this restriction is often ignored in practice. As a rough guide, the CEP is approximately 1.18 times the standard deviation of the linear errors in weapon impacts, although this convention is only justifiable if the errors in range and deflection are normally distributed (see paras 25-27).

## FREQUENCY DISTRIBUTIONS

## Introduction

10. It is very difficult to learn anything by examining unordered and unclassified data. Table 1 displays such raw data.

Table 1 Weekly Mileages Recorded by 60 Salesmen

| 504 | 592 | 671 | 498 | 601 | 532 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 623 | 548 | 467 | 487 | 399 | 482 |
| 507 | 477 | 501 | 562 | 555 | 642 |
| 477 | 522 | 627 | 556 | 622 | 521 |
| 429 | 491 | 497 | 510 | 603 | 547 |
| 535 | 517 | 612 | 491 | 432 | 508 |
| 577 | 444 | 556 | 639 | 444 | 723 |
| 562 | 685 | 432 | 642 | 562 | 662 |
| 688 | 492 | 486 | 467 | 474 | 433 |
| 417 | 512 | 563 | 612 | 375 | 578 |

Raw data is simply a list of data as received, in this case from sixty individual salesmen. Little of use can be learned from data presented in this form.

## Ungrouped Frequency Distribution

11. The first step in making the raw data more meaningful is to list the figures in order from the lowest mileage to the highest. At the same time, it may be convenient to annotate those figures that occur more than once with the frequency of occurrence. The result is the distribution shown in Table 2. The symbol for frequency is ' $f$. The sum of the frequencies ( $\sum f$ ) must equal the total number of items making up the raw data.

Table 2 Ungrouped Frequency Distribution

| Mileage | $\mathbf{f}$ | Mileage | $\mathbf{f}$ | Mileage | $\mathbf{f}$ | Mileage | $\mathbf{f}$ | Mileage | $\mathbf{f}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 375 | 1 | 482 | 1 | 508 | 1 | 555 | 1 | 622 | 1 |
| 399 | 1 | 486 | 1 | 510 | 1 | 556 | 2 | 623 | 1 |
| 417 | 1 | 487 | 1 | 512 | 1 | 562 | 3 | 627 | 1 |
| 429 | 1 | 491 | 1 | 517 | 1 | 563 | 1 | 639 | 1 |
| 432 | 2 | 492 | 1 | 521 | 1 | 577 | 1 | 642 | 2 |
| 433 | 1 | 497 | 1 | 522 | 1 | 578 | 1 | 662 | 1 |
| 444 | 2 | 498 | 1 | 532 | 1 | 592 | 1 | 671 | 1 |
| 467 | 2 | 501 | 1 | 535 | 2 | 601 | 1 | 685 | 1 |
| 474 | 1 | 504 | 1 | 547 | 1 | 603 | 1 | 688 | 1 |
| 477 | 2 | 507 | 1 | 548 | 1 | 612 | 2 | 723 | 1 |

## Grouped Frequency Distribution

12. Though the ungrouped frequency distribution is an improvement in the presentation, there are still too many figures for the mind to grasp the information effectively. More simplification is necessary in order to compress the data. This can be done by grouping the figures and showing the frequency of the group occurrence. The result is shown in Table 3.

Table 3 Grouped Frequency Distribution

| Mileage | f |
| :---: | :---: |
| 375 - under 425 | 3 |
| 425 - under 475 | 9 |
| 475 - under 525 | 18 |
| 525 - under 575 | 12 |
| 575 - under 625 | 9 |
| 625 - under 675 | 6 |
| 675 - under 725 | 3 |
| Total Records | 60 |

13. Effect of Grouping. As a result of grouping, it is possible to see from Table 3 that mileages cluster around 475-525. Although grouping highlights the pattern of a distribution it does lead to the loss of information about where in the group the 18 occurrences lie. The increased significance of the table has therefore been paid for but the cost is worthwhile. The loss of information also means that calculations made from grouped frequency distribution cannot be exact.

## Class Limits

14. The boundaries of a class are called the class limits. Care must be taken in deciding class limits to ensure that there is no overlapping of classes or gaps between them. For example if the class limits in table 3 had been 375-425 and 425-450 which group would a mileage of 425 have gone into? Likewise, if the class limits had been $375-424$ and $425-449$ a mileage of $4241 / 2$ would have no group to fit into.
15. Discrete and Continuous Data. In defining class limits, it should be remembered that discrete data increases in jumps. For instance, data relating to the number of children in families will be in whole units because $1 \frac{1}{2}$ and $21 / 4$ children are not possible. Continuous data, however, may include fractions.
16. Class Intervals. Class intervals define the width of a class. If the class intervals are equal, the distribution is said to be an equal class interval distribution.
17. Unequal Class Intervals. Some data is such that if equal class intervals were used a very few classes would contain all the occurrences whilst the majority would be empty. An example of this is the distribution of salaries using a class interval of $£ 1000$. In this situation, the class intervals should be arranged so that over-full classes are subdivided and near empty ones grouped together.
18. Choice of Classes. The construction of a grouped frequency distribution always involves a decision as to what classes to use. The following suggestions should be borne in mind:
a. Class intervals should be equal wherever possible.
b. Class intervals of 5,10 , or multiples of 10 are more convenient than, say, 7 or 11 .
c. Classes should be chosen so that occurrences within the classes tend to balance around the mid point.

## Construction of a Grouped Frequency Distribution

19. To construct a grouped frequency distribution directly from raw data the following steps should be taken:
a. Pick out the highest and lowest figures (375 and 723) and on the basis of these decide upon the list and the classes.
b. Take each figure in the raw data and insert a check mark (1) against the class into which it falls.
c. Total the check marks to find the frequency of each class (see Table 4).

Table 4 Direct Construction of Grouped Frequency Distribution

| Class | Check Marks | f |
| :---: | :---: | :---: |
| 375 - under 425 | 111 | 3 |
| 425 - under 475 | 11111111 | 9 |
| 475 - under 525 | $\begin{array}{lllll}1111 & 1111 & 1111 & 111\end{array}$ | 18 |
| 525 - under 575 | $\begin{array}{llll}1111 & 1111\end{array}$ | 12 |
| 575 - under 625 | 11111111 | 9 |
| 625 - under 675 | 11111 | 6 |
| 675 - under 725 | 111 | 3 |
|  | Total Records | 60 |

## GRAPHS OF OBSERVATIONS

## The Histogram

20. A Histogram is a graph of a frequency distribution. It is shown in Fig 1 and is constructed as follows:
a. The horizontal axis is a continuous scale running from one end of the distribution to the other. The axis should be labelled with the name of the variable and the unit of the measurement.
b. For each class in the distribution a vertical column is constructed with its base extending from one class limit to the other and its area proportional to the frequency of the class.

## 13-17 Fig 1 Histogram of Data from Table 3



## The Frequency Polygon

21. If the mid points of the tops of the blocks of a frequency histogram are joined by straight lines, the resulting figure is a frequency polygon as illustrated in Fig 2. The area enclosed by the horizontal axis, the polygon, and any two ordinates represents approximately the number of observations in the corresponding range. The total area enclosed by the polygon represents the total number of observations if the frequency scale is used, and unity if the probability scale is used.

## 13-17 Fig 2 Frequency Polygon



## The Frequency Curve

22. If the number of observations is greatly increased and the size of the class interval is correspondingly reduced, then the frequency histogram and the frequency polygon will tend to a smooth curve as in Fig 3. By increasing the number of observations indefinitely, the whole population instead of just a sample will have been analysed. Thus, the smooth curve obtained by this process represents the distribution of the complete population in the same way as the histogram or frequency polygon represents the distribution of the sample.

## 13-17 Fig 3 Frequency Curve


23. If the original sample is random, and large enough, then it is unlikely that the smooth curve will be very different from drawing a smooth curve through the mid points of the histogram blocks. Such a smooth curve is known as a frequency curve if the frequency scale is used or a probability curve if the probability scale is used. The area under the curve between any two ordinates represents, as accurately as any estimate can, the number of observations within the corresponding range, while the same area for a probability curve represents the probability that a single observation, taken at random from the complete population, will lie within the corresponding range. The latter idea is more useful since it applies to the whole population and is no longer confined to the sample.
24. Histograms frequently display a pattern in which there is a high column in the centre with decreasing columns spread symmetrically either side. If the class interval is small enough, the frequency curve looks like the cross section of a bell. This pattern occurs frequently in statistical work.

## The Normal Curve of Distribution

25. The task of handling statistical data can be simplified if a mathematical curve can be found approximating to that which would be produced by plotting the actual data on a graph. By substituting the mathematical graph for the real one, it is possible to make calculations to reveal facts about the distribution of the raw data which would otherwise have been difficult to determine. The normal curve of distribution satisfies this requirement. It has the following features:
a. It is symmetrical
b. It is bell shaped
c. Its mean lies at the peak of the curve
d. The two tails continuously approach, but never cross, the horizontal axis (x) (see Fig 4).

The formula for the curve can be ignored because any mathematical data relating to it can be found in mathematical tables.

## 13-17 Fig 4 Normal Curve of Distribution


26. Areas Below the Normal Curve of Distribution. The 'y' axis in Fig 4 is at the peak of the curve and passes through the mean value of the distribution. The total area beneath the curve is unity, representing the fact that a random variable is certain to lie between $+\infty$ and $-\infty$. If $1 \sigma$ lengths are now marked off on the x-axis from the mean value of the curve, the area enclosed by the curve and the $1 \sigma$ boundaries is $68.26 \%$ of the total area. Tables are available to give the areas lying under the curve between any two lines on the x-axis designated in terms of $\sigma$.
27. Areas and Frequencies. Areas under the normal distribution curve are proportional to frequencies. An area of $95 \%$ of the total area is equivalent to a frequency figure which indicates that $95 \%$ of all occurrences lie between the two $2 \sigma$ values. Similarly, $99.75 \%$ of all occurrences fall within the $3 \sigma$ values (see Fig 5). The height of the curve at a particular point has no practical relevance - areas under the curve must always be used to give the frequency of occurrence in a specified range of values.

## 13-17 Fig 5 Approximate Areas Beneath the Normal Curve



## CHAPTER 18 - ELEMENTARY THEORY OF PROBABILITY

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## Mathematical Definition of Probability

1. Suppose that a certain experiment can have just $n$ possible results, all of them equally likely (e.g. in drawing a card from a pack there are just 52 possible results and all of them are equally likely) and suppose that a certain event can occur as $m$ of these $n$ possible results (e.g. picking an ace can occur as 4 of the 52 possible results). Then the probability of the event occurring in any one experiment is defined as $\mathrm{m} / \mathrm{n}$.
2. Thus the probability of drawing an ace is $4 / 52$ or $1 / 13$. We may also say that the chances are 1 in 13 or that the odds are 12 to 1 against.
3. If the probability that an event will occur is $m / n$, then the probability that it will fail to occur is $(n-m) / n=1-m / n$. Thus if $p$ is the probability of success and $q$ the probability of failure, we have

$$
q=1-p \text { or } p=1-q \text { or } p+q=1
$$

4. It is certain that an event will either occur or fail to occur. The probability of either a success or a failure is $n / n=1$. Hence probability $=1$ denotes certainty and, similarly, probability $=0$ denotes impossibility.

## Interdependence of Events

5. 

a. Mutually Exclusive Events. Two events are mutually exclusive if the occurrence of one prevents the occurrence of the other. If a die is thrown, the occurrence of a 6 prevents the occurrence of a 5.
b. Independent Events. Two events are independent if the occurrence or non-occurrence of one has no effect on the probability of the occurrence of the other. If two dice are thrown, the result of one throw has no effect on the result of the other.
c. Dependent Events. Two events are dependent when the occurrence or non-occurrence of one event has some effect on the probability of occurrence of the other. If two cards are drawn from a pack, the probability that the second card is an ace is $3 / 51$ or $4 / 51$ depending on whether the first card was or was not an ace.

Notice that mutually exclusive events occur as alternative results of the same experiment whereas independent and dependent events occur as the simultaneous or consecutive results of different experiments.

## Calculation of Probabilities

6. Theorem I - Addition of Probabilities. If two events are mutually exclusive, then the probability of either one or the other event occurring is the sum of the probabilities of the individual events.

Proof. Let the probability of $E 1$ be $m 1 / n$ and let the probability of $E_{2}$ be $m_{2} / n$. Then out of $n$ equally likely events, $m_{1}$ are $E_{1}$ and $m_{2}$ are $E_{2}$, ie $m_{1}+m_{2}$ events out of $n$ are either $E_{1}$ or $E_{2}$. Hence the probability of either $E_{1}$ or $E_{2}$ occurring is:

$$
\frac{m_{1}+m_{2}}{n}=\frac{m_{1}}{n}+\frac{m_{2}}{n}
$$

## 7. Theorem II - Multiplication of Probabilities.

a. Independent Events. If two events are independent, then the probability of both one and the other happening is the product of the probabilities of the individual events.
b. Dependent Events. If two events are dependent, then the probability of both the first event and the second event happening is the product of the probability of the first event and the conditional probability of the second event on the assumption that the first event has happened.

