UNIVERSITY OF WARWICK

Summer Examinations 2015/16

Linear Algebra

Time allowed: 1.5 hours

Answer THREE questions. Each question is worth 25 marks.

Calculators are not needed and are not permitted in this examination.

A formula sheet is provided at the end of the exam paper.

Read carefully the instructions on the answer book provided and make sure that the particulars required are entered on each answer book. If you answer more questions than are required and do not indicate which answers should be ignored, we will mark the requisite number of questions in the order in which they appear in the answer book(s): answers beyond that number will not be considered.

---

1. (a) Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a set of vectors in $\mathbb{R}^n$.

(i) Define what is meant by a *linear combination* of the vectors in S. **(2 marks)**

(ii) Define the *span* of S. **(2 marks)**

(iii) State a necessary condition on the number of vectors in S for S to be a basis for $\mathbb{R}^n$. **(2 marks)**

(iv) Show that this condition is not sufficient by writing down a set of vectors in $\mathbb{R}^3$ that satisfies this condition but doesn't form a basis. **(2 marks)**

(b) Let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \text{and} \qquad T = \left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}.$$

(i) Show that T is a basis for $\mathbb{R}^3$. **(4 marks)**

(ii) Write down the change of basis matrix Q that converts from the nonstandard basis T to the standard basis S. **(2 marks)**
Calculate the inverse matrix Q^{-1} that converts from S to T. **(3 marks)**

(iii) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $f(x, y, z) = (2x - y, 2y - z, 2z - x)$. Show that this is a linear transformation. **(3 marks)**

(iv) Write down a matrix representing f relative to the basis S. **(2 marks)**

(v) Using your answers to part (b)(ii), find a matrix B that represents f relative to the nonstandard basis T. **(3 marks)**

(Continued overleaf)

2. Consider the following system of simultaneous linear equations:

$$\left.\begin{aligned} w + 2x - 3y + 2z &= 2 \\ 2w + 5x - 8y + 6z &= 5 \\ 3w + 4x - 5y + 2z &= 4 \end{aligned}\right\} \qquad (*)$$

(a) Write down the coefficient matrix A and the augmented matrix A' for the system $(*)$. **(3 marks)**

(b) By a sequence of elementary row operations, convert A' to reduced row echelon form, and thereby calculate its rank. **(6 marks)**

(c) Using your answer to part (b), write down the solution set for $(*)$. **(3 marks)**

Now consider the coefficient matrix A as a linear map from $\mathbb{R}^5$ to $\mathbb{R}^3$.

(d) Find a basis for the kernel of A. **(3 marks)**

(e) Find a basis for the row space of A. **(3 marks)**

(f) Find a basis for the column space of A. **(3 marks)**

(g) Express the vector $\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$ as a linear combination of the basis vectors in part (f), and explain the relevance of this to the solubility of $(*)$. **(4 marks)**

---

3. (a) Let M be an $n \times n$ square matrix.

(i) Say what it means to *diagonalise* M, and give a necessary and sufficient condition for this to be possible. **(3 marks)**

(ii) Show that the trace of M is equal to the sum of its eigenvalues. **(5 marks)**

(b) Consider the following second order linear difference equation:

$$u_{n+1} = u_n + 2u_{n-1} \qquad (\dagger)$$

(i) Write $(\dagger)$ as a pair of first order linear difference equations. **(2 marks)**

(ii) Write down the coefficient matrix A of these equations. **(2 marks)**

(iii) Diagonalise A. **(6 marks)**

(iv) Use your answer to part (b)(iii) to solve the difference equation $(\dagger)$ in the case where $u_0 = u_1 = 1$. **(4 marks)**

(v) By considering the dominant eigenvector of A, say what happens to the ratio u_{n+1}/u_n as $n \to \infty$. **(3 marks)**

---

4. Let $P_3(\mathbb{Q})$ denote the vector space of cubic polynomials with rational coefficients, whose field of scalars is the rational number field $\mathbb{Q}$:

$$P_3(\mathbb{Q}) = \{a + bx + cx^2 + dx^3 : a, b, c, d \in \mathbb{Q}\}$$

(a) (i) Define what it means for a subset W of a vector space V to be a *subspace* of V. **(2 marks)**

