Imperial College London

BSc/MSci EXAMINATION April 2014

This paper is also taken for the relevant Examination for the Associateship

MECHANICS, VIBRATIONS AND WAVES

For 1st-Year Physics Students

Tuesday, 29th April 2014: 14:00 to 16:00

Answer **ALL** *questions.*
Marks will be awarded for showing working and words of explanation throughout
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 4 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 4 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) A point particle of mass 570 g is projected at 12 ms^{-1} into a resistive medium that acts such that the drag force is given by $0.32v$ N in the opposite direction to the particle's velocity which is of magnitude v ms^{-1}. No other forces act on the particle. Find the speed of the particle after 2.0 s.

[5 marks]

(ii) A particle moving in free space makes an elastic collision with a stationary particle of the same mass. The collision is not head-on so both particles move following the collision. Show that the angle between the trajectories of the two particles after the collision is $\frac{\pi}{2}$ radians.

[5 marks]

(iii) The potential energy, U, shared between two water molecules is found empirically to be given by the formula

$$U = 5 \times 10^{-21} \left(\frac{10^{-114}}{r^{12}} - \frac{10^{-57}}{r^{6}} \right) \text{ joules}$$

where r is the distance between the centres of the molecules in metres. Use this formula to estimate the equilibrium separation of two water molecules and the amount of energy required to completely separate them from the equilibrium.

[5 marks]

(iv) A point mass m is located a distance d from a point mass $2m$. Where should an object be placed on the line between the two point masses so that the net gravitational force on the object due to the point masses is zero? Describe whether the object is in stable or unstable equilibrium, and whether it is in a state of compression, tension or neither.

[5 marks]

(v) What is the moment of inertia of a particle of mass m a distance r from an axis of rotation? Starting from first principles, show that the moment of inertia of a thin rod of length L and mass M through its centre is $\frac{1}{12}ML^2$. Show all working and use a sketch if necessary.

[5 marks]

[Total 25 marks]

2. The 1-D wave equation for shallow water waves (wavelength $\gg$ depth) is

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{1}{hg}\frac{\partial^2 \eta}{\partial t^2}$$

where η is the longitudinal displacement, h is the depth of the water and g is the gravitational acceleration.

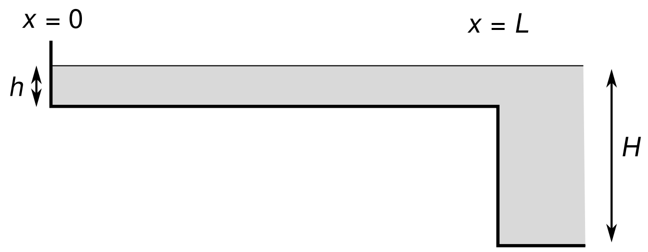
(i) Explain what is meant by dispersion and what are normal and anomalous dispersion. Use the solution $\eta = e^{i(\omega t - kx)}$ to derive the dispersion relationship for shallow water waves and hence show that the phase velocity is $v_p = \sqrt{hg}$. What dispersion do these waves exhibit and how would a pulse of such waves change on propagation? [6 marks]

(ii) Treating the water as an incompressible fluid, show that for $\psi \ll h$, the vertical displacement ψ of the surface due to the wave is related to the longitudinal displacement by

$$\psi \approx -h\frac{\partial \eta}{\partial x}$$

(Hint: consider the effect of the wave on the thickness and height of an infinitesimal element δx and hence write down an expression for the volume of such an element.) [5 marks]

A channel of water has length L and depth h. It is closed at $x = 0$. At $x = L$ it opens onto to a region of deeper water with depth $H \gg h$. The boundary condition at $x = L$ is that $\psi = 0$.



(iii) Describe the principle of superposition and explain why it applies for shallow water waves. A wave $\eta_1(x, t) = Ae^{i(\omega t + kx)}$ is incident to the closed end of the water channel. Write down a general expression for the reflected wave. Write down the boundary condition for $x = 0$ and hence show that the resulting disturbance is given by

$$\eta(x, t) = 2Aie^{i\omega t}\sin(kx)$$

[6 marks]

(iv) By considering the boundary at $x = L$, derive an expression for the frequencies of the standing wave modes and show that they are in the ratios 1 : 3 : 5 : ....

[6 marks]

(v) On a single set of axes carefully sketch both the longitudinal displacement η and the vertical displacement ψ against x for the fundamental mode. On a second set of axes do the same for the second harmonic. Explain what are meant by

[This question continues on the next page . . .]

longitudinal and transverse waves and discuss the motion of the water waves with reference to your plots.

[7 marks]

(vi) The tidal range around the south west coast of England is among the largest in the world, a consequence of the fundamental standing wave resonance over the continental shelf coinciding with the tidal driving period of 12 hours. The sea is of approximately uniform depth from the coast up to the edge of the continental shelf whereupon the sea floor drops away rapidly to the oceanic abyss. If the continental shelf extends to 300 km from the coast, estimate its depth.

[5 marks]

[Total 35 marks]

3. This question concerns kinematics and motion in a uniform gravitational field.

(i) Define (a) velocity and (b) acceleration. Give two examples of when a constant acceleration is observed in nature.

[5 marks]

(ii) A tap (faucet) pushes out a stream of water of circular cross section of radius r_0 directly downwards at an exit speed u.

(a) Given that water is an incompressible fluid use a purely kinematic argument to explain why the radius of a water column gets narrower with distance below the tap.

(b) Assuming a constant downward acceleration, g, of water, show that the radius of the water column and a distance s below the tap is given by

$$r = r_0 \left(1 + \frac{2gs}{u^2}\right)^{-\frac{1}{4}}$$

State any assumptions you make. Hint: find an expression for the volume of water per unit time exiting the tap and consider what volume per unit time will pass a point at the distance s below.

[10 marks]

(iii) Use the equation derived in 3(ii)(b) to explain why the stream from a tap narrows very quickly as it exits the tap before forming a more uniform column far below. Go on to provide a qualitative explanation as to why the stream does not get infinitesimally narrow but eventually splits into individual droplets.

[5 marks]

[Total 20 marks]

4. This question is about gyroscopes and precession.

A simple gyroscope is a flywheel allowed to spin freely about a spindle. If the flywheel spins with a high enough angular velocity about the spindle and one end is placed on a raised pivot then the gyroscope can be seen to precess around the vertical axis.

(i) Sketch a free body diagram for a gyroscope when it is steadily precessing. Explain the nature of each of the forces.

[5 marks]

(ii) Consider the case when a gyroscope precesses steadily in the plane with the spindle remaining parallel to the floor. By considering the changing angular momentum of the gyroscope, and with use of vector diagrams, show that the gyroscope precesses with an angular velocity along the vertical axis of magnitude

$$\frac{mgr}{I\omega}$$

where m is the flywheel's mass, I its moment of inertia about the spindle, ω the magnitude of its angular velocity along the spindle axis and r the distance from the pivot to the flywheel. The mass of the spindle can be assumed to be much smaller than the mass of the flywheel.

Use the formula to discuss how the flywheel's size and mass affect the precession rate for a given flywheel rotation rate.

[10 marks]

(iii) As the flywheel slows down through friction the formula suggests the precession rate increases. Explain where the necessary energy for this comes from and comment on the angular momentum of the system along the vertical axis.

[5 marks]

[Total 20 marks]