Proof. Let the probability of $E_{1}$ be $m_{1} / n_{1}$ and let the probability of $E_{2}$ be $m_{2} / n_{2}$. If $E_{1}$ and $E_{2}$ are independent, then the probability of $E_{2}$ is independent of the success or failure of $E_{1}$, but if they are dependent then the probability $m_{2} / n_{2}$ is to be regarded as the conditional probability of $E_{2}$ on the assumption that $E_{1}$ has happened. Then the total number of possible results of the two experiments together is $n_{1} n_{2}$. Of these possible results, $E_{1}$ and $E_{2}$ can occur together in $m_{1} m_{2}$ ways. Hence probability of both $E_{1}$ and $E_{2}$ occurring is:

$$
\frac{\mathrm{m}_{1} \mathrm{~m}_{2}}{\mathrm{n}_{1} \mathrm{n}_{2}}=\frac{\mathrm{m}_{1}}{\mathrm{n}_{1}} \times \frac{\mathrm{m}_{2}}{\mathrm{n}_{2}}
$$

8. Conclusion both Theorem I and Theorem II extend to any number of events.

Example 1. A card is drawn from a pack. What is the probability of it being either an ace, the king of clubs or a red queen?

The probability of drawing an ace is $\frac{4}{52}$
The probability of drawing the king of clubs is $\frac{1}{52}$
The probability of drawing a red queen is $\frac{2}{52}$
Hence the required probability is:

$$
\frac{4}{52}+\frac{1}{52}+\frac{2}{52}=\frac{7}{52}
$$

Example 2. Two dice are thrown and a card is drawn from a pack. What is the probability that both dice will show sixes and that the card will be the ace of spades?

The probability of a six in one throw is $\frac{1}{6}$ and the probability of drawing the ace of spades is $\frac{1}{52}$ Hence, the required probability is:

$$
\frac{1}{6} \times \frac{1}{6} \times \frac{1}{52}=\frac{1}{1872}
$$

Example 3. Two dice are thrown. What is the probability of the total throw being 10? The possible successes are $(4,6),(5,5),(6,4)$. The result of one die is independent of the result of the other.

$$
\text { Probability of }(4,6) \text { is } \frac{1}{6} \times \frac{1}{6}=\frac{1}{36}
$$

Probability of $(5,5)$ is $\quad \frac{1}{6} \times \frac{1}{6}=\frac{1}{36}$

Probability of $(6,4)$ is $\frac{1}{6} \times \frac{1}{6}=\frac{1}{36}$

Any combination of the pair excludes every other combination. Hence the required probability is:

$$
\frac{1}{36}+\frac{1}{36}+\frac{1}{36}=\frac{1}{12}
$$

Example 4. Two cards are drawn from a pack. What is the probability that they will both be aces?

The probability that the first card is an ace is $\frac{4}{52}$ and the conditional probability that the second card shall also be an ace is $\frac{3}{51}$. Hence, the required probability is:

$$
\frac{1}{13} \times \frac{1}{17}=\frac{1}{221}
$$

Example 5. Five balls are drawn from a bag containing 6 white balls and 4 black balls. What is the probability that 3 white balls and 2 black balls are drawn?

Although the rules for combining probabilities are important, it sometimes pays to work from first principles, i.e. direct from the mathematical definition of probability as in this example.

5 balls can be selected from 10 in $\binom{10}{5}$ ways, that is in
$\frac{10 \cdot 9 \cdot 8 \cdot 7.6}{1 \cdot 2 \cdot 3 \cdot 4.5}=\frac{2 \cdot 9 \cdot 7 \cdot 2}{1}=2.9 \cdot 7.2$ ways

This is the total number of possible selections.

Three white balls can be selected from 6 in $\binom{6}{3}$ ways, i.e.
$\frac{6.5 .4}{1 \cdot 2.3}=5.4$ ways
2 black balls can be selected from 4 in $\binom{4}{2}$ ways, ie

$$
\frac{4.3}{1.2}=2.3 \text { ways }
$$

Hence, the number of ways in which 3 white balls and 2 black balls can be selected is 5.4.2.3, from which the required probability is:

$$
\frac{5 \cdot 4 \cdot 2 \cdot 3}{2 \cdot 9 \cdot 7 \cdot 2}=\frac{10}{21}
$$

## Limitations of Mathematical Definition

9. It is not always possible to use the mathematical definition of probability because the definition requires that the experiment should have a finite number of possible different results. In practice, there are often an infinite number of possible results (e.g. in dropping a bomb on a target). Moreover, the definition depends on the results being equally likely, which to a certain extent begs the question unless we can satisfy ourselves intuitively that the results are equally likely. Thus, when the number of possible results is infinite or when we cannot be sure that they are all equally likely, we cannot use the mathematical definition, and instead we use the following one.

## Statistical Definition of Probability

10. Suppose that a certain experiment is carried out $n$ times and that a certain event occurs as $m$ of the n results. Then the probability of the event occurring as the result of any one experiment is defined as:

$$
\mathrm{p}=\lim _{\mathrm{n} \rightarrow \infty} \frac{\mathrm{~m}}{\mathrm{n}}
$$

In practice, we cannot let $n \rightarrow \infty$, but instead we take $n$ to be as large as we conveniently can. The resulting value obtained for the probability is then the best available estimate in cases where the mathematical definition cannot be used.
11. It can be shown that there is little theoretical difference between the two definitions and that therefore, the theorems proved on the basis of the mathematical definition still hold for probabilities obtained statistically.

Example 6. The operational requirement for a guided weapon demands that the weapon should have a reliability of $90 \%$. If the weapon can be broken down into 120 functionally-tested components of equal complexity and reliability, determine the reliability demanded from each component.

The reliability of a weapon or a component is the probability that the weapon or component will be completely serviceable. In this case, the statistical definition of probability clearly applies. Now if $\mathrm{R}_{\mathrm{C}}$
is the reliability of a component, then the probability that 120 such components will all be serviceable together may be obtained using the Multiplication Theorem for independent events.
$0.90=\mathrm{R}_{\mathrm{C}}^{120}$, or $\mathrm{R}_{\mathrm{C}}=\sqrt[120]{0.90}=0.9991$

Thus, each component must have a reliability of 99.91 \%.

## Probability of At Least One Success

12. It is important to distinguish between:
a. The probability of at least one success.
b. The probability of one success only.

Suppose it is required to get a minimum of one hit on a target. It is not sufficient to calculate the probability of one hit only because this would exclude the possibility of 2,3 , or more hits which must also count as successes. The criterion of success is at least one hit.

Example 7. Three ballistic missiles are launched at a certain target. From the statistical analysis of performance trials of the missile, it is estimated that the probability of hitting the target with a single missile is $\frac{1}{6}$. Calculate the chance of scoring: a. One hit only; b. At least one hit.

For a: The chance of a hit with the first missile but not with the other two is:
$\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}=\frac{25}{216}$

The chance of a hit with the second missile but not with the other two is:
$\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}=\frac{25}{216}$

The chance of a hit with the third missile but not with the other two is:
$\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}=\frac{25}{216}$

Hence the chance of one hit only is:

$$
\frac{25}{216}+\frac{25}{216}+\frac{25}{216}=\frac{75}{216}
$$

For b: In the same way, the chance of two hits only is:
$\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times 3=\frac{15}{216}$
and the chance of three hits is:
$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}=\frac{1}{216}$

Hence the chance of at least one hit is:
$\frac{75}{216}+\frac{15}{216}+\frac{1}{216}=\frac{91}{216}$

Alternatively, since the chance of a hit with one missile is $\frac{1}{6}$, the chance of missing with one missile is $1-\frac{1}{6}=\frac{5}{6}$ Thus, the chance of missing with all three missiles is $\left(\frac{5}{6}\right)^{3}=\frac{125}{216}$, and the chance of failing to miss with all three missiles, that is the chance of at least one hit, is:
$1-\frac{125}{216}=\frac{91}{216}$

To generalize, let the chance of success in one attempt be p, then the chance of failure in one attempt is $(1-p)$, ie the chance of failure in $n$ attempts $=(1-p)^{n}$, and finally, the chance of at least one success in $n$ attempts is $1-(1-p)^{n}$.

Example 8. If the probability of obtaining a hit with a single missile is assessed as $\frac{1}{20}$, how many missiles must be launched to give a $75 \%$ chance of at least one hit?

If n is the number of missiles which must be launched we require that:

$$
1-\left(1-\frac{1}{20}\right)^{\mathrm{n}}=0.75
$$

$$
\text { ie }\left(\frac{19}{20}\right)^{\mathrm{n}}=0.25
$$

$$
\text { or } 0.95^{\mathrm{n}}=0.25
$$

Taking logarithms,
$n \log 0.95=\log 0.25$
$\therefore \mathrm{n}=\frac{\log 0.25}{\log 0.95}=27$

Example 9. The data given below refers to an interceptor fighter armed with 2 air-to-air guided weapons. Determine the overall effectiveness of the weapon system.

| Aircraft serviceability | 0.90 |
| :--- | :--- |
| Aircraft reliability in flight | 0.80 |
| Missile reliability | 0.70 |
| Missile lethality | 0.50 |

The probability that the aircraft will be both serviceable and reliable in flight, and therefore able to deliver the weapon is:
$0.90 \times 0.80=0.72$

The probability that a single weapon will inflict the required damage on the target is:
$0.70 \times 0.50=0.35$

Thus, the probability that at least one weapon will inflict the required damage is:
$1-(1-0.35)^{2}=0.58$
i.e. the overall effectiveness is: $0.72 \times 0.58=0.42$ or $42 \%$

## Reliability

13. The accurate assessment of the reliability of complex and expensive systems or equipment can be a vitally important factor in planning the purchase or deployment of resources. In order to quantify reliability for analysis, it is necessary to be able to attach a numerical value to it. Reliability is defined as the probability that an item will not fail during a given period of time. The probability of an item not failing is denoted by $p$. The probability of an item failing is denoted by $q$. It follows that $p+q=1$. It is important that when a figure for reliability is quoted the time period to which it relates should also be quoted. It should be noted that reliability is a probability and is therefore expressed as a fraction of 1 or a percentage.
14. Combination. The overall reliability ( R ) of a combination of components may be calculated on the assumption that the quantities $p$ and $q$ are independent of the other components in the system. This being so, probabilities must be combined by the multiplication rule to give the probability that all will occur together.

## a. Components in Series.

--- (1) --- (2) --- ..... --- (n) ---

The system will survive only if all the components survive.
$\therefore R=p_{1} \times p_{2} \times \ldots \ldots \ldots p_{n}$

## b. Components in Parallel.



The system will fail only if all the components fail.

$$
\begin{array}{ll}
\therefore & 1-R=q_{1} \times q_{2} \times \ldots \ldots . . q_{n} \\
\text { or } & R=1-\left(q_{1} \times q_{2} \times \ldots \ldots \ldots q_{n}\right) \\
\text { or } & R=1-\left(1-p_{1}\right)\left(1-p_{2}\right) \ldots \ldots\left(1-p_{n}\right)
\end{array}
$$

A parallel system may be referred to as a redundant system.

## c. Components may be combined partly in series and partly in parallel.



These systems may be reduced to a system of units in series by obtaining the overall reliability of each parallel branch. Thus in the above example, if R23 is the overall reliability of the parallel components 2 and 3
$R 23=1-q_{2} \times q_{3}$
and the system reduces to:
--- (I) --- (R23) --- (4)

$$
\text { whence, } \begin{aligned}
R & =p_{1} \times R 23 \times p_{4} \\
& =p_{1} \times p_{4} \times\left[1-\left(1-p_{2}\right)(1-p 3)\right]
\end{aligned}
$$

15. Where redundant components are used on a mutually exclusive basis either one is used or another, and in this case the addition rule for combining the probabilities must be used. For example, a navigation system consists of mode 1 (reliability $p_{1}$ ) and mode $2\left(p_{2}\right)$ which will only be switched in by a switch, of reliability $p_{3}$, if mode 1 fails. The system will survive only if mode 1 survives or if mode 1 fails and the switch operates and mode 2 survives. In terms of reliabilities, the switch and mode 2 are only called upon to survive if mode 1 fails.


The term $\left(1-p_{1}\right)\left(p_{3}\right)\left(p_{2}\right)$ is the reliability contribution of the standby equipment. The overall reliability of the system can be worked out by the series/parallel calculation, giving:

$$
R=1-\left(1-p_{1}\right)\left(1-p_{3} p_{2}\right)=p_{1}+p_{2} p_{3}-p_{1} p_{2} p_{3} .
$$

## CHAPTER 19 - THE NATURE OF HEAT

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## Temperature and Heat

1. Heat is a form of energy possessed by a body by virtue of its molecular agitation. The heat content of an object is not measured simply by its temperature; heat content is also a function of mass. One unit of heat is the kilocalorie (kcal), and may be defined as the quantity of heat required to raise the temperature of 1 kilogram of pure water from $14.5^{\circ} \mathrm{C}$ to $15.5^{\circ} \mathrm{C}$ at a pressure of 1 atmosphere. The temperature range needs to be specified, as the quantity of heat required to raise the temperature by $1^{\circ} \mathrm{C}$ depends slightly on which ${ }^{\circ} \mathrm{C}$ is chosen.
2. Because heat is a form of energy, it may equally, and perhaps more properly, be expressed in the SI unit of energy, the joule ( J ). Indeed, by international agreement the kilocalorie is now defined as 4186.8 joules.