(ii) Let $P_3(\mathbb{Z})$ denote the set of cubic polynomials with integer coefficients. Is $P_3(\mathbb{Z})$ a subspace of $P_3(\mathbb{Q})$? If so, show that it satisfies the required conditions; if not, say which conditions aren't satisfied. **(4 marks)**

(b) Let $\phi\colon P_3(\mathbb{Q}) \to P_3(\mathbb{Q})$ be the function that maps a cubic polynomial $f(x) = a + bx + cx^2 + dx^3$ to the polynomial $f(x-1)$ obtained by substituting $(x-1)$ for x.

(i) Write down the standard basis for $P_3(\mathbb{Q})$. **(2 marks)**

(ii) What is $\dim P_3(\mathbb{Q})$? **(1 mark)**

(iii) Show that ϕ is a linear map. **(4 marks)**

(iv) Write down a matrix representing this linear map ϕ relative to the standard basis for $P_3(\mathbb{Q})$. **(4 marks)**

(v) What is the kernel $\ker(\phi)$ of ϕ? **(4 marks)**

(vi) Let $\psi\colon P_3(\mathbb{Q}) \to P_3(\mathbb{Q})$ be the function that maps a quartic polynomial $f(x)$ to $f(x+1)$. Write down a matrix representing ψ relative to the standard basis and show it is the inverse of the matrix in part (b)(iv). **(4 marks)**

---

5. (a) (i) Define what it means for an $n \times n$ matrix to be *symmetric*. **(2 marks)**

(ii) Show that the eigenvalues of a symmetric real matrix are all real. **(5 marks)**

(b) Let $Q(x, y, z) = 3x^2 + 3y^2 + 3z^2 - 2xz$.

(i) Find a symmetric 3×3 matrix A such that Q can be written as $\mathbf{x}^T A \mathbf{x}$ where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. **(2 marks)**

(ii) Find the eigenvalues and eigenvectors of A. **(6 marks)**

(iii) Verify that the eigenvectors are pairwise orthogonal. **(3 marks)**

(iv) Normalise the eigenvectors and thereby construct an orthogonal matrix $\hat{P}$ and a diagonal matrix D such that $A = \hat{P} D \hat{P}^T$. **(3 marks)**

(v) Use the orthogonal map $\mathbf{x} = \hat{P}\mathbf{X}$, where $\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$, to diagonalise the quadratic form Q. **(3 marks)**

(vi) What is the sign property (definiteness) of Q, and why? **(1 mark)**

FORMULA SHEET

Axioms for a field

A1	$a + (b + c) = (a + b) + c$	Associative law for addition
A2	$a + 0 = 0 + a = a$	Existence of an additive identity
A3	$a + (-a) = (-a) + a = 0$	Existence of an additive inverse
A4	$a + b = b + a$	Commutative law for addition
M1	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$	Associative law for multiplication
M2	$a \cdot 1 = 1 \cdot a = a,\ a \neq 0$	Existence of a multiplicative identity
M3	$a \cdot a^{-1} = a^{-1} \cdot a = 1,\ a \neq 0$	Existence of a multiplicative inverse
M4	$a \cdot b = b \cdot a$	Commutative law for multiplication
D1	$a \cdot (b + c) = a \cdot b + a \cdot c$	Distributive law

Axioms for a vector space

V0	$\mathbf{u} + \mathbf{v} \in V$	Closure under vector addition
V1	$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative law for vector addition
V2	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative law for vector addition
V3	$\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$	Existence of zero vector (additive identity)
V4	$\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}$	Existence of negative vectors (additive inverses)
S0	$k\mathbf{v} \in V$	Closure under scalar multiplication
S1	$h(k\mathbf{v}) = (hk)\mathbf{v}$	Associative law for scalar multiplication
S2	$1\mathbf{v} = \mathbf{v}$	Multiplicative identity for scalar multiplication
S3	$(h + k)\mathbf{v} = h\mathbf{v} + k\mathbf{v}$	First distributive law
S4	$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$	Second distributive law