UNIVERSITY OF WARWICK

Summer Examinations 2015/16

Mathematical Analysis

Time allowed: 1.5 hours

Answer THREE questions. Each question is worth 25 marks.

Calculators are not needed and are not permitted in this examination.

A formula sheet is provided at the end of the exam paper.

Read carefully the instructions on the answer book provided and make sure that the particulars required are entered on each answer book. If you answer more questions than are required and do not indicate which answers should be ignored, we will mark the requisite number of questions in the order in which they appear in the answer book(s): answers beyond that number will not be considered.

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1. (a) (i) Define the *Cartesian product* $A \times B$ of two sets A and B. **(2 marks)**

(ii) Define the *difference* $A \setminus B$ of two sets A and B. **(2 marks)**

(iii) Suppose A and B are both finite sets, with $|A| = m$ and $|B| = n$.
What is $|A \times B|$? **(1 mark)**
What are the minimum and maximum possible values of $|A \setminus B|$? **(2 marks)**

(b) (i) Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$, and let $f \colon \mathbb{R} \to \mathbb{R}_+$ with $f(x) = e^x$. Prove that f is a bijection. **(4 marks)**

(ii) For a set A, recall that the *power set* of A is the set

$$P(A) = \{B : B \subseteq A\}$$

of all subsets of A. Let $g \colon \mathbb{N} \to P(\mathbb{N})$ by $g(n) = \{n\}$. Is g injective? Is it surjective? (In each case, either prove the statement or provide a counterexample.) **(4 marks)**

(c) (i) Define what it means for a set A to be *countable*. **(2 marks)**

(ii) Show that the set $\mathbb{Z}$ of integers is countable. **(3 marks)**

(iii) Show that the set $\mathbb{Q}_+$ of positive rational numbers is countable. **(5 marks)**

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2. (a) (i) Let n be an integer. Show that if n^3 is divisible by 5, then so is n. **(3 marks)**

(ii) Show that $\sqrt[3]{5}$ is irrational. **(4 marks)**

(b) Let $p(z) = z^6 - 729$.

(i) Find all six roots of p in polar form. **(3 marks)**

(ii) Convert these roots to Cartesian form and verify that any complex roots occur as conjugate pairs. **(3 marks)**

(iii) Use your answer to part (ii) to factorise p as a product of linear and quadratic polynomials with real coefficients. **(3 marks)**

(c) (i) Prove by induction that $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$. **(3 marks)**

(ii) Prove by induction that $\sum_{k=1}^{n}(2k-1) = n^2$ for all $n \in \mathbb{N}$. **(2 marks)**

(iii) Use your answer to part (ii) to show that

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots = \frac{1+3+\cdots+(2n-1)}{(2n+1)+\cdots+(4n-1)}$$

for any $n \in \mathbb{N}$. **(4 marks)**

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3. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function, and suppose that $a, b \in \mathbb{R}$ are two fixed real numbers. Define what it means for f to be:

(i) *continuous* at $x = a$; **(2 marks)**

(ii) *continuous* in the closed interval $[a, b]$. **(3 marks)**

(iii) Suppose that

$$g(x) = \begin{cases} \sin(x) & \text{if } x < 0, \\ x & \text{if } x \geqslant 0. \end{cases}$$

Show that g is continuous and differentiable at $x = 0$. **(4 marks)**

(b) State L'Hôpital's Rule. **(2 marks)**

Use it to calculate the following limits:

(i) $\lim_{x\to 0} \frac{x - \sin x}{x^3}$. **(2 marks)**

(ii) $\lim_{x\to\infty} \frac{x^2+9x+4}{3x^2+7x+8}$. **(2 marks)**

(iii) $\lim_{x\to 0}(1 + \sin 2x)^{\cot 3x}$. **(3 marks)**

(c) State Cauchy's Mean Value Theorem. **(3 marks)**

Use it to prove L'Hôpital's Rule. **(4 marks)**

(Continued overleaf)

4. (a) Consider the polynomial $p(x) = x^3 - 2x - 2$.

(i) Use the Intermediate Value Theorem to show that p has at least one root in the interval $(1, 2)$. **(3 marks)**

(ii) Use Rolle's Theorem to show that p has at most one root in the interval $(1, 2)$. **(4 marks)**

(b) (i) State Taylor's Theorem. **(3 marks)**

(ii) Calculate the Taylor–Maclaurin series for $\sin^{-1}(x)$, about $x = 0$, up to and including the term in x^5. **(4 marks)**

(iii) Use your answer to part (ii) to obtain a (not very accurate) rational approximation to $\frac{\pi}{2}$. **(2 marks)**

(iv) Again using your answer to part (ii), write down the Taylor–Maclaurin series for $\frac{1}{\sqrt{1-x^2}}$, about $x = 0$, up to and including the term in x^4. **(3 marks)**

(c) Calculate the following improper integrals:

(i) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$. **(3 marks)**

(ii) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$. **(3 marks)**

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5. (a) Find the general solution of the following differential equation: **(6 marks)**

$$\frac{dx}{dt} + 4x = t + e^{-2t}$$

Find the specific solution with initial conditions $x = 0$ when $t = 0$. **(2 marks)**

(b) Find the general solution of the following differential equation: **(6 marks)**

$$\frac{dy}{dx} = \frac{1 - x^2}{y^2}$$

Find the specific solution with initial conditions $y = 1$ when $x = 0$. **(2 marks)**

(c) Solve the following differential equation: **(6 marks)**

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \sin(3x)$$

Find the specific solution for which $y = \frac{dy}{dx} = 0$ when $x = 0$. **(3 marks)**

FORMULA SHEET

Short table of derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
k	0	x^a	ax^{a-1}
$\cos(ax)$	$-a\sin(ax)$	$\sin(ax)$	$a\cos(ax)$
e^{ax}	ae^{ax}	$\ln(ax)$	$\frac{1}{x}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$-\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\tan(ax)$	$a\sec^2(ax)$	$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2+x^2}$

Trigonometric identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin\alpha\sin\beta = \tfrac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)) \qquad \cos\alpha\cos\beta = \tfrac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$$

Values of trigonometric functions

$$\sin\left(\tfrac{\pi}{4}\right) = \tfrac{1}{\sqrt{2}} = \tfrac{\sqrt{2}}{2} \qquad \cos\left(\tfrac{\pi}{4}\right) = \tfrac{1}{\sqrt{2}} = \tfrac{\sqrt{2}}{2} \qquad \tan\left(\tfrac{\pi}{4}\right) = 1$$
$$\sin\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{3}}{2} \qquad \cos\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2} \qquad \tan\left(\tfrac{\pi}{3}\right) = \sqrt{3}$$
$$\sin\left(\tfrac{\pi}{6}\right) = \tfrac{1}{2} \qquad \cos\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2} \qquad \tan\left(\tfrac{\pi}{6}\right) = \tfrac{1}{\sqrt{3}} = \tfrac{\sqrt{3}}{3}$$

Integration by parts

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Taylor series about $x = a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \cdots$$

valid for some interval I containing x and a.