## Specific Heat

3. Materials other than water require different quantities of heat to change their temperature by $1^{\circ} \mathrm{C}$. All materials may thus be attributed a value, known as the specific heat, which reflects this variation and is defined as the amount of heat required to raise the temperature of 1 kilogram of the substance by $1^{\circ} \mathrm{C}$. This may be expressed in the equation:
```
Q = mct joules
where Q = quantity of heat
m}=\mathrm{ mass of substance (kg)
c = specific heat ( }\mp@subsup{\textrm{J kg}}{}{-1}\mp@subsup{\textrm{O}}{}{-1}
t = change in temperature ( }\mp@subsup{}{}{\circ}\textrm{C}
```

4. The value of specific heat depends upon the external conditions under which the heat is applied. Two variables are normally taken into consideration, leading to two values, one at constant pressure and one at constant volume. In the case of solids and liquids which are generally heated at constant pressure, then only the constant pressure value is normally quoted, and in any case the difference between the two values is negligible for all normal purposes. However, in the case of gases the two specific heats are quite different.

## Change of State

5. When a material changes state from a solid to a liquid or from a liquid to a gas, or vice versa, then energy must either be added to the substance or be released from the substance. For example, in order to change 1 kg of ice into water, approximately $335 \times 10^{3} \mathrm{~J}$ of heat needs to be added. During the change of state the temperature does not rise, i.e. 1 kg of ice at $0^{\circ} \mathrm{C}$ changes to 1 kg of water at $0^{\circ} \mathrm{C}$. Conversely, to freeze 1 kg of water then approximately $335 \times 10^{3} \mathrm{~J}$ of heat needs to be removed. The heat which is required to change the state of a substance, without any temperature change, is known as latent heat. Where the change is between solid and liquid it is known as the latent heat of fusion; where the change is between liquid and gas it is known as latent heat of vaporization. In both cases, the values quoted refer to 1 kg of the substance at the normal melting and boiling points. (Heat energy which causes a change of temperature without giving rise to a change of state is defined as sensible heat.)
6. Supercooling. If a liquid is cooled slowly and is kept motionless, its temperature can be reduced to well below its normal freezing point. This is known as supercooling. A supercooled liquid is in an unstable state and any disturbance will cause some of the liquid to solidify, thereby releasing latent heat. The temperature of the supercooled liquid is raised to its freezing point by the release of this latent heat and the normal process of solidification takes place.

## Heat Transfer

7. Heat may be transferred from one place to another by three mechanisms:
a. Conduction. If one end of, say, a metal rod is heated, then the atoms and electrons at that end will acquire higher kinetic and potential energy than those in other parts of the rod. In random collisions, these energetic atoms and electrons will transfer energy to their neighbours which in turn will become more energetic, collide with their neighbours, and transfer some of their energy. Thus in this way the thermal energy which was applied at one end of the rod will be transferred along the rod. This process is known as conduction.
b. Convection. Convection is the transfer of heat from one place to another occasioned by the movement of the heated substance. Convection only occurs in liquids and gases.
c. Radiation. In radiation, the heat is transferred in the form of electromagnetic waves. The intervening medium plays no part in the process and indeed radiation can take place in a vacuum. Heat is radiated more efficiently by dull surfaces and dark colours than by polished ones and light colours; thus, the most efficient radiating surface is matt black. Correspondingly, radiated heat energy is most efficiently absorbed by matt black surfaces, and most efficiently reflected by light shiny ones.

## CHAPTER 20 - TEMPERATURE AND EXPANSION

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## The Concept of Temperature

1. Temperature may be defined as the degree of hotness of a body measured according to some fixed scale. The sense of touch readily distinguishes between hot and cold bodies and thus between higher and lower temperatures. Temperature may also be regarded in another way; if two bodies are placed in thermal contact, then heat will flow from the body at higher temperature to that at a lower temperature.
2. The direction in which the heat flows does not depend on the quantity of heat in either body. For example, the water in a tank may be at a lower temperature than a hot soldering iron, but owing to its greater volume it may contain a greater quantity of heat. If the soldering iron is immersed in the water, heat will pass from the iron to the water. Once the temperatures are equal no more heat will be transferred.

## Temperature Scales

3. The temperature of an object is measured by a thermometer which makes use of those properties of liquids, gases and other substances, which vary continuously with temperature and are independent of previous treatment. Common methods involve the measurement of the expansion of solids, liquids or gases as they are heated, (liquid-in-glass thermometers and bi-metallic strips), the measurement of gas pressure as a gas is heated under constant volume (constant volume gas thermometer), and the change in colour of emitted light as an object is heated (optical pyrometer).
4. All of these thermometers require to be calibrated according to a defined scale. Historically two fixed temperature points have been defined; zero degrees Celsius (originally centigrade), and 100 degrees Celsius, or their Fahrenheit equivalents of $32{ }^{\circ} \mathrm{F}$ and $212{ }^{\circ} \mathrm{F}$.
5. The lower point was defined by the temperature at which pure water and ice exist in thermal equilibrium; the upper point at which pure water and steam exist in thermal equilibrium. Both points were defined at a pressure of 1 atmosphere $\left(1.01325 \times 10^{5} \mathrm{Nm}^{-2}\right)$.
6. Temperature scales are now defined relative to the Kelvin scale. The temperature at which all molecular agitation due to heat energy ceases is defined as Absolute Zero. This corresponds to a temperature of approximately $-273^{\circ}$ on the Celsius scale and it is assigned a value of 0 K . The size of the degree Kelvin is identical to that of the degree Celsius, thus the conventional fixed points of $0^{\circ} \mathrm{C}$ and $100^{\circ} \mathrm{C}$ are 273 K and 373 K respectively. Conventionally, the degrees sign is omitted when referring to Kelvin temperatures.
7. Temperature Conversion. Conversion of temperature values between Celsius and Fahrenheit scales may be accomplished using the following two equations:

$$
\begin{aligned}
& \left(X^{\circ} \mathrm{C} \times \frac{9}{5}\right)+32=Y^{\circ} \mathrm{F} \\
& \frac{5}{9} \times\left(Y^{\circ} \mathrm{F}-32\right)=X^{\circ} \mathrm{C}
\end{aligned}
$$

## Temperature Measurement

8. Temperature is measured using a thermometer (or a pyrometer for high temperatures), various types of which are described in the following paragraphs.
9. Liquid-in-glass Thermometers. The liquid-in-glass thermometer is the simplest instrument for the measurement of temperature. It depends on the fact that as the temperature of a liquid changes, so the volume of that liquid expands or contracts. For most purposes, mercury is used as the liquid. However, as mercury solidifies at $-39^{\circ} \mathrm{C}$, there is a limit to the lower end of its useful range. Alcohol may be used for lower temperature measurement, but it is limited at the higher temperatures as it has a boiling point of $78^{\circ} \mathrm{C}$.
10. Bi-metallic Strip. The bi-metallic strip thermometer consists of two strips of dissimilar metals welded together, and usually formed into a helix. One end of the helix is fixed while the other is free to rotate. As the two metals have different coefficients of expansion, they will expand or contract at different rates as temperature changes. This will be manifested in the helix coiling and uncoiling in response to temperature changes. A pointer is attached to the free end of the helix and this moves over a graduated scale. This type of thermometer is frequently used for outside air temperature measurement. A non-coiled version is often used as the sensing element in thermostatic controls in which the bending of the bi-metallic strip makes or breaks an electrical circuit.
11. Pyrometers. Conventional thermometers are not suitable for the measurement of very high temperatures, such as those found in jet pipes. The instrument used for high temperature measurement is called a pyrometer and three types are described below:
a. Thermocouple. The principle of operation of the thermocouple is illustrated in Fig 1. A and B are junctions of dissimilar metals, $G$ is a sensitive galvanometer. If the temperature of the two junctions is different, a current will flow from the iron to the copper at the colder junction and from the copper to the iron at the hotter junction. The size of the current is measured by the galvanometer; a higher current indicates a greater temperature difference between the two junctions. The advantage of this system is that the cold junction and the galvanometer can be remote from the hot, sensing, junction. The main disadvantage of the system is that it has to be calibrated, both to relate the current to the temperature difference and to determine the cold junction temperature, so that actual, rather than relative, temperatures can be determined. The two metals used in the junctions can be varied to suit the temperature range to be measured. Thermocouples are typically used in the measurement of jet pipe temperatures (see also Volume 5, Chapter 26).

## 13-20 Fig 1 Thermocouple


b. Optical Radiation Pyrometer. The radiation pyrometer relies on the principle that materials change colour and brightness as they are heated. A simple type of optical pyrometer is used to measure the temperature of kilns and furnaces. The instrument has an electrically heated filament which is placed between the eye of the observer and the bright interior of the furnace. The current flowing through the filament is adjusted until the filament brightness matches that of the furnace interior. As with the thermocouple, calibration against known temperature sources enables the measured current to be related to temperature. For automatic use, in a more hostile environment such as an aero-engine, more sophisticated instruments are available. These use photo-voltaic cells and amplifiers to measure the emitted radiation which is then converted to a measured temperature.
c. Resistance Wire. The electrical resistance pyrometer relies on the fact that the electrical resistance of materials varies with temperature. The resistance of metals increases with temperature increases while the resistance of non-metals decreases with temperature increases. Over moderate temperature ranges the resistance change is proportional to temperature change.

## The Behaviour of Gases

12. It can be shown that, for a given amount of gas (say $n$ moles), the pressure ( p ), the volume $(\mathrm{V})$, and the temperature $(T)$, of the gas are related by the ideal gas equation:
$\mathrm{pV}=\mathrm{nRT}$, where R is the universal gas constant.
13. This equation incorporates two laws as follows:
a. Boyle's Law. This law asserts that if the temperature of a gas is kept constant then the product of the pressure and the volume remains constant as a given amount of gas is compressed or expanded:
i.e. $p V=$ constant.
b. Charles' Law. This law asserts that if the pressure of a gas is kept constant then the ratio of the volume to the temperature remains constant as a given amount of gas is heated or cooled:

$$
\text { i.e. } \frac{V}{T}=\text { constant }
$$

This behaviour of gases is used as the basis for the two types of gas thermometer: the constant volume and the constant pressure thermometers.
14. Isothermal and Adiabatic Changes. Fig 2 illustrates the two ways in which the pressure of a gas changes as the volume is changed. The difference depends upon whether the temperature changes concurrently. If a gas is compressed slowly such that there is time for the energy transferred to it to be dissipated through the container, then the temperature of the gas will remain constant, and the change of state is said to be isothermal. On the other hand if no energy transfer between the gas and the surroundings is permitted, then its temperature will rise and the change in state is termed adiabatic.

## 13-20 Fig 2 Isothermal and Adiabatic Changes



## Expansion of Solids and Liquids

15. Solids expand as they are heated. The thermal expansion of a solid is usually best described by the increase in linear dimension. The increment is directly proportional to the temperature change and to the original length. Thus:

$$
\Delta L=\alpha L \Delta T
$$

where $L=$ original length
$\Delta L=$ change of length
$\Delta T=$ change in temperature

The constant, $\alpha$, is called the coefficient of linear expansion. Its value for any material varies slightly with the initial temperature.
16. The volumetric expansion of a solid can be expressed in an analogous equation:

$$
\text { where } \begin{aligned}
\Delta V & =\beta V \Delta T \\
V & =\text { original volume } \\
\Delta V & =\text { change in volume } \\
\Delta T & =\text { change in temperature }
\end{aligned}
$$

and the constant, $\beta$, is the coefficient of volume expansion $(\beta=3 \alpha)$.
17. Most liquids expand when they are heated, so that their density reduces with increasing temperature. However, water exhibits a somewhat unusual variation. From $0^{\circ} \mathrm{C}$ to $4^{\circ} \mathrm{C}$ the volume decreases, non-linearly, as temperature increases. Above $4{ }^{\circ} \mathrm{C}$, the volume increases with temperature. Thus, water has its maximum density at $4^{\circ} \mathrm{C}$.

## CHAPTER 21 - THE NATURE OF LIGHT

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## Introduction

1. The true nature of light is a question which has taxed scientists for many generations. Most theories treat light as the transport of energy, either as a wave motion or as a stream of particles. Each of these ideas can be used to explain certain phenomena associated with light but neither can satisfactorily explain them all. For example the wave theory can explain why crossed light beams do not scatter each other, whereas the particle concept would not allow this to happen. Conversely the wave theory cannot be used to explain the photoelectric effect - only the particle theory is satisfactory. Neither idea really defines what light actually is; rather each is a model which can be used to describe and predict the behaviour of light under a particular set of circumstances. Thus it is necessary to choose the appropriate model for the task in hand. In general, where the behaviour of light in motion is being studied, the wave model is more useful, while the particle model is to be preferred when studying the interaction of light with other matter, eg absorption and emission.

## The Wave Model

2. When a pebble is tossed into a pond it sets into motion the water particles with which it comes into contact. These particles set neighbouring particles in motion and so on until the disturbance reaches the edge of the pond. In fact any particular particle only oscillates vertically about a mean position. It does not itself move to the edge of the pond; only the disturbance moves through the water. This is a typical characteristic of wave motion. The oscillations are at right angles to the direction of propagation of the wave, and such a wave motion is known as a transverse wave. Light waves are also transverse waves, but rather than representing the motion of particles in a medium, the wave represents the variations in electric and magnetic field strength. Neither of these fields can exist without the other and they are interdependent. The fields are mutually perpendicular and both are at right angles to the direction of wave motion. This arrangement is illustrated in Fig 1. Unlike water waves, which are constrained to move along the water surface, light waves can propagate in any direction and are not dependent upon the medium; indeed, they can move through a vacuum.

## 13-21 Fig 1 Light Wave


3. The waves generated by tossing a pebble into a pond are small compared with, for example, sea waves. Also whereas the distance between the crests of the pond waves may be only a few centimetres, with sea waves the separation may be several metres. Similar variations occur in light waves and these parameters are summarized in Fig. 2, where it will be seen that light waves have a sinusoidal form. The distance between the subsequent crests of a wave (or between any other corresponding points) is known as the wavelength and is usually given the symbol $\lambda$. The vertical size of the wave is measured from the mean level and is known as the amplitude. The time taken for
corresponding points on a wave to pass a fixed point is known as the period, and the number of corresponding points passing in unit time is the frequency (f). The wave travels at a speed (C), which depends upon the medium through which it is passing. In free space the speed of light is approximately $3 \times 10^{8}$ metres per second ( $\mathrm{m} / \mathrm{s}$ or $\mathrm{ms}^{-1}$ ) or 186,000 miles per second. From this, it will be seen that speed, frequency, and wavelength are related by the equation: $\mathrm{C}=\mathrm{f} \lambda$.

## 13-21 Fig 2 Wave Parameters



## Polarization

4. In general the electric and magnetic fields of a light wave are free to vibrate in any of the infinite planes at right angles to the direction of propagation. Any ordinary light source consists of the superposition of a number of plane waves each with a random plane of vibration. Such a light beam is said to be unpolarized. If, however, the electric field is constrained to lie in a particular plane then the light is said to be polarized. Polarization can be achieved by passing the light through a suitable filter.

## The Electromagnetic Spectrum

5. Light waves can be shown to have essentially the same characteristics as many other types of radiation such as radio waves, microwaves and X-rays. Indeed all of these are examples of electomagnetic radiation, differing primarily in their frequencies. The electromagnetic spectrum describes a large range of such waves, illustrated in Fig 3, in which visible light occupies only a very small band. White light consists of a graded spectrum containing the colours Red, Orange, Yellow, Green, Blue, Indigo and Violet.

## 13-21 Fig 3 The Electromagnetic Spectrum



## Wave-fronts and Rays

6. From a point source, light is propagated in all directions. Travelling out from the source, wave maxima will be encountered at regular intervals, corresponding to the wavelength. Similarly at different positions, but still with the same spacing, minima will occur and the same will apply for any intermediate value of amplitude. Lines can be drawn joining points of equal amplitude, analogous to contour lines, and these are known as wave-fronts. As the radiation occurs in three dimensions these wave-fronts are in fact spherical surfaces, but it is often satisfactory to treat the radiation as if it were planar in which case the wave-fronts reduce to circles. As a further practical simplification, if the wave-front is at a large distance from the source, and if only a small sector is investigated, then the wave-front may be approximated by a straight line.
7. The direction of propagation of light is at $90^{\circ}$ to the wave-fronts and a line representing this direction is known as a ray. Rays are often used in diagrams where only the direction of propagation is of concern. Wave-fronts and rays are illustrated in Fig 4.

## 13-21 Fig 4 Wave-fronts and Rays



## Superposition and Interference

8. When two or more sinusoidal waves act at the same place then the result can be found by adding algebraically the individual amplitudes. Four examples of this principle of superposition are shown in Figs 5, 6, 7 and 8 . In Fig 5, the two waves have the same amplitude and frequency and are in phase, ie the maxima and minima of each wave occur at the same time and place. In this case the waves reinforce each other, thus resulting in a doubling of the amplitude. This is known as constructive interference. In Fig 6, the same two waves are completely out of phase. At every point the amplitudes have the same magnitude but the opposite sign, and so cancel each other - a situation known as destructive interference. In Fig 7, three waves having the same frequency but different phases add to produce another sinusoid with the same frequency. Fig 8 shows the general case where two waves have different amplitudes and frequencies.

13-21 Fig 5 Two Sinusoidal Waves, Same Amplitude, Frequency and Phase



13-21 Fig 6 Two Sinusoidal Waves, Same Amplitude and Frequency, $180^{\circ}$ Out of Phase


## 13-21 Fig 7 Three Sinusoidal Waves, Same Amplitude, and Frequency, Different Phases



13-21 Fig 8 Two Sinusoidal Waves, Different Amplitude and Frequency

9. If two close sources radiate light of the same frequency and if the two radiations are in phase then, as shown in Fig 9, at some points there will be constructive interference whilst at others there will be destructive interference. The result is a radially symmetrical interference pattern.

## 13-21 Fig 9 Constructive and Destructive Interference


10. It is a characteristic feature of waves that they will deflect around the edges of obstacles placed in their path and spread into the shadow zone behind the obstacle. This effect can often be seen where for example water waves pass through a narrow harbour entrance or impinge upon a breakwater. There is no complete 'shadow' behind the wall, rather the waves appear to bend around the obstacle. This phenomenon is known as diffraction. In general the amount of diffraction is related to the wavelength and the size of the gap through which the waves must pass. The deviations of the wave are quite large when the size of the obstacle or gap is of the same order as the wavelength. Light, behaving as a wave form, will also experience diffraction but as the wavelength of light is very small the effect is only pronounced when the obstacle or gap is very small.
11. The effect of diffraction is closely associated with the ideas of constructive and destructive interference developed in paragraphs 7 to 9 . An insight into the effect can be gained by studying the passage of light through a pair of closely spaced narrow slits.
12. Fig 10 shows plane waves arriving at a screen in which there are two narrow slits which are perpendicular to the page. The two slits act as sources of light and therefore two series of circular wavefronts can be constructed, one set from each slit. In the diagram the wave-fronts represent the crests of the waves. As the slits were illuminated by the same incident light the light leaving each slit has the same wavelength, amplitude and phase. The principle of superposition can be used to predict the effects which will be observed. At point $P$ crests of waves from each slit arrive simultaneously and therefore constructive interference takes place and a reinforcement of amplitude will be seen. The same argument can be applied to any point where crests intersect. Such points have been joined by bold lines in the diagram and strong waves would be expected to be seen radiating along these lines. If the distance between point $P$ and each slit is measured it will be seen to be different by one wavelength (1 $\lambda$ ). The same is true for any point along the bold line through $P$. Other lines of constructive interference will each demonstrate a different value for the difference in distance and these values have been indicated in the diagram. In each case, the value is an integral multiple of a wavelength ( $n \lambda$ ).

## 13-21 Fig 10 Diffraction by Two Slits


13. Point $Q$ corresponds to the point of intersection of a crest from one slit and a trough from the other and therefore destructive interference will occur. The dashed lines represent the lines along which destructive interference occurs. In this case, the difference in distance from any point on a dashed line to each slit is an integral multiple of half a wavelength ( $n \lambda / 2$ ).
14. Points like $P$ and $Q$ represent the extremes of complete constructive interference and complete destructive interference. In between these points will be points where the interference is partially constructive, ie the resultant amplitude is greater then zero but less than twice the amplitude of each wave.
15. Consider now Fig 11a, in which point $R$ is a large distance from an opaque screen in which there are two slits $S_{1}$ and $S 2$. There is a small angle between the lines joining the slits to point $R$, however, if the point is sufficiently far away then this angle becomes so small that it may be safely ignored and the lines can be considered to be parallel. This situation is reflected in Fig 11b which is a magnification of the area close to the screen in which the lines joining the slits to R are drawn parallel. The lines are inclined at an angle $\theta$ to lines normal to the screen.

13-21 Fig 11 Determination of Angles for Constructive and Destructive Interference

16. A line $S_{2} M$ is drawn which is perpendicular to the lines from each slit to $R$. Point $M$ and the slit $S_{2}$ are the same distance from $R$ and the difference between distances $I_{1}$ and $I_{2}$ is equal to the distance between slit $S_{1}$ and $M$.
17. It has been shown in paragraphs 11 and 12 that the factor determining whether constructive or destructive interference occurs at $R$ is the difference between the distances $I_{1}$ and $I_{2}$. Thus, it is now necessary to relate the distance $\mathrm{S}_{1} \mathrm{M}$ to the angle $\theta$.

|  | angle $\theta+$ angle $\alpha=90^{\circ}$ |
| ---: | :--- |
| and | angle $\beta+$ angle $\alpha=90^{\circ}$ |
| Thus, | angle $\beta=$ angle $\theta$ |
| Now, | $\sin \beta=S_{1} M / d$ |
| Thus, | $\sin \theta=S_{1} M / d$ |
| So, | $S_{1} M=d \sin \theta$ |

For constructive interference to occur the difference between the distances $I_{1}$ and $I_{2}$ must be equal to an integral number of wavelengths so:

$$
I_{1}-I_{2}=S_{1} M=d \sin \theta=n \lambda
$$

or, $\sin \theta=n \lambda / d$, where $n$ is any integer.

This equation determines the angles at which constructive interference occurs in terms of wavelength and slit spacing. It can be shown that the equivalent equation for destructive interference is:

$$
\sin \theta=(n+1 / 2) \lambda / d
$$

18. As the number of equally spaced slits in the screen (and hence interference sources) is increased, the destructive interference regions widen, as shown in Fig 12. A screen with many equally spaced slits is known as a diffraction grating.

## 13-21 Fig 12 Diffraction Patterns


c 100 Slits


## Huygens' Principle

19. A knowledge of the manner in which a wave-front propagates is necessary in order to explain the phenomena of reflection and refraction. In 1690 the Dutch physicist Huygens proposed the following method:

To find the change of position of a wave-front in a small interval, t , draw many small spheres of radius [wave speed] $\times t$ with centres on the old wave-front. The new wave-front is the surface of tangency to those spheres.

It should be remembered that this is only a model which enables predictions to be made, it is not meant to be a description of reality. The small spheres employed in this construction are known as wavelets. Clearly most diagrams are constrained to showing phenomena in two dimensions only in which case the spherical wavelets reduce to circles. Fig 13 shows how Huygens' principle is used to predict the new position of a planar wave-front after a short interval.

## 13-21 Fig 13 Huygens' Principle of Wave-front Propagation



## Reflection

20. The case of a plane wave incident on a plane mirror is shown in Fig 14. Fig 14a shows the wavefronts approaching a reflecting surface. One edge of the leading wave-front is just touching the surface at point $P$. The situation a short time later is shown in Fig 14b where some Huygens' wavelets have been constructed (the portion of the wavelets below the surface have been omitted as irrelevant). The new wave-front touches the surface at point $P^{\prime}$. To the right of $P^{\prime}$ the new wave-front is parallel to the old wave-front as it has not yet been reflected. In order to find the position of the new wave-front to the left of $\mathrm{P}^{\prime}$, a straight line is drawn starting from $\mathrm{P}^{\prime}$, and tangential to the wavelet centred on P (Huygens' Principle). This straight line represents the part of the wave that has been reflected. The two right-angled triangles are congruent as they have a common side, $\mathrm{PP}^{\prime}$, and the short sides, PQ and $\mathrm{P}^{\prime} \mathrm{Q}^{\prime}$ are equal (being radii of the wavelets). Thus the angles $\theta$ and $\theta^{\prime}$ are equal. In studying reflection it is usually more convenient to deal with rays rather than with wave-fronts, since they more readily show the direction of propagation.

## 13-21 Fig 14 Reflection Using Huygens' Principle



Fig 15 shows the reflection of the rays corresponding to the wave-fronts of Fig 14 c .

## 13-21 Fig 15 Reflection of a Ray



In summary there are two Laws of Reflection as follows:
a. The angle of incidence equals the angle of reflection.
b. The incident and reflected rays and the normal to the reflecting surface lie in the same plane.

## Refraction

21. Huygens' Principle can also be used to predict the behaviour of light when it is transmitted at a boundary between two mediums rather than being reflected. Consider Fig 16. AB represents a wavefront arriving at such an interface at an angle $\theta_{i}$, the point $A$ arriving before point $B$. To find the position of the wave-front after a short time interval, $t$, Huygens' wavelets are constructed emanating from points $A$ and $B$. From point $B$, which can be assumed to be in air, the light travels at velocity $v_{1}$, and a wavelet can be drawn representing the time taken for the light to travel from $B$ to $B^{\prime}$. In the same time interval the light from $A$ is travelling in the transparent medium (glass say) at a slower velocity $\left(v_{2}\right)$. Thus the same time interval will correspond to a smaller wavelet. The new wave-front is now drawn from point $B^{\prime}$ to be tangential to the wavelet centred on $A$. Thus the light wave has been deviated as it changes from one medium to another in which the velocity of light is different. This phenomenon is known as refraction.

## 13-21 Fig 16 Refraction Using Huygens' Principle


22. From Fig 16:
and $\quad \begin{aligned} \mathrm{BB}^{\prime} & =\mathrm{v}_{1} \mathrm{t} \\ \mathrm{AA}^{\prime} & =\mathrm{v}_{2} \mathrm{t}\end{aligned}$
Thus, $\quad \frac{\mathrm{BB}^{\prime}}{\mathrm{AB}^{\prime}}=\frac{\mathrm{v}_{1}}{\mathrm{v}_{2}}$
$\sin \theta_{i}=\frac{\mathrm{BB}^{\prime}}{\mathrm{AB}^{\prime}}$
and $\sin \theta_{r}=\frac{\mathrm{AA}^{\prime}}{\mathrm{AB}^{\prime}}$
Hence, $\frac{\sin \theta_{\mathrm{i}}}{\sin \theta_{\mathrm{r}}}=\frac{\mathrm{BB}^{\prime}}{\mathrm{AA}^{\prime}}$
Noting that $\theta_{\mathrm{i}}=\mathrm{i}$ and $\theta_{\mathrm{r}}=\mathrm{r}$ and comparing with equation (1):

$$
\begin{equation*}
\frac{\sin \mathrm{i}}{\sin \mathrm{r}}=\frac{\mathrm{v}_{1}}{\mathrm{v}_{2}} \tag{2}
\end{equation*}
$$

23. It is normal to express the velocities of light in the two media as fractions of $c$, the velocity of light in a vacuum. Hence, $v_{1}=c / n_{1}$, and, $v_{2}=c / n_{2}$. The numbers $n_{1}$ and $n_{2}$ are known as the refractive indices of medium 1 and medium 2 respectively, and equation (2) may be rewritten as:

$$
\begin{equation*}
\frac{\sin \mathrm{i}}{\sin \mathrm{r}}=\frac{\mathrm{n}_{2}}{\mathrm{n}_{1}} \tag{3}
\end{equation*}
$$

If the incident wave is travelling in a vacuum, (or, for nearly all practical purposes, in air), then $n_{1}=1$ and equation (3) reduces to:

$$
\frac{\sin \mathrm{i}}{\sin \mathrm{r}}=\mathrm{n}
$$

where n is the refractive index of the second medium. This relationship between the angle of incidence, angle of refraction and the refractive indices of the media is known as Snell's Law of refraction. When a wave passes from a medium of high velocity to one of lower velocity then it is refracted towards the normal and conversely when passing from a 'slower' medium to a 'faster' medium it is refracted away from the normal. As with reflection the two rays and the normal to the interface all lie in the same plane. It is also evident that there is a corresponding change in wavelength.

## Total Internal Reflection

24. Consider rays of light travelling from a slower velocity medium, such as water to a faster velocity medium (say air) as illustrated in Fig 17. Ray 1, emitted normal to the surface of the water continues into the air normal to the surface whereas ray 2 , at an angle of incidence $i$, is refracted through the angle $r$.

## 13-21 Fig 17 Internal Reflection



As $i$ is increased, $r$ eventually becomes $90^{\circ}$ (ray 3). The value of the angle of incidence obtained in this case is called the critical angle. The change from refraction to reflection is not sudden, for some reflection always takes place at the surface of separation when $i$ is less than critical. However, when $i$ is greater than the critical angle all of the incident light is reflected at the boundary. This behaviour is known as total internal reflection. The phenomenon has a number of practical applications, perhaps the most important of which has been the development of the optical fibre in which the light is transmitted along the fibre experiencing total internal reflection when it impinges upon the fibre walls. Since a refracted ray cannot be deviated more than $90^{\circ}$ from the normal to the interface, when $r=90^{\circ}$, $\sin i$ has its maximum possible value and, as $\sin r=1$ :

$$
\sin i=\frac{v_{1}}{v_{2}}
$$

## Dispersion

25. Unlike reflection, refraction is frequency dependent. This is because the velocity of a wave in a medium changes as the frequency of the wave changes. Thus the 'bending power' of a given material is dependent upon the frequency; an effect known as dispersion. Thus for example if a ray of light containing a mix of frequencies is refracted by a medium then each of the component frequencies, or colours, will emerge at a different angle. Traditionally a prism has been used to demonstrate this effect and to analyse the component frequencies of a light source. Fig 18 illustrates the arrangement. The incident light ray experiences refraction at each glass/air interface with those components at the red end of the spectrum experiencing less refraction than those at the violet end.

13-21 Fig 18 Dispersion of Light by a Prism


## The Particle Model - The Photoelectric Effect

26. Although the wave model has provided a good description of many of the ways in which light behaves, it fails to explain the photoelectric effect.
27. The photoelectric effect involves the conversion of light into electricity and is used in solar cells and photographic light meters for example. A simple device to illustrate the effect is shown in Fig 19. Light from a lamp illuminates a metal electrode enclosed in an evacuated tube. Electrons are ejected from this electrode, travel to the collecting electrode (A) and then flow around the circuit in which an ammeter can measure the current. The kinetic energy of the ejected electrons can be determined by applying a potential difference between the emitting and collecting electrodes using an adjustable source. With the polarity as shown the collector exerts a repulsive force on the electrons. A potential can therefore be applied which will just stop the flow of electrons from the emitter to the collector. This potential is known as the stopping voltage.

## 13-21 Fig 19 Apparatus for the Investigation of the Photoelectric Effect


28. It can be shown experimentally that the kinetic energy of the electrons increases linearly with the frequency of the incident light as shown in Fig 20. Below a certain frequency, the light is incapable of ejecting electrons; this frequency, $\mathrm{f}_{1}$, in Fig 20 is called the threshold frequency.

## 13-21 Fig 20 Variation of Kinetic Energy with Frequency


29. The energy of waves depends upon intensity and not frequency and, therefore, the wave model would predict that the kinetic energy of the ejected electrons would increase with increasing intensity. Thus, the wave model is at variance with the experimental result.
30. The particle model assumes that monochromatic light of frequency $f$ comprises identical particles each carrying energy hf where h is a constant known as Planck's constant. The particles of light (known as photons) collide with electrons in the metal and the energy hf carried by a photon is transferred to an electron. When the frequency of the light is below the threshold frequency, the energy carried by the photon is insufficient to free even the most weakly bound electrons from the metal and all of the photon's energy is converted into heat. Once the frequency exceeds the threshold frequency then the energy is sufficient to free electrons from the metal and also to give the electrons some kinetic energy. The higher the frequency, the higher the photon energy, and thus the higher will be the ejected electron's kinetic energy.

## The Measurement of Light

31. Visible light is part of the electromagnetic spectrum, just like radar and X-rays (see Fig 3). As with all electromagnetic radiation, it can be measured in terms of both frequency $\left(4.3 \times 10^{14} \mathrm{~Hz}\right.$ to $\left.7.5 \times 10^{14} \mathrm{~Hz}\right)$ and wavelength $\left(0.4 \times 10^{-6} \mathrm{~Hz}\right.$ to $\left.0.7 \times 10^{-6} \mathrm{~Hz}\right)$. However, valid though these measurements are, they do not convey information about how light is perceived by the human eye. The concepts of power, brightness, and illumination need to be considered.
32. It is beyond the scope of AP 3456 to discuss photometrics (the study and measurement of visible light) in great detail. The definitions given in the following paragraphs represent a simplified approach to light which should suffice for aircrew purposes.
33. The 'power' of traditional electric filament light bulbs was usually stated in watts, the most common types being 60 watts and 100 watts. However, the watt rating only refers to how much power the bulbs consume, not how much light they give out. Over $80 \%$ of the energy consumed is used to heat the filament to make it glow and emit light. Modern compact fluorescent (CF) bulbs generate light by 'gas discharge' and are much more energy efficient. For the same light output, CF bulbs consume about a quarter of the energy of filament bulbs. Typical light output for a conventional 100 watt bulb is about 1750 lumens. This same output can be generated by a CF bulb rated at 27 watts.
34. The 'lumen' is the SI unit of light flow or 'luminous flux' and is the most common measurement of light output. The relationship between lumens, lux and candela is shown in Fig 21. The candela is the SI unit of luminous intensity and, in very simple terms, is the amount of light generated by a 'standard' candle. A typical 100 watt incandescent filament light bulb has a luminous intensity of about 120 candelas. Light level, illumination or illuminance is measured in lux (or millilux, where 1 lux $=1000$ millilux). In simple terms, 1 lux is the light level obtained when a candle is held one meter from a subject in a darkened room. If the candle is held one foot away from the subject, the light level obtained is obviously higher (this is the old imperial measure of illuminance, known as one 'footcandle', which is approximately equal to 10 lux). To light a surface of one square meter evenly at 1 lux requires 1 lumen of total light, i.e. 1 lux $=1$ lumen $/ \mathrm{m}^{2}$.

## 13-21 Fig 21 Illustration of Common Lighting Measurements


35. Outside on a clear summer day, in the UK, the light level is about 10,000 lux. Outdoor light levels for different conditions are shown in Table 1.

Table 1 Typical Outdoor Light Levels

| Condition | Illumination (lux) |
| :--- | :--- |
| Full daylight | 10,000 |
| Overcast day | 1,000 |
| Very dark day | 100 |
| Twilight | 10 |
| Full Moon | 0.1 |
| Starlight | 0.001 (1 millilux) |
| Overcast night | 0.0001 |

36. In buildings, light levels are considerably reduced and artificial lighting may be required depending on the tasks being undertaken. Table 2 shows recommended light levels for various indoor situations.

Table 2 Recommended Indoor Light Levels

| Location/Activity | Illumination (lux) |
| :--- | :---: |
| Warehouses, Homes, Theatres | 150 |
| Normal Office Work, Library, Showrooms, Laboratories | 500 |
| Supermarkets, Mechanical Workshops | 750 |
| Operating Theatres, Normal Drawing Work | 1,000 |
| Detailed Drawing Work, Very Detailed Mechanical Work | $1500-2000$ |

## CHAPTER 22 - MIRRORS AND LENSES

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## Introduction

1. This chapter will deal with the formation of images by mirrors and lenses. Images will be described as upright or inverted, magnified or diminished, real or virtual, and sometimes reversed. Whereas most of these terms are straightforward, the terms 'real' and 'virtual' need some explanation.
2. If rays of light coming from a point (the object) are caused to converge to a second point, the second point is called the image and is a real image. If, however, rays of light coming from a point are made to appear to diverge from a second point, the second point is a virtual image. It will become apparent that an image formed by a mirror will be real if object and image are on the same side of the mirror, whereas with a lens the image is real if it occurs on the opposite side of the lens to the object. Further differences are that a real image can be projected on to a screen whereas a virtual image cannot, and real images are inverted whilst virtual images are upright.

## MIRRORS

## Plane Mirrors

3. Fig 1 shows a straight-line object, $A B$, being reflected in a plane mirror $M M^{\prime}$. The image of $A B$ can be determined by considering its end points. Two incident rays are drawn from each point to the mirror and produced according to the laws of reflection. Both pairs of reflected rays are then extended behind themirror. Thus, the reflected pair of rays coming from $A$ converge at $A^{\prime}$ and the pair from $B$ converge at $B^{\prime}$. If the points $A^{\prime}$ and $B^{\prime}$ are joined by a straight line then the complete image of $A B$ is produced.

## 13-22 Fig 1 Reflection of an Object in a Plane Mirror


4. Fig 2 shows the paths of the rays of light required for an eye to see the image in a mirror. Measurement will show that the image is as far behind the mirror as the object is in front and that the size of the image is the same as that of the object. The image is reversed and virtual. Everyday experience in the use of plane mirrors will confirm that the image is upright.

13-22 Fig 2 Viewing an Image in a Plane Mirror


## Curved Mirrors

5. The commonest types of curved mirrors are those consisting of a portion of the surface of a sphere; they may be either concave or convex. The centre of the sphere, of which the curved mirror is a part, is called the centre of curvature, and its radius is called the radius of curvature. A line drawn from the centre of curvature to the centre of the mirror surface is called the principal axis. These terms are illustrated in Fig 3.

## 13-22 Fig 3 Features of a Spherical Mirror


6. Consider Fig 4. If the rays of light falling on a curved mirror are parallel to the principal axis, they are reflected from a concave mirror so that they converge at one point, and from a convex mirror such that they appear to diverge from one point. This point is called the principal focus (F). The distance from the principal focus to the centre of the mirror is called the focal length and is approximately equal to half the radius of curvature.

13-22 Fig 4 Spherical Mirror - Reflection of Rays Parallel to the Principal Axis

7. If a source of light is at or near the principal focus of a spherical concave mirror, the light rays striking the mirror near its centre are reflected parallel to the principal axis. The rays striking the edge of the mirror are reflected so that they diverge. Parallel rays could be produced from all points on the mirror by altering the shape to a parabola, as shown in Fig 5.

13-22 Fig 5 Comparison of Spherical and Parabolic Curved Mirrors


Sypherical Curve


Parabolic Curve

## Images in Curved Mirrors

8. The position and size of an image produced by a curved mirror can be determined by the technique of ray tracing. The behaviour of certain rays from an object can be easily determined and drawn as follows:
a. Rays parallel to the principal axis will be reflected through, or appear to diverge from, the principal focus.
b. Rays passing through the centre of curvature will be reflected along the same line.
9. Fig 6 illustrates the principle applied to an object reflected in a convex mirror. F is the principal focus and $C$ is the centre of curvature. Rays from points $A$ and $B$ on the object, parallel to the principal axis, are reflected as if they diverged from $F$. Rays from $A$ and $B$, which would pass through $C$ if extended behind the mirror, are reflected back along the same path. The image is virtual, upright and smaller than the object. These characteristics are true regardless of the object's distance from the mirror.

## 13-22 Fig 6 Formation of Image in a Convex Mirror


10. Images in concave mirrors are real unless the object is placed between the principal focus and the mirror. The image increases in size as the object is brought from infinity towards the mirror, attaining the same size as the object when the object reaches the centre of curvature and being magnified when the object is inside the centre of curvature.
11. Fig 7 shows how ray tracing can be used to determine the position and size of an image produced by a concave mirror. In Fig 7a the object is outside the centre of curvature; in Fig 7b the object is inside the principal focus.

## 13-22 Fig 7 Formation of Images in Concave Mirrors


12. The position of the image produced by a spherical curved mirror can be determined from the equation:

$$
\frac{1}{\mathrm{v}}+\frac{1}{\mathrm{u}}=\frac{1}{\mathrm{f}}
$$

where $u=$ object distance,$v=$ image distance and $f=$ focal length.
13. In order for the equation to differentiate between real and virtual images a sign convention is necessary. The 'real is positive' convention is normally used, and the following rules apply:
a. A concave mirror has a positive focal length.
b. A convex mirror has a negative focal length.
c. Real objects are assigned a positive u value.
d. Real images have positive $v$ values.
e. Virtual images have negative $v$ values.

The formula will automatically generate the correct sign for any derived distance.
14. Magnification. The linear magnification due to a mirror is the ratio of the height of the image to the height of the object. In Fig 8, where $A^{\prime} B^{\prime}$ is the image of $A B$ and the triangles $A B P$ and $A^{\prime} B^{\prime} P$ are similar:

$$
\mathrm{m}=\frac{\mathrm{A}^{\prime} \mathrm{B}^{\prime}}{\mathrm{AB}}=\frac{\mathrm{PB}^{\prime}}{\mathrm{PB}}=\frac{\mathrm{v}}{\mathrm{u}}
$$

Thus when the image is further from the mirror than the object is, the magnification will be greater than one, and vice versa. When a real object produces a real image the magnification is positive whilst if a
virtual image is produced the magnification is negative. By rearranging the formula in para 12, it can be shown that magnification can be expressed in terms of $v$ and $f$, or $u$ and $f$ as follows:

$$
m=\frac{v-f}{f}, \text { or } m=\frac{f}{u-f}
$$

## 13-22 Fig 8 Magnification in a Concave Mirror



## LENSES

## Description

15. A lens is a portion of a transparent medium bounded by two curved surfaces. Most lenses are made of glass or plastic, and their surfaces are portions of spheres or cylinders. Only spherical lenses (of which there are two basic types), will be described here:
a. Convex Lenses. These are thicker at the centre than at the edges and are known as converging lenses (Fig 9a).
b. Concave Lenses. These are thinner at the centre than at the edges and are known as diverging lenses (Fig 9b).

13-22 Fig 9 Convex and Concave Lenses
a Convex Lenses
b Concave Lenses

16. Lenses have two surfaces, each of which may be considered to be part of a spherical surface, and therefore have a centre of curvature. A straight line joining the two centres of curvature is called the principal axis and is perpendicular to the surfaces where it passes through them as, shown in Fig 10.

## 13-22 Fig 10 Principal Axis of Convex Lens


17. The principal focus is the point on the principal axis to which all rays which are close to and parallel to the axis converge, or from which they appear to diverge, after refraction. The optical centre is a fixed point for any particular lens and coincides with the geometric centre of a symmetrical lens (see para 20). The distance from the principal focus to the optical centre of a lens is called the focal length. These terms are illustrated in Fig 11.

## 13-22 Fig 11 Lens Terminology


18. Refraction by Convex Lenses. A convex lens is approximately the same as two prisms placed base to base. Fig 12 shows parallel rays of light falling on a pair of prisms. At $O$ the ray $A O$ is refracted towards the normal NF. As it leaves the prism at $B$ it is refracted away from the normal $B E$ along the line $B C$. The feature to be noted is that light is bent towards the base or thicker part of the prism. Similarly, when light rays parallel to the principal axis fall on a convex lens they are refracted towards the thick part of the lens as shown in Fig 13.

## 13-22 Fig 12 Refraction by Two Prisms



8-23 Fig 1 Refraction by Convex Lens

19. Refraction by Concave Lenses. A concave lens is approximately the same as the prism arrangement shown in Fig 14. The parallel rays of light are refracted towards the base of each prism and therefore diverge. Similarly, the lens in Fig 15 causes light rays parallel to the principal axis to diverge, apparently from a point $F$ which is called the virtual focus.

13-22 Fig 13 Refraction by Two Prisms


## 13-22 Fig 14 Refraction by a Concave Lens



## Images Formed by Lenses

20. The ray tracing technique can equally be applied to the formation of images by lenses. The rays of use are similar to those employed in the case of mirrors. A ray parallel to the principal axis will emerge to pass through the principal focus in the case of a converging lens or will appear to have passed through the principal focus in the case of a diverging lens. As light paths are reversible, a ray which passes through the principal focus before entering a convex lens, or which would have passed through the principal focus had it not been intercepted by a concave lens, will emerge parallel to the principal axis. Finally a ray coincident with the principal axis will be undeviated; in practice provided that the thickness and diameter of the lens are small compared with its focal length, and provided that the location of the object or image point is not too far from the axis, then this non deviation rule can be generalized to include all rays passing through the optical centre.
21. Fig 16 shows the formation of an image by a concave lens. The image is always virtual, upright and smaller than the object. Fig 17 illustrates the formation of a real image by a convex lens and Fig 18 shows the formation of a virtual image by a convex lens.

## 13-22 Fig 15 Formation of Virtual Image of Concave Lens



## 13-22 Fig 16 Formation of Real Image by Convex Lens



## 13-22 Fig 17 Formation of Virtual Image by Convex Lens


22. The formula: $\frac{1}{\mathrm{v}}+\frac{1}{\mathrm{u}}=\frac{1}{\mathrm{f}}$ is equally applicable to lenses as it is to mirrors, using the same sign convention; convex lenses having positive focal lengths, concave lenses having negative focal lengths. The magnification formula is also the same as for mirrors.

## Lens Power

23. A thick lens with sharply curved surfaces bends light rays more than a thin flat lens does; it has a shorter focal length. The ability of a lens to refract light rays is a measure of its power. The power is measured in dioptres (symbol D) and if the focal length (f) is measured in metres then:

$$
\mathrm{D}=\frac{1}{\mathrm{f}}
$$

The power of a convex lens is positive and that of a concave lens is negative.
24. If two lenses are placed in contact then the resultant power can be obtained by summing the powers of the individual components lenses.

## Lens Defects

25. Spherical Aberration. Particularly if a lens has a wide aperture, rays parallel to the principal axis and passing through the periphery of the lens converge to a point which is nearer to the lens than the point to which a narrow central beam of parallel rays converge. At P in Fig 19, the central parts of the object are blurred whilst the peripheral portions are distinct. At $F$ the situation will be the reverse. The distance PF is called the longitudinal spherical aberration. Spherical aberration is more pronounced in thick lenses with short focal lengths than in thin lenses with long focal lengths.

## 13-22 Fig 18 Spherical Aberration


26. Chromatic Aberration. Since the refractive index of a prism or lens is greater for violet light than for red light, the lens may be considered as having a different focal length for each colour as shown in Fig 20.

## 13-22 Fig 19 Chromatic Aberration


27. Correcting Aberrations. Spherical aberration can be prevented by placing an adjustable diaphragm in front of the lens thus eliminating peripheral light rays. Alternatively a compound lens can be used to correct for both spherical and chromatic aberration.

## CHAPTER 23 - INFRA-RED RADIATION

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## Characteristics of Infra-red Radiation

1. Infra-red (IR) radiation is electro-magnetic radiation and occupies that part of the electromagnetic spectrum between visible light and microwaves. The IR part of the spectrum is sub-divided into Near IR, Middle IR, Far IR and Extreme IR. The position and division of the IR band, together with the appropriate wavelengths and frequencies, is shown in Fig 1.

## 13-23 Fig 1 Infra-red in the Electromagnetic Spectrum



Note: One micron ()$=10^{-6}$ metres and is now known as one micro-metre in the SI system.
2. All bodies with a temperature greater than absolute zero $\left(0 \mathrm{~K},-273{ }^{\circ} \mathrm{C}\right)$ emit IR radiation and it may be propagated both in a vacuum and in a physical medium. As a part of the electro-magnetic spectrum it shares many of the attributes of, for example, light and radio waves; thus it can be reflected, refracted, diffracted and polarized, and it can be transmitted through many materials which are opaque to visible light.

## Absorption and Emission

3. Black Body. The radiation incident upon a body can be absorbed, reflected or transmitted by that body. If a body absorbs all of the incident radiation then it is termed a 'black body'. A black body is also an ideal emitter in that the radiation from a black body is greater than that from any other similar body at the same temperature.
4. Emissivity ( $\varepsilon$ ). In $I R$, the black body is used as a standard and its absorbing and emitting efficiency is said to be unity; i.e. $\varepsilon=1$. Objects which are less efficient radiators, $(\varepsilon<1)$, are termed 'grey bodies'. Emissivity is a function of the type of material and its surface finish, and it can vary with wavelength and temperature. When $\varepsilon$ varies with wavelength the body is termed a selective radiator. The $\varepsilon$ for metals is low, typically 0.1 , and increases with increasing temperature; the $\varepsilon$ for non-metals is high, typically 0.9 , and decreases with increasing temperature.

## Spectral Emittance

5. Planck's Law. A black body whose temperature is above absolute zero emits IR radiation over a range of wavelengths with different amounts of energy radiated at each wavelength. A description of this energy distribution is provided by the spectral emittance, $W \lambda$, which is the power emitted by unit area of the radiating surface, per unit interval of wavelength. Max Planck determined that the distribution of energy is governed by the equation:

$$
\mathrm{W} \lambda=\frac{2 \pi \mathrm{c}^{2} \mathrm{~h}}{\lambda^{5}}\left(\mathrm{e}^{\mathrm{hc} / \mathrm{kT} \lambda}-1\right)^{-1}
$$


6. Temperature/Emittance Relationship. This rather complex relationship is best shown graphically, as in Fig 2 in which the spectral emittance is plotted against wavelength for a variety of temperatures. It will be seen that the total emittance, which is given by the area under the curve, increases rapidly with increasing temperature and that the wavelength of maximum emittance shifts towards the shorter wavelengths as the temperature is increased.

## 13-23 Fig 2 Distribution of IR Energy with Temperature


7. Stefan-Boltzmann Law. The total emittance of a black body is obtained by integrating the Max Planck equation which gives the result:

$$
W=\sigma T^{4}
$$

$$
\text { where } \begin{aligned}
\mathrm{W} & =\text { Total emittance } \\
\sigma & =\text { Stefan Boltzmann constant } \\
\mathrm{T} & =\text { Absolute temperature }
\end{aligned}
$$

For a grey body, the total radiant emittance is modified by the emissivity, thus:

$$
\mathrm{W}=\varepsilon \sigma \mathrm{T}^{4}
$$

8. Wien's Displacement Law. The wavelength corresponding to the peak of radiation is governed by Wien's displacement law which states that the wavelength of peak radiation $\left(\lambda_{m}\right)$, multiplied by the absolute temperature is a constant. Thus:

$$
\lambda_{m} T=2900 \mu \mathrm{~K}
$$

By substituting $\lambda_{m}=2900 / T$ into Planck's expression it is found that:

$$
W \lambda_{m}=1.3 \times 10^{-15} \mathrm{~T}^{5} \text { expressed in Watts } \mathrm{cm}^{-2} \mu^{-1}
$$

ie the maximum spectral radiant emittance depends upon the fifth power of the temperature.

## Geometric Spreading

9. The laws so far discussed relate to the radiation intensity at the surface of the radiating object. In general, radiation is detected at some distance from the object and the radiation intensity decreases with distance from the source as it spreads into an ever-increasing volume of space. Two types of source are of interest; the point source and the plane extended source.
10. Point Source. A point source radiates uniformly into a spherical volume. In this case the intensity of radiation varies as the inverse square of the distance between source and detector.
11. Plane Extended Source. When the radiating surface is a plane of finite dimensions radiating uniformly from all parts of the surface then the radiant intensity received by a detector varies with the angle between the line of sight and the normal to the surface. For a source of area A the total radiant emittance is WA. The radiant emittance received at a distance $d$ and at an angle $\theta$ from the normal is given by:

$$
\frac{\text { WA }}{2 \pi \mathrm{~d}^{2}} \cos \theta
$$

## IR Sources

12. It is convenient to classify IR sources by the part they play in IR systems; ie as targets, as background, or as controlled sources. A target is an object which is to be detected, located or identified by means of IR techniques, while a background is any distribution or pattern of radiation, external to the observing equipment, which is capable of interfering with the desired observations. Clearly what might be considered a target in one situation could be regarded as background in another. As an example terrain features would be regarded as targets in a reconnaissance application but would be background in a low-level air intercept situation. Controlled sources are those which supply the power required for active IR systems (e.g. communications), or provide the standard for calibrating IR devices.

## Targets

13. Aircraft Target. A supersonic aircraft generates three principle sources of detectable and usable IR energy. The typical jet pipe temperature of 773 K produces a peak of radiation, (from Wien's law), at $3.75 \mu$. The exhaust plume produces two peaks generated by the gas constituents; one at 2.5 to $3.2 \mu$ due to carbon dioxide, the other at 4.2 to $4.5 \mu$ due to water vapour. The third source is due to leading edge kinetic heating giving a typical temperature of 338 K with a corresponding radiation peak at about $7 \mu$.
14. Reconnaissance. Terrestrial IR reconnaissance and imaging relies on the IR radiation from the Earth which has a typical temperature of 300 K . The peak of radiation corresponding to this temperature is about $10 \mu$ and so systems must be designed to work at this wavelength.

## Background Sources

15. Regardless of the nature of the target source, a certain amount of background or interfering radiation will be present, appearing in the detection system as noise. The natural sources which produce this background radiation may be broadly classified as terrestrial or atmospheric and celestial.
16. Terrestrial Sources. Whenever an IR system is looking below the horizon it encounters the terrestrial background radiation. As all terrestrial constituents are above absolute zero they will radiate in the infra-red, and in addition IR radiation from the sun will be reflected. Green vegetation is a particularly strong reflector which accounts for its bright image in IR photographs or imaging systems. Conversely, water, which is a good reflector in the visible part of the spectrum, is a good absorber of IR, and therefore appears dark in IR images.
17. Atmospheric and Celestial Sources. Whenever an IR device looks above the horizon the sky provides the background radiation. The radiation characteristics of celestial sources depend on the source temperature together with modifications by the atmosphere.
a. The Sun. The sun approximates to a black body radiator at a temperature of $6,000 \mathrm{~K}$ and thus has a peak of radiation at $0.5 \mu$, which corresponds to yellow-green light. The distribution of energy is shown in Fig 3 from which it will be seen that half of the radiant power occurs in the infra-red. The Earth's atmosphere changes the spectrum by absorption, scattering and some re-
radiation such that although the distribution curve has essentially the same shape, the intensity is decreased and the shorter, ultraviolet, wavelengths are filtered out. The proportion of IR energy remains the same or perhaps may be slightly higher. Sunlight reflected from clouds, terrain and sea shows a similar energy distribution.

13-23 Fig 3 Spectral Distribution of Solar Radiation

b. The Moon. The bulk of the energy received from the moon is re-radiated solar radiation, modified by reflection from the lunar surface, slight absorption by any lunar atmosphere and by the Earth's atmosphere. The moon is also a natural radiating source with a lunar daytime surface temperature up to 373 K and lunar night time temperature of about 120 K . The near sub-surface temperature remains constant at 230 K , corresponding to peak radiation at $12.6 \mu$.
c. Sky. Fig 4 shows a comparison of the spectral distribution due to a clear day and a clear night sky. At night, the short wavelength background radiation caused by the scattering of sunlight by air molecules, dust and other particles, disappears. At night there is a tendency for the Earth's surface and the atmosphere to blend with a loss of horizon since both are at the same temperature and have similar emissivities.

## 13-23 Fig 4 Spectral Energy Distribution of Background Radiation from the Sky


d. Clouds. Clouds produce considerable variation in sky background, both by day and by night, with the greatest effect occurring at wavelengths shorter than $3 \mu$ due to solar radiation reflected from cloud surfaces. At wavelengths longer than $3 \mu$, the background radiation intensity caused by clouds is higher than that of the clear sky. Low bright clouds produce a larger increase in background radiation intensity at this wavelength than do darker or higher clouds. As the cloud formation changes the sky background changes and the IR observer is presented with a varying background both in time and space. The most serious cloud effect on IR detection systems is that of the bright cloud edge. A small local area of IR radiation is produced which may be comparable in area to that of the target, and also brighter. Early IR homing missiles showed a greater affinity for cumulus cloud types than the target aircraft. Discrimination from this background effect requires the use of spectral and spatial filtering.

## IR Transmission in the Atmosphere

18. Atmospheric Absorption. The periodic motions of the electrons in the atoms of a substance, vibrating and rotating at certain frequencies, give rise to the radiation of electro-magnetic waves at the same frequencies. However, the constituents of the Earth's atmosphere also contain electrons which have certain natural frequencies. When these natural frequencies are matched by those of the radiation which strikes them, resonance absorption occurs and the energy is re-radiated in all directions. The effect of this phenomenon is to attenuate certain IR frequencies. Water vapour and carbon dioxide are the principle attenuators of $I R$ radiation in the atmosphere. Figs $5 \mathrm{a}, 5 \mathrm{~b}$ and 5 c show the transmission characteristics of the atmosphere at sea-level, at 30,000 ft and at 40,000 ft.
19. Scattering. The amount of scattering depends upon particle size and particles in the atmosphere are rarely bigger than $0.5 \mu$, and thus they have little effect on wavelengths of $3 \mu$ or greater. However, once moisture condenses on to the particles to form fog or clouds, the droplet size can range between 0.5 and $80 \mu$, with the peak of the size distribution between 5 and $15 \mu$. Thus fog and cloud particles are comparable in size to IR wavelengths and transmittance becomes poor. Raindrops are considerably larger than IR wavelengths and consequently scattering is not so pronounced. Rain, however, tends to even out the temperature difference between a target and its surroundings.

## 13-23 Fig 5 Atmospheric Transmittance vs. Altitude

a Transmittance at Sea Level

b Transmittance at $30,000 \mathrm{ft}$

c Transmittance at $40,000 \mathrm{ft}$

20. Scintillation. Where a beam of IR passes through regions of temperature variation it is refracted from its original direction. Since such regions of air are unstable, the deviation of the beam is a random, time varying quantity. The effect is most pronounced when the line of sight passes close to the earth and gives rise to unwanted modulations of the signal, and incorrect direction information for distant targets.

## CHAPTER 24 - LASERS

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## Introduction

1. The laser is a device that emits an extremely intense beam of energy in the form of electromagnetic radiation in the near ultra-violet, visible, or infra-red part of the electromagnetic spectrum. The word LASER is an acronym derived from the definition of its function, Light Amplification by Stimulated Emission of Radiation. The word has been so integrated into the English language that it is no longer written in capital letters, as are most acronyms. Indeed, in technical circles, its use has spawned a verb, to lase, which describes the action of using a laser. Unlike the radiation from other sources, laser light is monochromatic (single wavelength), coherent (all waves in phase), and highly collimated (near parallel beam). Since the first laser was constructed in 1960 in California, the development has been rapid and uses have been found in a wide variety of civil and military spheres, including surgery, communications, holography and target marking and range-finding. In order to understand the principle of operation of a laser it is first necessary to appreciate some aspects of atomic structure and energy levels.

## Atomic Energy Levels

2. The atom consists of a central nucleus containing positively charged protons and neutral neutrons. Surrounding the nucleus are negatively charged electrons. The number of protons and electrons are equal thus resulting in a net zero charge on the atom as a whole. The electrons have a certain energy level due to the sum of their kinetic energy and electrostatic potential energy. However electrons within an atom are constrained to exist in one of a series of discrete energy levels. In normal circumstances the electrons will adopt the minimum energy levels permitted and the atom is then said to be in its ground state. In order for electrons to enter a higher energy level, energy in one form or another has to be supplied. If such a transition to a higher energy level occurs the atom is said to be in an excited state.
3. Conventionally, the energy states of an atom can be shown on an energy level diagram as shown in Fig 1. The horizontal lines represent the permitted energy levels, increasing upwards, separated by varying energy differences, $\Delta \mathrm{E}$. The horizontal extent of the lines has no significance. The base line is the ground state i.e. the lowest energy level in which atoms will normally be found (A in Fig 1). By supplying energy, it may be possible to excite an atom (B in Fig 1) into a higher energy level. This process is known as absorption.

13-24 Fig 1 Atomic Energy Levels


## Emissions

4. Spontaneous Emission. An atom in an excited state is unstable and will have a tendency to revert to the ground state. In doing so, it will emit the excess energy as a single quantum of energy known as a photon, a process known as spontaneous emission. If a large population of atoms are excited into higher states, as for example in a fluorescent lighting tube, then they will occupy a wide band of energy levels. On undergoing spontaneous emissions, some will revert to the ground state directly whilst others will drop via intermediate levels. In either case photons will be emitted with a wide range of energy levels corresponding to the various energy level differences. The frequency of the emitted energy is determined by the Planck-Einstein equation:

$$
E=h f
$$

where $E$ is the photon energy, f is the frequency and h is Planck's constant.
Thus in a fluorescent tube, as there are a wide variety of energy level transitions, there will be a wide variety of frequencies in the emitted light giving the impression of white light. It should be noted that what transitions occur and when they occur is a random process. Equally the direction in which the emitted photon is radiated is also random. Thus the radiation generated by spontaneous emission is isotropic (i.e. radiating in all directions), non-coherent and covers a wide frequency band.
5. Stimulated Emission. As early as 1917 Einstein predicted on theoretical grounds that the downward transition of an atom could be stimulated to occur by an incident photon of exactly the same energy as the difference between the energy levels. It is this type of emission that is exploited in lasers. This process is shown in Fig 2. It should be noted that the incident photon is not absorbed and so for each incident photon, two photons are emitted, each of which can stimulate further emissions providing that there are atoms in the higher energy level. Furthermore, these emitted photons have the same energy, and therefore frequency, the same phase and are emitted in the same direction as the incident photons. These are, of course, the characteristics of laser radiation.

## 13-24 Fig 2 Stimulated Emission



Ground State
6. Population Inversion. In the normal course of events, however, most atoms are in the ground state and so incident photons are more likely to excite a ground state atom than to induce stimulated emission. It is therefore necessary to ensure that there are more atoms in the appropriate higher energy level than in the ground state, a situation known as a population inversion. The process by which this is achieved will be described with reference to the ruby laser which was the first lasing medium to be used.
7. Optical Pumping. Fig 3a illustrates the normal configuration with respect to the chromium atoms within a ruby crystal. The diagram shows a number of atoms in the ground state and a number of as yet unoccupied higher energy levels. It should be noted that the energy levels in Fig 3 refer to the energy of the atom as a whole and not to the energy levels of the constituent electrons. At the start of the process the ruby is subjected to a burst of intense white light generated by a system similar to a photographic electronic flash gun. As the white light comprises a wide range of frequencies then a whole range of energies will be imparted to the ground state atoms. Some of these atoms will therefore be excited to a range of higher energy levels (Fig 3b); a process known as optical pumping.

## 13-24 Fig 3 The Stages of Operation in a Ruby Laser

a Optical Pumping

c Population Inversion

b Atomic Excitation

d Stimulated Emission


8. The Metastable State. From these higher energy levels spontaneous emissions will occur but whereas some will be due to transitions to the ground state, in the case of chromium the majority will decay to an intermediate level known as a metastable state as shown in Fig 3c from which atoms may emit photons at random. Nevertheless, this state, apart from being a preferential level, has the additional feature that atoms tend to remain there for a longer time (by a factor of some 1000s) than they do in any other level other than the ground state. In this way a population inversion is achieved i.e. there are more atoms in the metastable state than in the ground state.
9. Lasing Action. Inevitably at some time an atom in the metastable state will make a spontaneous transition to the ground state with the emission of a photon. This photon can now do one of two things; it can either excite a ground state atom into a higher level or it can stimulate an excited atom in the metastable state to make a transition to the ground state. Since a population inversion has been achieved, then on balance it is more likely to stimulate emission than to be absorbed by a ground state atom (Fig 3d). Thus lasing will be initiated. At the end of this process all of the atoms will be back in the ground state ready for further optical pumping to start the cycle again.
10. Other Techniques. Optical pumping is not the only means of achieving a population inversion. The helium-neon laser, for example, uses a different method. The medium in this case is a mixture of helium and neon gases of which the neon is responsible for lasing. Energy is input to the helium by means of an electrical discharge and the energized helium atoms transmit their excess energy not by radiation but in collisions with neon atoms. The neon atoms are excited to a high energy level such that there is a population inversion between this level and an intermediate level rather than with respect to the ground state. Stimulated emission therefore occurs between these two higher levels. This process is illustrated in Fig 4. Atoms in the bottom lasing level eventually decay spontaneously back to the ground state.

## 13-24 Fig 4 The Processes in a Helium-Neon Laser


11. The system so far described produces monochromatic and coherent radiation, however it is not very intense and is not emitted as a beam. This is because the stimulating photons are incident upon the atoms from random directions and so the emitted photons follow, likewise, random directions. In addition there will of course be a proportion of random spontaneous emissions. It is therefore necessary to ensure that as many photons as possible are travelling in the required direction. This is achieved by having the lasing medium within an optical resonant cavity.

## The Laser

12. The features of the working laser are shown in Fig 5. The optical resonant cavity is achieved by placing mirrors at each end of the lasing medium. These mirrors are separated by an integral number of $1 / 2$-wavelengths of the laser radiation and are accurately aligned perpendicular to the laser axis. One of the mirrors is semi-transparent. Photons travelling normal to the mirrors will be reflected backwards and forwards through the cavity and in the process will stimulate further emissions which will radiate in the same direction. The $\mathrm{n} \times 1 / 2$-wavelength nature of the mirror separation ensures that the radiation stays in phase. Off axis radiation will soon be lost to the system through the side walls allowing the axial radiation to increase rapidly in relation to the non-axial radiation. The semi-silvered mirror allows the highly directional beam to leave the cavity.

## 13-24 Fig 5 Laser Schematic


13. Q-switching. A typical ruby laser as described will have a nominal output power of several kW and a pulse length in the order of a millisecond. For many applications it would be beneficial to increase the power by reducing the pulse length. The technique used to achieve this is known as Qswitching. Between the lasing medium and the fully silvered mirror is a glass cell containing a green dye. Although the lasing action starts once the pumping commences, the green dye absorbs the red laser light preventing the build up of energy in the resonant cavity. In doing so the molecules in the dye are raised to an excited state. The concentration of the dye is arranged so that the dye molecules are all excited coincidently with the maximum number of atoms of chromium being in the metastable level. At this point the dye becomes transparent to the laser wavelength and there is then a very rapid build up of lasing action. The pulse of laser radiation is delivered in about 10 nanoseconds, before the dye molecules return to the ground state and shut off the laser. The output power can be increased to the order of hundreds of mW by this technique.

## CHAPTER 25 - THE NATURE OF SOUND

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## Introduction

1. A Definition of Sound. Sound is the name given to the sensation perceived by the human ear. Every sound is produced by the vibration of the object from which it originates; thus, any explanation of the nature of sound must include a discussion of vibratory motion. The audible sound spectrum is generally acknowledged to cover the frequency range from 15 to 20,000 hertz.
2. Simple Harmonic Motion. The simplest form of vibratory motion can be represented by the oscillation of a pendulum bob swinging through a small angle. If the displacement of the bob from the central position is plotted on a graph against time, the variation of the displacement with time gives rise to a sine curve as shown in Fig 1. This motion is called simple harmonic motion, and could equally describe the vibration of a tuning fork. The displacement of the bob can be described by the equation:

$$
\begin{equation*}
\mathrm{d}=\mathrm{a} \sin \frac{2 \pi \mathrm{t}}{\mathrm{~T}} \tag{1}
\end{equation*}
$$

where a and T are constants as shown in Fig 1.

## 13-25 Fig 1 Simple Harmonic Motion


3. Frequency, Amplitude and Period. In Fig 1, the cycle represents one complete swing of the pendulum. The frequency of the oscillation is defined as the number of cycles per second ( 1 cycle per second $=1$ hertz (Hz)). During one cycle, the bob twice attains maximum deflection from the central position. The maximum distance displaced is called the amplitude, and in Fig 1 this is shown by the constant $a$. The period of a vibration is the time it takes to complete one cycle. The frequency can be expressed as:

$$
\begin{equation*}
\mathrm{f}=\frac{1}{\mathrm{~T}} . \tag{2}
\end{equation*}
$$

where $f$ is the frequency and $T$ is the period in seconds.
4. Fourier's Theorem. The vibration of a tuning fork is the nearest audible equivalent to the oscillation of a pendulum. It can be shown that any vibratory motion which repeats itself regularly can be represented as the resultant or combination of simple harmonic frequencies of suitably chosen amplitudes. These frequencies must also be integral multiples of the frequency with which the motion repeats itself (Fourier's Theorem). Hence, equation (1) is the basis for describing the vibrations of all sounding objects.
5. Medium of Transmission. If a sound source is placed in an airtight chamber which is slowly evacuated of air, the sound will gradually die away as the vacuum increases. Eventually the sound will cease, although it can be seen that the source is still vibrating. Such an experiment can be used to demonstrate that a material medium such as air, water, wood, glass, or metal is required for the sound to be transmitted.

## The Propagation of Sound

6. Transverse Waves and Longitudinal Waves. The wave motion observed when the surface of a pond is disturbed is called transverse wave motion, because the particles of water oscillate at right angles to the direction of propagation of the waves. Sound waves, however, are propagated as longitudinal waves. In this kind of wave motion the particles oscillate, each about a fixed point, in the direction of propagation of the waves. In Fig 2, the undisturbed particles of a medium are represented by equally spaced dots. A similar set of particles is shown in Fig 3 being disturbed by the passage of a sound wave through the medium. Each particle is displaced to the right and left of its undisturbed position as the wave passes through the medium. If the displacement of a single particle is plotted on the vertical axis of a time graph the familiar sine wave form of simple harmonic motion is produced as shown in Fig 4.

## 13-25 Fig 2 Undisturbed Particles in a Medium




## 13-25 Fig 3 Passage of a Sound Wave Through a Medium



## 13-25 Fig 4 Longitudinal Wave Plotted in Graphical Form


7. Pressure Variations. When the particles of a medium are displaced by the passage of a sound wave, there is a consequent local variation in pressure. It is these small changes in pressure which actuate the human ear and mechanical devices such as microphones. Fig 5 shows the pressure variations that accompany the passage of a sound wave; they consist of alternate compressions and rarefactions.

## 13-25 Fig 5 Pressure Variation in a Medium



## The Properties of Sound Waves

8. Reflection. Like light waves, sound waves are reflected from a plane surface such that the angle of reflection is equal to the angle of incidence. It can also be shown that sound waves come to a focus when they are incident on a concave reflector.
9. Reverberation. If sound is generated within a large enclosed space it can be heard directly from the source and indirectly from reflected and diffused (multiple reflection) sound waves. The indirect sounds are called reverberations and continue for a finite time after the sound source has been silenced.
10. Refraction. Sound waves travel faster in warm air than in cold air and are therefore refracted when there is a temperature gradient. Refraction also occurs in water and other media because of changes in the velocity of sound.
11. Interference. If two sound sources are of the same frequency and intensity, and are initially in phase (i.e. coherent) they will interfere with each other and will cancel or reinforce according to the path difference. If the path difference is an odd number of half wavelengths, cancellation occurs and no sound is heard. If it is an even number of half wavelengths, the sounds will reinforce and a louder sound is heard.
12. The Wave-front. If a single pulse of noise, such as an explosion, occurs in a medium, the paths followed by the sound waves can be traced by placing a number of recording microphones in the vicinity and noting the time taken for the sound to reach each microphone. If the microphones are located at specified ranges from the source, the points in space reached simultaneously by the sound can be plotted. These points are considered to lie on a surface called the wave-front, and, in a homogeneous medium, the direction of propagation is perpendicular to this surface. It can also be observed that the sound is propagated outwards at a constant velocity.
13. Diffraction. It is a common experience that it is possible to hear sound even when the source is behind an obstruction. This 'bending' of sound waves (or indeed any other type of wave) around such an obstacle is known as diffraction. Although the mathematical treatment is rather complex, a satisfactory explanation of the phenomenon can be made using Huygens' principle. Huygens' principle states that all points on a wave front can be considered as point sources from which secondary wavelets are generated. After a time interval, a new position of the wave-front will be established as the surface of tangency to these secondary wavelets. The way that this principle accounts for the 'bending' of sound around an obstacle is shown in Fig 6.

## 13-25 Fig 6 Huygens' Principle - 'Bending' Sound around an Obstacle



## The Velocity of Sound

14. Since the pressure variations produced by a sound wave are so rapid that no transfer of heat can occur, the process is considered to be adiabatic. The velocity of sound (c) in a gas is found to be given by:

$$
\begin{equation*}
\mathrm{c}=\sqrt{\frac{\gamma \mathrm{p}}{\rho}} \tag{3}
\end{equation*}
$$

where y is the ratio of the specific heat at constant temperature to that at constant volume, p is the pressure, and $\rho$ is the density.
15. Since $\frac{p}{\rho}=R T$ in an ideal gas, where $R$ is the specific gas constant and $T$ is the temperature in $K$, equation (3) can be rewritten as:

$$
\begin{equation*}
\mathrm{c}=\sqrt{\gamma \mathrm{RT}} \tag{4}
\end{equation*}
$$

Therefore in a given gas, since $\gamma$ and $R$ are constants, $c \propto R \sqrt{T}$ within the range in which the gas obeys the ideal gas equation. A working expression for the speed of sound in air at a temperature of $t^{\circ} \mathrm{C}$ is given by the equation:

$$
c_{t}=(330+0.61 t) \mathrm{ms}^{-1}
$$

In equation (3) both $\gamma$ and $\frac{p}{\rho}$ are constant for a given gas at a specific temperature, and from this it can be deduced that the velocity of sound in air is independent of pressure.
16. The velocity of sound in water is covered in Volume 13, Chapter 28.

## The Intensity of Sound

17. The intensity of sound at any place is defined as the rate of flow of energy across unit area perpendicular to the direction of propagation. If a sound source is emitting J joules of energy per second uniformly in all directions it can be calculated that the energy passing through unit area is proportional to the inverse square of the distance from the source. The intensity of sound is further attenuated by the absorption of energy by the medium through which it is propagated.

## The Doppler Effect

18. The Doppler effect occurs when there is a relative velocity between the sound source and the observer

## 13-25 Fig 7 The Doppler Effect



Fig 7 shows how the change of wavelength and hence frequency occurs when a source of sound is moving either towards or away from the observer. The circles represented by $A, B, C$ etc correspond to successive wave-fronts generated at $a, b, c$ etc by the moving source. It is clear that to the observer at $X$, passage of the wave-fronts will be more frequent (ie the observer will hear a higher pitched sound than was generated), while to the observer at $Y$, passage of the wave fronts will be less frequent (i.e. the observer will hear a lower pitched sound).
19. The velocity of sound in air of uniform temperature is constant, irrespective of any movement of the source or the observer. However, any movement of the source will alter the wavelength of a sound in air and hence change the pitch of the sound heard by the observer. If the component of velocity of the source towards the observer is $V_{s}$, then the frequency, $f^{\prime}$, of the note heard by the observer is given by:

$$
\mathrm{f}^{\prime}=\mathrm{f}_{0} \cdot \frac{\mathrm{c}}{\mathrm{c}-\mathrm{V}_{\mathrm{s}}}
$$

where $f_{0}=$ frequency of the note if heard from a stationary source, and $c=$ velocity of sound in air. Any movement of the observer alters the velocity of the sound relative to the observer, and this also results in a change of pitch. If $\mathrm{V}_{0}$ is the component of velocity of the observer towards the source, then the frequency, $\mathrm{f}^{\prime}$, of the note heard by the observer is given by:

$$
\mathrm{f}^{\prime}=\mathrm{f}_{0} \cdot \frac{\mathrm{c}+\mathrm{V}_{0}}{\mathrm{c}}
$$

If both the source and the observer are moving then the frequency, $\mathrm{f}^{\prime \prime}$, of the note heard by the observer is given by:

$$
\mathrm{f}^{\prime \prime}=\mathrm{f}_{0} \cdot \frac{\mathrm{c}+\mathrm{V}_{0}}{\mathrm{c}-\mathrm{V}_{\mathrm{s}}}
$$

